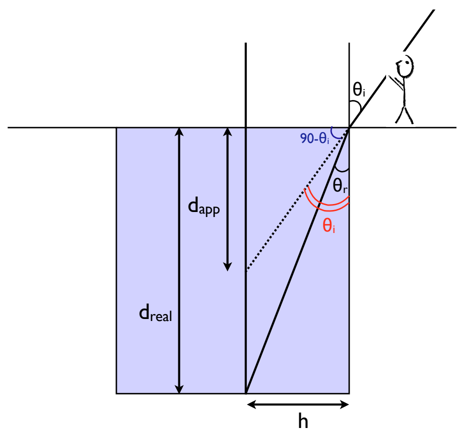
University of Alabama
Department of Physics and Astronomy

PH 102 / LeClair Summer II 2012

Problem Set 5: Solutions

1. What is the apparent depth of a swimming pool in which there is water of depth 3 m: **(a)** when viewed from normal incidence, and **(b)** when viewed at an angle of 60° with respect to the surface normal? The refractive index of water is 1.33.

As always, we first need to draw a little picture of the situation at hand.



Note that the angle of incidence θ_i is with respect to the *normal* of the water's surface itself, rather than with respect to the air-water interface, as that is our usual convention. That means we are interested in the incident angles for the observer of 90° (normal) and 60°. The depth of the pool will be $d_{\text{real}} = 3\,\text{m}$. If an observer views the bottom of the pool with an angle θ_i with respect to the surface normal, refracted rays from the bottom of the pool will be bent away from the surface normal on the way to their eyes. That is, rays emanating from the bottom of the pool will make an angle $\theta_r < \theta_i$ with respect to the surface normal, and rays exiting the pool will make an angle θ_i with the surface normal. This is owing to the fact that the light will be bent *toward* the normal in the faster medium, the air, on exiting the water.

What depth does the observer actually see? They see what light would do in the absence of refraction, the path that light rays would appear to take if the rays were not "bent" by the water. In this case, that means that the observer standing next to the pool would think they saw the light rays coming from an angle θ_i with respect to the surface normal (dotted line in the pool). The *lateral* position of the bottom of the pool would remain unchanged. If the real light rays intersect the bottom of the pool a distance h from the edge, then the apparent bottom of the pool is also a distance h from the edge of the pool. Try demonstrating this with a drinking straw in a glass of water!

So what to do? First off, we can apply Snell's law. If the index of refraction of air is n_a, and the water has an index of refraction n_w, then

$$n_w \sin\theta_r = n_a \sin\theta_i \tag{1}$$

We can also use the triangle defined by d_{real} and h:

$$\tan\theta_r = \frac{h}{d_{\text{real}}} \tag{2}$$

as well as the triangle defined by d_{real} and h[i]:

$$\tan(90-\theta_i) = \frac{d_{\text{app}}}{h} = \frac{1}{\tan\theta_i} \tag{3}$$

Solving the last two equations for h,

$$h = d_{\text{real}} \tan\theta_r = d_{\text{app}} \tan\theta_i \tag{4}$$

$$\implies \quad d_{\text{app}} = d_{\text{real}} \left[\frac{\tan\theta_r}{\tan\theta_i}\right] \tag{5}$$

From Snell's law, we already have a relationship between θ_r and θ_i already:

$$\theta_r = \sin^{-1}\left[\frac{n_a \sin\theta_i}{n_w}\right] \tag{6}$$

Putting everything together,

$$d_{\text{app}} = \frac{d_{\text{real}}}{\tan\theta_i} \tan\theta_r = \frac{d_{\text{real}}}{\tan\theta_i} \left[\tan\left(\sin^{-1}\left[\frac{n_a \sin\theta_i}{n_w}\right]\right)\right] \tag{7}$$

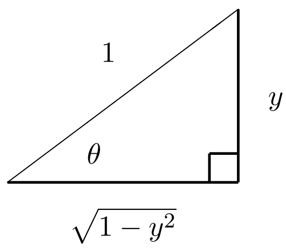
If you just plug in the numbers at this point, you would be able to solve part (a) of the question. Do that, and you should find $d_{\text{app}} \approx 1.49\,\text{m}$.

