University of Alabama
Department of Physics and Astronomy

PH 102-2 / LeClair Spring 2008

In-class exercise: Special Relativity
Solutions

Instructions:

1. Answer all questions below.
2. Show your work for full credit, attach sheets as necessary.
3. You are encouraged to work in groups, turn in one paper per group
4. *List all group members names.*

1. Just to be clear, we will label quantities measured in the earth's reference frame with primes ($\prime$), and quantities without primes are with respect to the probe's reference frame. The relative velocity between the earth and the probe is the same from both reference frames, $v = v'$. From the probe's (and its generators') reference frame, it is the observers on earth that are moving. The observers on earth should then see a *longer* time interval compared to the proper time measured on the probe:

$$\Delta t' = \gamma \Delta_p = \frac{15\,\text{yrs}}{\sqrt{1 - \frac{(0.76c)^2}{c^2}}} \approx 23\,\text{yrs}$$

According to observers on earth, the generators should fail after a period of $\Delta t'$. Also according to them, the probe should have traveled a distance $d' = v'\Delta t'$ - the earth-bound observers watched the probe travel for an interval $\Delta t'$ at a constant velocity of v' in their reference frame:

$$d' = v'\Delta t' = (23\,\text{yrs})\left(3 \times 10^8\,\text{m/s}\right) \approx 2.2 \times 10^{17}\,\text{m}$$

Alternatively, we could express the distance in light years - the distance light travels in one year. To do that, we just have to realize that $0.76c$ means the probe travels at 76% of the speed of light:

$$d' = (0.76\,\text{light speed})\,(23\,\text{yrs}) \approx 18\,\text{light-years}$$

Finally, how about the distance traveled according to the probe? That is just the relative velocity multiplied by the elapsed time *from the probe's reference frame, i.e.,* the proper time:

$$d = v\Delta t = (15\,\text{yrs})\left(3 \times 10^8\,\text{m/s}\right)(0.76) = 1.1 \times 10^{17}\,\text{m} = 11\,\text{light-years}$$

2. The rest energy of a particle is by definition independent of its velocity. What would make this problem easier is knowing the mass of the proton in units of MeV/c^2 ... but we'll pretend we don't know that. All we know is the conversion between eV and J, and the proton mass.

$$
\begin{aligned}
E_R &= mc^2 = \left(1.67\times10^{-27}\,\mathrm{kg}\right)\left(3\times10^{8}\,\mathrm{m/s}\right)^2 = 1.50\times10^{-10}\,\mathrm{J}\\
1.50\times10^{-10}\,\mathrm{J} &= \left(1.50\times10^{-10}\,\mathrm{J}\right)\left(\frac{1\,\mathrm{eV}}{1.60\times10^{-19}\,\mathrm{J}}\right) = 9.39\times10^{8}\,\mathrm{eV} = 939\,\mathrm{MeV}
\end{aligned}
$$

Now that we have calculated the rest energy mc^2, the total energy is easy to calculate, once we find γ for $v = 0.95c$. First, let's find γ, and then E_{tot}:

$$\gamma = \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}} \approx 3.2$$

Of course, you could have looked this up in the notes, there is a handy table just for this!

$$E_{\mathrm{tot}} = \gamma mc^2 = 3.2\,(939\,\mathrm{MeV}) = 3007\,\mathrm{MeV} \approx 3.01\,\mathrm{GeV}$$

Finally, remember that the *total* energy is the sum of the rest and kinetic energies: $E_{\mathrm{tot}} = E_R + KE$. We can find kinetic energy by just subtracting the rest energy from the total energy:

$$KE = E_{\mathrm{tot}} - E_R = 3007\,\mathrm{MeV} - 939\,\mathrm{MeV} \approx 2070\,\mathrm{MeV} = 2.07\,\mathrm{GeV}$$

Recall that the prefix "G" means 10^9.

3. Straight out of the practice problems in the notes. This is just a problem of relativistically adding velocities, if we can keep them all straight. Let the unprimed system denote velocities measured relative to the earth, and the primed system those measured relative to the enterprise. We have, then:

$$
\begin{aligned}
v_e = 0.900c &= \text{Enterprise relative to earth}\\
v_k = 0.700c &= \text{Klingon ship relative to earth}\\
v_k' = ? \quad &= \text{Klingon ship, relative to Enterprise}
\end{aligned}
$$

Since the Enterprise is moving faster relative to the earth than the Klingon ship, that means that from the Enterprise's point of view, the Klingons are actually moving backwards toward them. If we plug what we know into the velocity addition formula ...

$$v_k = \frac{v_e + v_k'}{1 + \frac{v_e v_k'}{c^2}}$$

It takes a bit of algebra, but we can readily solve this for v_k':

$$v_k' = \frac{v_e - v_k}{1 - \frac{v_e v_k}{c^2}}$$

Not so surprisingly, what we have just done is to re-write the 'velocity addition formula' as a 'velocity subtraction formula.' It is just rearranging same formula (you can verify that both equations above are equivalent ...), but the second form is far more convenient for our present purposes. In fact, both forms are given in the notes as well, but it is worth seeing again that you really only need one of them, they are not both unique.

Anyway: we can find the velocity of the Klingon ship relative to the enterprise in terms of both ships' velocities relative to the earth. In the limit that both velocities are much smaller than c, we see that $v_k' \approx v_e - v_k = 0.200c$, just as we would expect from normal Newtonian physics. Since in this case, neither velocity is negligible compared to c, the actual v_k' will be significantly larger. At this point, we can just plug in the numbers we have and see:

$$\begin{aligned} v_k' &= \frac{v_e - v_k}{1 - \frac{v_e v_k}{c^2}} \\ &= \frac{0.900c - 0.700c}{1 - \frac{(0.900c)(0.700c)}{c^2}} \\ &= \frac{0.200c}{1 - (0.900)(0.700)} = \frac{0.200c}{0.37} \\ v_k' &\approx 0.541c \end{aligned}$$

So, as far as the crew of the Enterprise is concerned, they are overtaking the Klingon ship at a rate of $0.541c$.

4. This problem is not as hard as it seems, we just need to use our expressions for kinetic energy and rest energy and set one equal to $\frac{2}{3}$ of the other:

$$\begin{aligned} KE &= \frac{2}{3}E_R \\ (\gamma - 1)\,mc^2 &= \frac{2}{3}mc^2 \\ (\gamma - 1)\,\cancel{mc^2} &= \frac{2}{3}\cancel{mc^2} \\ \Longrightarrow \gamma - 1 &= \frac{2}{3} \\ \Longrightarrow \gamma &= \frac{5}{3} \end{aligned}$$

Now we just need to use the definition of γ and solve for v. If your algebra comes out right, you should get this:

$$v = c\sqrt{1 - \frac{1}{\gamma^2}} \approx 0.8c \tag{1}$$

5. The space traveler wants to be traveling along the *longer* dimension of the billboard, since only lengths along the direction of motion will appear shorter. If the billboard is to appear square, the longer dimension should be contracted by a factor 2. This means we need $\gamma = 2$, since the length according to the observer in motion appears shorter by a factor γ. Using the equation for v in terms of γ above, you should find $v = 0.87c$.

6. Once again, we would like a length contraction of a factor 2, *i.e.*, $\gamma = 2$, so $v \approx 0.87c$