University of Alabama
Department of Physics and Astronomy

PH 102 / LeClair Summer II 2012

Problem Set 1: Solutions

1. A classic "paradox" involving length contraction and the relativity of simultaneity is as follows: Suppose a runner moving at $0.75c$ carries a horizontal pole 15 m long toward a barn that is 10 m long. The barn has front and rear doors. An observer on the ground can instantly and simultaneously open and close the two doors by remote control. When the runner and the pole are inside the barn, the ground observer closes and then opens both doors so that the runner and pole are momentarily captured inside the barn and then proceed to exit the barn from the back door. Do both the runner and the ground observer agree that the runner makes it safely through the barn? Justify your answer. *We will go over this in class on Monday.*

Given the relative velocity between the reference frames $v = 0.75c$, we will have to account for length contraction and the relativity of simultaneity. One point to make clear: in the barn's reference frame, the doors close and then open immediately, and at the same time. They do not stay closed.

From the point of view of the person in the barn, the pole is moving toward them at velocity $v = 0.75c$, and its length is contracted by a factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1.51 \tag{1}$$

Thus, instead of 15 m the person in the barn sees the pole as having a length $15/\gamma \approx 9.9\,\text{m}$, and observes the runner to make it safely through.

From the point of view of the runner, the barn is in motion, and appears shortened by a factor γ to $10/\gamma \approx 6.6\,\text{m}$. When do the doors close from the runner's point of view? The runner will not see the doors close at the same time, since the runner is in motion relative to the doors and they are spatially separated. If the person in the barn sees the doors closing with time delay $\Delta t = 0$ (i.e., simultaneously), the runner sees a time delay $\Delta t'$ governed by the Lorentz transformation:

$$\Delta t' = \gamma \left(\Delta t + \frac{v \Delta x}{c^2} \right) \tag{2}$$

Here Δx is the separation between events – the front door closing and the back door closing – in the barn's reference frame, i.e., the length of the barn L_B. If we call the front door opening the first event and the back door closing the second event, then we are implying the runner's heading is positive, and that makes the distance Δx negative. This means that the runner sees the *front door close first* and then the back:

$$\Delta t' = -\frac{\gamma v L_B}{c^2} \approx 37.8\,\text{ns} \tag{3}$$

From the person in the barn's point of view, he will see the runner reach the front door, then close it immediately. At that point, the runner also thinks the pole is at the front door, and that part of its 15 m length (according to him) is still sticking out of the back of the barn (which is 6.6 m long according to the runner). The runner then has time $\Delta t'$ to get that much of the pole in through the back door before it closes. The runner doesn't need to get out of the barn completely – remember, the doors will close and then open again immediately, the runner just can't be caught in the middle of either door while that happens. So, the runner must go the length of the pole minus the length of the barn before $\Delta t'$ passes. That length is:

$$L'_p - L'_B = 15\,\text{m} - \frac{10\,\text{m}}{\gamma} \approx 8.38\,\text{m} \tag{4}$$

At velocity v, going this length takes $8.38\,\text{m}/0.75c \approx 37.2\,\text{ns}$, leaving 0.6 ns to spare before the rear door closes. Thus, both the runner and the observer in the barn agree that the runner makes it through. Thus, both observers agree that the pole does not smash into the doors, but makes it through the barn safely.

We have only given a quick discussion here, one can perform a much more rigorous analysis. For further discussion see, for example:

`http://hyperphysics.phy-astr.gsu.edu/hbase/relativ/polebarn.html`
`http://en.wikipedia.org/wiki/Ladder_paradox`
`http://www.xs4all.nl/~johanw/PhysFAQ/Relativity/SR/barn_pole.html`

And an applet for good measure:

`http://webphysics.davidson.edu/physlet_resources/special_relativity/ex1.html`

(The links should be clickable.)

2. An astronaut takes a trip to Sirius, which is located a distance of 8 lightyears from the Earth. The astronaut measures the time of the one-way journey to be 6 yr. If the spaceship moves at a constant speed of $0.8c$, how can the 8-ly distance be reconciled with the 6-yr trip time measured by the astronaut?

The 8 light-year distance is that measured according to the stationary observers, viz., the earthlings. According to the astronaut, who is in motion relative to Earth and Sirius, the distance is shortened by a factor γ:

$$L_{\text{astronaut}} = \frac{1}{\gamma}L_{\text{earthlings}} = (8\,\text{light-years})\left(\sqrt{1-(0.8c)^2/c^2}\right) = (8\,\text{light-years})\,(0.6) = 4.8\,\text{light-years} \tag{5}$$

The astronaut measures the trip to take 6 yr, which means the astronaut would report a velocity of

$$v = \frac{4.8\,\text{light-years}}{6\,\text{yr}} = 0.8c \tag{6}$$

Thus, there is no paradox: the difference in measured times is due to time dilation/length contraction. More to the point: we can't divide one observers distance by another observer's time and expect to get sensible answers unless they are in the same reference frame!

3. A proton is accelerated to a velocity $v = 0.999c$ and sent down an evacuated metal tube 100 m long. Take the speed of light as $c = 3.0 \times 10^8$ m/s.

(a) In the protons reference frame, how long is the tube? **(b)** In the protons frame, how long does it take to traverse the length of the tube? **(c)** In the laboratory frame, how long does it take for the proton to traverse the length of the tube?

Since the proton is moving with respect to the tube, it will appear shortened from the proton's point of view by a factor γ from its proper rest length of $L = 100$ m:

$$\gamma = \frac{1}{\sqrt{1-0.999^2}} \approx 22.4 \tag{7}$$

$$L'_{\text{proton}} = \frac{L}{\gamma} = \frac{100\,\text{m}}{\gamma} \approx 4.46\,\text{m} \tag{8}$$

In the proton's rest frame, we would find the time taken by dividing the distance according to the proton by the velocity according to the proton. The proton sees the tube as having a distance L'_{proton}, and travel at velocity $v = 0.999c$ relative to the tube:

$$t'_{\text{proton}} = \frac{L'_{\text{proton}}}{v} \approx 1.48 \times 10^{-8}\,\text{s} = 14.8\,\text{ns} \tag{9}$$

In the laboratory frame, the proton travels at $0.999c$ and must traverse 100 m, so the time elapsed is

$$t_{\text{proton}} = \frac{L}{v} \approx 3.34 \times 10^{-7}\,\text{s} = 0.334\,\mu\text{s} \tag{10}$$

4. An atomic clock aboard a spaceship runs slow compared to an Earth-based atomic clock at a rate of 2.0 seconds per day. What is the speed of the spaceship?

The proper time t_p is that measured on earth, while a dilated time $t = \gamma t_p$ is measured on the ship. If the clock aboard the ship is 2 s per day slow, then using $1\,\text{day} = 86400\,\text{s}$

$$t - t_p = \gamma t_p - t_p = (\gamma - 1)\,t_p = (\gamma - 1)\,(86400\,\text{s}) = 2\,\text{s} \tag{11}$$

This gives

$$\gamma = \frac{2}{86400} + 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1.00002315 \tag{12}$$

Solving for v, we find $v \approx 0.0068c$.