For part (b), you would have a problem using this expression as-is. One of the angles is $\theta_i = 0$, normal incidence, which means we have to divide by zero in the expression above. Dividing by zero is worse than drowning kittens, far worse. Thankfully, we know enough trigonometry to save the

[i]Along with an identify for $\tan\theta$, *viz.*, $\tan(90-\theta) = 1/\tan\theta$

poor kittens.

We can save the kittens by remembering an identity for $\tan\left[\sin^{-1}(y)\right]$. If we have an equation like $\theta = \sin^{-1} y$, it implies $\sin\theta = \theta$. This means θ is an angle whose sine is y. If θ in a right triangle, it has an opposite side y and a hypotenuse 1 so that $\sin\theta = 1$, making the adjacent side $\sqrt{1-y^2}$. The tangent of angle θ must then be $y/\sqrt{1-y^2}$. Something like this:



$$\theta = \sin^{-1}(y)$$
$$\tan\theta = y/\sqrt{1-y^2}$$

Thus, going back to our original example $\theta = \sin^{-1} y$, it must be true that

$$\tan\left[\sin^{-1}(y)\right] = \frac{y}{\sqrt{1-y^2}} \tag{8}$$

Using this identity in our equation for d_{app},

$$d_{\text{app}} = \frac{d_{\text{real}}}{\tan\theta_i}\left[\frac{n_a\sin\theta_i}{n_w\sqrt{1-\left[\frac{n_a\sin\theta_i}{n_w}\right]^2}}\right] = n_a\frac{d_{\text{real}}}{\tan\theta_i}\left[\frac{\sin\theta_i}{\sqrt{n_w^2-n_a^2\sin^2\theta_i}}\right] = \frac{n_a d_{\text{real}}\cos\theta_i}{\sqrt{n_w^2-n_a^2\sin^2\theta_i}} \tag{9}$$

Viewed from normal incidence with respect to the surface means $\theta_i = 0$ – looking straight down at the surface of the water. In this case, $\sin\theta_i = 0$, and the result is simple:

$$d_{\text{app}} = d_{\text{real}}\frac{n_a}{n_w} \approx 2.26\,\text{m} \tag{10}$$

Viewed from 60° with respect to the normal gives, as before,

$$d_{\text{app}} = d_{\text{real}}\cos 60\left[\frac{1}{\sqrt{1.33^2-\sin^2 60}}\right] \approx 1.49\,\text{m} \tag{11}$$

There are easier ways to solve the normal incidence problem, without endangering any kittens whatsoever. Solving that problem, however, is a special case, and of limited utility. You would still have to solve the case of 60° incidence separately. I wanted to show you here that solving the general problem just once is all you need to do, so long as you are careful enough. The easier way to solve *only* part (b) would be to use the result we developed for a flat refracting surface, $q = -(n_2/n_1)p$. In the present case, treat the apparent depth as q, the real depth as p, which implies n_2 is air and n_1 is water. Since q is on the same side as p, it is *negative*, so the *depth* would be $-q$ away from the interface. That leads us to the following:

$$-q = d_{\text{app}} = \frac{n_a}{n_w} p = \frac{n_a}{n_w} d_{\text{real}} \qquad \text{normal incidence only} \tag{12}$$

which is in perfect agreement with our expression above, derived from the more general problem.

2. A point source of light is placed at a fixed distance l from a screen. A thin convex lens of focal length f is placed somewhere between the source and screen, a distance q from the screen and p from the source. The lens is moved back and forth between the source and screen, but both screen and source remain fixed, thus $p + q = l$ at all times. What is the minimum value of l such that a focused image will be formed at two different positions of the lens?

