University of Alabama
Department of Physics and Astronomy

PH 102-2 / LeClair — Spring 2008

Problem Set 1 SOLUTIONS

1. $0.87c$**.** The proper length of a meter stick, measured in its own reference frame, is obviously 1 m. For a moving observer to see the meter stick as only $L'=0.5\,\mathrm{m}$ long, we need a length contraction of a factor 2:

$$L' = \frac{L_p}{\gamma} \quad \Longrightarrow \quad \frac{L_p}{L'} = \gamma = 2$$

Thus, for the meter stick to be contracted by a factor 2, we need $\gamma=2$. Using the equation for v in terms of γ above, you should find $v=0.87c$:

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\\
\sqrt{1-\frac{v^2}{c^2}} &= \frac{1}{\gamma}\\
1-\frac{v^2}{c^2} &= \frac{1}{\gamma^2}\\
v^2 &= c^2\left(1-\gamma^2\right)\\
v &= c\sqrt{1-\frac{1}{\gamma^2}} = c\sqrt{1-\frac{1}{4}} = \frac{c\sqrt{3}}{2} \approx 0.87c
\end{aligned}$$

2. $0.305c$**.** We can use the result of the last problem here - once again, we know γ, and want to find the corresponding v. Really, just an exercise to make sure you have your algebra down cold ...

$$v = c\sqrt{1-\frac{1}{\gamma^2}} = c\sqrt{1-\frac{1}{1.05^2}} \approx 0.305c$$

3. **4.59 s.** The proper time is that measured by in the reference frame of the pendulum itself, $\Delta t_p = 2.00\,\mathrm{sec}$. The moving observer has to observe a *longer* period $\Delta t'$ for the pendulum, since from the observer's point of view, the pendulum is moving relative to it. Observers always perceive clocks moving relative to them as running slow. The factor between the two times is just γ:

$$\Delta t' = \gamma \Delta t_p = \frac{2.0\,\mathrm{s}}{\sqrt{1-\frac{0.900^2 c^2}{c^2}}} = \frac{2.0\,\mathrm{s}}{\sqrt{1-0.81}} = \frac{2.0\,\mathrm{s}}{\sqrt{0.436}} \approx 4.59\,\mathrm{s}$$

4. $0.87c$ Once again, the factor between the two times is just γ. The clock at rest measures the proper time Δt_p. If the moving clock runs only half as fast, its time intervals $\Delta t'$ are twice as long $\Delta t' = 2\Delta t_p$. Thus:

$$\frac{\Delta t'}{\Delta t_p} = \gamma = 2$$

From the first problem, we know that $\gamma = 2$ occurs when $v \approx 0.87c$

5. $23.4\,\mathbf{m}$ We presume that the motion is purely along the direction of the spaceship's length. Since length contraction occurs only along the direction of motion, the width is unaffected, it still appears to be 25.0 m for the external observer. Along the direction of motion, the length should appear contracted by a factor γ. remember that the proper length is that measured at rest. As usual, the primed quantities are for the external observer.

$$\begin{aligned} L' &= \frac{L_p}{\gamma} \\ \gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.95^2}} \approx 3.2 \\ \Longrightarrow \quad L' &= \frac{75\,\text{m}}{3.2} \approx 23.4\,\text{m} \end{aligned}$$

6. $0.99c$ The first thing we need to do in order to avoid confusion is label everything properly. We will say the observer on earth is in the unprimed reference frame, and those in the first ship are in the primed frame. Since the spacecraft are moving in opposite directions, *one of them has to be negative.* Let us say that spaceship 1 is moving in the positive direction, so that spaceship 2 has a negative velocity.

$$\begin{aligned} v_1 &= \text{velocity of first ship observed from earth} = 0.8c \\ v_2 &= \text{velocity of second ship observed from earth} = -0.9c \\ v_2' &= \text{velocity of second ship observed from first} = \,? \end{aligned}$$

What we want to find is v_2', the relative velocity of the two ships. If we completely ignore relativity just for a minute, what would the answer be? The relative velocity of the two ships would just be the velocity of one minus the other - we subtract the two velocities as measured from earth to get their relative velocity. Now, to do it correctly, we just need to use our relativistic velocity subtraction formula, taking care that one of them is negative:

$$\begin{aligned} v_2' &= \frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}} \\ &= \frac{-0.9c - 0.8c}{1-(0.8)(-0.9)} = \frac{-1.7c}{1.72} \approx -0.99c \end{aligned}$$

The overall answer comes out negative, which makes sense: the velocity of ship 2 is still in the negative direction when viewed from ship 1.

7. $0.98c$ Just like the last problem, let us first label what we know. Let the observer on the ground be in the unprimed frame, and the passenger in the car the primed frame:

$$
\begin{aligned}
v_b &= \text{velocity of the ball relative to the ground} = ? \\
v_c &= \text{velocity of the car relative to the ground} = 0.9c \\
v_b' &= \text{velocity of the ball relative to the car} = 0.7c
\end{aligned}
$$

Again, ask yourself how you would figure this out without relativity first, and that will help you pick the proper relativistic formula. Without relativity, you would just add the velocity of the car relative to the ground and the velocity of the ball relative to the car. Thus, all we need to do use our correct relativistic velocity addition formula:

$$
\begin{aligned}
v_b &= \frac{v_c + v_b'}{1 + \frac{v_c v_b'}{c^2}} \\
&= \frac{1.6c}{1 + (0.9)\,(0.7)} \approx 0.98c
\end{aligned}
$$

8. 15.4 μ**s; 649 m** Let the earth be in reference frame O' (primed frame), and the muon itself in O (unprimed frame). First, since we know we will need it, for $v = 0.990c$, $\gamma = 7.09$. Next, the numbers we are given are measured in the earth's reference frame, so it will be easiest to calculate the time in the earth's frame first. The muon, according to earthbound observers, travels 4600 m at a speed of $0.990c$, so the apparent decay time is just distance divided by velocity.

$$
\Delta t'_{\text{earth}} = \frac{4600\,\text{m}}{0.990\,(3\times 10^8\,\text{m/s})} \approx 1.54\times 10^{-5}\,\text{s} = 15.4\,\mu\text{s}
$$

This is *not* the proper time - proper time is measured in the muon's own frame. According to the muon, the earth is moving toward them! Given γ and time measured on earth, we can find the proper time in the muon's frame easily:

$$
\Delta t_p = \frac{\Delta t'_{\text{earth}}}{\gamma} \approx \frac{1.54\,\mu\text{s}}{7.09} = 2.18\,\mu\text{s}
$$

This makes sense - since the people on earth are the moving observers in this case, they should see a longer time interval. About seven times longer, in this case, since $\gamma \approx 7$. The muon is at rest in its own frame, and measures the shorter proper time interval. Now we have the proper time, meaured in the muon's reference frame, and the relative velocity, so we can calculate the distance from the muon's point of view using quantities valid in its reference frame.

$$
d_\mu = v\Delta t_p = v\frac{\Delta t'_{\text{earth}}}{\gamma} \approx 649\,\text{m}
$$

9. $v \leq 0.14c$**,** $v \leq 0.31c$**.** First of all, what do we mean by error? You want to find percent error between momentum calculated with the relativistic formula (*viz.*, $|\vec{\mathbf{p}}_{\text{rel}}| = \gamma m |\vec{\mathbf{v}}|$) and the classical formula (*viz.*, $|\vec{\mathbf{p}}_{\text{calss}} = m|\vec{\mathbf{v}}|$). First, we will drop the vector notation now, since error in momentum will only be in magnitude, not direction. Let - $p_{\text{rel}} \equiv |\vec{\mathbf{p}}_{\text{rel}}|$ and $p_{\text{class}} \equiv |\vec{\mathbf{p}}_{\text{class}}|$. The definition of error you want is the difference between the two, divided by the correct one - the relativistic formula.

$$100\% \cdot \left|\frac{p_{\text{rel}} - p_{\text{class}}}{p_{\text{rel}}}\right| \leq \text{error desired}$$

For the last line, we drop the percent Now we can just plug in what we know:

$$\left|\frac{p_{\text{rel}} - p_{\text{class}}}{p_{\text{rel}}}\right| = \left|\frac{\gamma mv - mv}{\gamma mv}\right| = \left|\frac{\gamma \cancel{mv} - \cancel{mv}}{\gamma \cancel{mv}}\right| = \left|\frac{\gamma - 1}{\gamma}\right| \leq \text{error}$$

We can further simplify this:

$$\left|\frac{\gamma - 1}{\gamma}\right| = \left|1 - \frac{1}{\gamma}\right| \leq \text{error}$$
$$\left|1 - \text{error}\right| \leq \left|\frac{1}{\gamma}\right|$$

What we really want is v. Remember the equation for v in terms of γ from problem 2? Take that, and plug in the expression above:

$$v = c\sqrt{1 - \frac{1}{\gamma^2}} \leq c\sqrt{1 - \left|(1 - \text{error})\right|^2}$$

Now all we need to do is plug in the desired minimum errors - 1% or 0.01 for **(a)**, and 5% or 0.05 for **(b)**:

(a) $v \leq c\sqrt{1 - \left|(1 - \text{error})\right|^2} = c\sqrt{1 - \left|(1 - 0.01)\right|^2} \approx c\sqrt{0.02} \approx 0.14c$

(b) $v \leq c\sqrt{1 - \left|(1 - \text{error})\right|^2} = c\sqrt{1 - \left|(1 - 0.05)\right|^2} \approx c\sqrt{0.098} \approx 0.31c$