What we are told is that $p + q = l$ at all times. We can also use the lens equation which relates p, q, and the focal length f: $1/p + 1/q = 1/f$. What will determine the minimum value for l? If we want to focus an image, we need to have a real image, not a virtual one. For a convex lens, this will happen only when $p > f$. We also know that q must be positive for the image to be real. Combining these two facts, certainly we must have $l = p+q > f$ since it must be separately true that $p > f$ and $q > 0$. Thus, we need only make sure that q is real and positive in order to have a real image formed – if q is negative, the image is virtual. If q is imaginary, so is the image . . .

Mathematically, what we want to do is combine the two expressions we have into a single one that relates q to f and l (by eliminating p), since f and l are the fixed constants in this problem. We then enforce that q is real and positive, and the conditions under which this is true will give the minimal value of l. There are several ways to combine the equations and solve for q. Here are two ways to go about it, each giving the same result in the end:

Method 1

$$l = p + q \quad \Longrightarrow \quad p = l - q \tag{13}$$

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{1}{f} - \frac{1}{l-q} \tag{14}$$

$$\frac{1}{q} = \frac{l-q}{f\,(l-q)} - \frac{f}{f\,(l-q)} \tag{15}$$

$$\frac{1}{q} = \frac{l-q-f}{f\,(l-q)} \tag{16}$$

$$fl - fq = ql - q^2 - qf \tag{17}$$

$$q^2 - lq + fl = 0 \tag{18}$$

Method 2

$$l = p + q \quad \Longrightarrow \quad p = l - q \tag{19}$$

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq} = \frac{l}{pq} \tag{20}$$

$$\frac{1}{f} = \frac{l}{pq} = \frac{l}{q\,(l-q)} \tag{21}$$

$$ql - q^2 = fl \tag{22}$$

$$q^2 - lq + fl = 0 \tag{23}$$

Both cases lead us to:

$$q = \frac{1}{2}\left(l \pm \sqrt{l^2 - 4fl}\right) \tag{24}$$

From the solution to the quadratic above, we can see that there are two real image positions q_1 and q_2 when the factor under the square root is positive, which occurs when $l^2 > 4fl$ or $l > 4f$. When the length l is exactly four times the focal length, $l = 4f$, there is only one solution to the quadratic, and this is the minimum l which gives q real and positive. Thus, the critical position is when $l = 4f$, which results in $q = \frac{l}{2}$ and $q = p$.

3. Referring to the previous question, it is found that at one position the image height is a, while at the second, the image height is b. Show that the height of the object is $\sqrt{ab}$.

In order to find the image height, we need the magnification factor. To get the magnification factor, we need p and q for both image positions. The previous problem gives us two image positions q_1 and q_2 for any $l > 4f$:

$$\{q_1, q_2\} = \frac{1}{2}\left(l \pm \sqrt{l^2 - 4fl}\right) \tag{25}$$

or more explicitly,

$$q_1 = \frac{1}{2}\left(l + \sqrt{l^2 - 4fl}\right) \tag{26}$$

$$q_2 = \frac{1}{2}\left(l - \sqrt{l^2 - 4fl}\right) \tag{27}$$

There will be corresponding values for the object positions p in each case, given by $p_i = l - q_i$.

$$p_1 = l - q_1 = \frac{1}{2}\left(l - \sqrt{l^2 - 4fl}\right) \tag{28}$$

$$p_2 = l - q_2 = \frac{1}{2}\left(l + \sqrt{l^2 - 4fl}\right) \tag{29}$$

In fact, $p_1 = q_2$ and $p_2 = q_1$, and this is no accident: the problem is completely symmetric if we swap image and object, so it must come out this way.

We know we have two positions for image and object, and we know that the image height is different in each case (while of course the object height is constant). Knowing the magnification factor for a spherical lens is $M = -q/p$, we can relate the two heights in each case. Let the images have heights a and b for cases 1 and 2, respectively, and call the object height h. Then

$$M_1 = \frac{a}{h} = -\frac{q_1}{p_1} \tag{30}$$

$$M_2 = \frac{b}{h} = -\frac{q_2}{p_2} \tag{31}$$

This doesn't do much good yet, as we know neither h, M_1, nor M_2. However, we can multiply the two equations we have:

$$\frac{ab}{h^2} = \frac{q_1 q_2}{p_1 p_2} \tag{32}$$

Now remember that we noted $p_1 = q_2$ and $p_2 = q_1$, and we are done:

$$\frac{ab}{h^2} = \frac{q_1 q_2}{p_1 p_2} = \frac{q_1 q_2}{q_2 q_1} = 1 \tag{33}$$

$$\Longrightarrow \qquad h = \sqrt{ab} \tag{34}$$

If you didn't notice this, you can work it out explicitly, it isn't much more work. First, $q_1 q_2$. Note that the product has the form $(x+y)(x-y) = x^2 - y^2$ to make things easier.

$$q_1 q_2 = \left[\frac{1}{2}\left(l + \sqrt{l^2 - 4fl}\right)\right]\left[\frac{1}{2}\left(l - \sqrt{l^2 - 4fl}\right)\right] = \frac{1}{4}\left(l + \sqrt{l^2 - 4fl}\right)\left(l - \sqrt{l^2 - 4fl}\right) \tag{35}$$

$$= \frac{1}{4}\left[l^2 - \left(l^2 - 4fl\right)\right] = fl \tag{36}$$

Not as messy as it looks. For the denominator, $p_1 p_2$, we know the result will be the same since $p_1 = q_2$ and $p_2 = q_1$:

$$p_1 p_2 = \left[\frac{1}{2}\left(l - \sqrt{l^2 - 4fl}\right)\right]\left[\frac{1}{2}\left(l + \sqrt{l^2 - 4fl}\right)\right] = \frac{1}{4}\left[l^2 - \left(l^2 - 4fl\right)\right] = fl \tag{37}$$

Different method, same result: $q_1 q_2 / p_1 p_2 = 1$, and thus $h = \sqrt{ab}$.

4. A spherical mirror which forms only virtual images has a radius of curvature of $R = 0.5\,\text{m}$. **(a)** Is the mirror concave or convex? What is the focal length of the mirror? **(b)** Where should an object be placed to obtain a magnification of $+0.5$?

If the images are *always* virtual, the mirror must be **convex**. A concave mirror will only form virtual images if $p < f$. The focal length is $|f| = R/2$ in general. When we form a virtual mirror with a convex lens, we are forming the image on the far side of the mirror, where both q and the focal length f are negative. This means $f = -R/2 = -0.25\,\text{m}$.

We desire $M = +0.5$, and we know for a spherical mirror $M = -q/p$. This tells us that $q = -Mp = -0.5p$. As noted above, since the image is virtual and formed behind the mirror, $q < 0$, and this is consistent with both M and p being positive (as p must be if the object is in front of the mirror). We can use this in the mirror equation to find p:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p} - \frac{1}{Mp} = \frac{1}{p}\left(1 - \frac{1}{M}\right) = \frac{1}{f} = -\frac{2}{R} \tag{38}$$

$$\implies \qquad p = -\frac{R}{2}\left(1 - \frac{1}{M}\right) = \frac{R}{2}\left(\frac{1}{M} - 1\right) = \frac{0.25\,\mathrm{m}}{2}\left(\frac{1}{0.5} - 1\right) = 0.25\,\mathrm{m} \tag{39}$$

Thus, the object should be placed at $p = 0.25\,\mathrm{m}$, which gives $q = -0.125$ with $M = +0.5$, consistent with $f = -R/2 = -0.25\,\mathrm{m}$.