University of Alabama
Department of Physics and Astronomy

PH 102-2 / LeClair Spring 2008

Problem Set 2 SOLUTIONS

1. 5 points. Why must hospital personnel wear special conducting shoes while working around oxygen in an operating room? What might happen if they wore shoes with rubber soles?

Conducting shoes are worn to avoid building up static charge while walking. Rubber-soled shoes as they rub against the floor build up a static charge (conduction charging), which could discharge with a spark. The spark could cause a fire or explosion in an oxygen-rich environment. Conducting shoes provide a constant connection to the earth, "grounding" the wearer and allowing any excess charge to leak away.

2. 10 points. Two solid spheres, both of radius R, carry identical total charges, Q. One sphere is a good conductor while the other is an insulator. If the charge on the insulating sphere is uniformly distributed throughout its interior volume, how do the electric fields outside these two spheres compare? Are the fields identical inside the two spheres?

First off: if this problem didn't make any sense at all, you may want to re-read Sect. 3.8 in the notes covering Gauss' law, that is the key to the whole problem.

Outside of the two spheres, we have only to remember our key result from Gauss' law: the field from spherically symmetric charge distributions is equivalent to that of a point charge. The insulating sphere has a uniform charge distribution, and is therefore spherically symmetric. We know that for any isolated conductor, all excess charge must reside on the surface, and must be uniformly distributed, so the conducting sphere also has a spherically symmetric charge distribution. Since both have a spherically symmetric charge distribution and contain a total charge Q, outside the spheres at a distance r the electric field is the same for both, and the same as for a point charge Q:

$$E = \frac{k_e Q}{r^2} \qquad r \geq R \quad \text{both spheres}$$

Inside the spheres, the two situations are qualitatively different. We know that inside any conductor in electrostatic equilibrium, the electric field is zero, since all excess charge resides on the surface.

$$E = 0 \qquad r < R \quad \text{conducting sphere}$$

For the insulating sphere, this is not true. The charge is distributed over the whole volume. Based on Gauss' law, for any radius $r < R$, we know that only the charge that resides *inside* the radius r contributes to the electric field (see the notes, Sect. 3.8.3). The fraction of the total charge Q that resides inside a radius r is just the volume fraction of a sphere radius r to the total volume:

$$\text{fraction of charge within } r = Q \cdot \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = Q \cdot \frac{r^3}{R^3} \qquad r \leq R$$

This makes sense - when we are right at the radius of the sphere, $R\!=\!r$, we have the full charge Q, and when we are at the dead center ($r\!=\!0$) there is no charge effectively. The electric field inside the insulating sphere at radius r is just Coulomb's law for the fraction of charge within the radius r:

$$E = \frac{k_e Q}{r^2}\frac{r^3}{R^3} = \frac{k_e Q r}{R^3} \qquad r \leq R \quad \text{insulating sphere}$$

Again, this is sensible - the electric field increases linearly from zero at the center of the sphere as more and more charge is outside the radius r.

3. 10 points. Two charges of 15 μC and 10 μC, respectively, lie along the x axis 1.0 m apart. Where can a third *negative* charge be placed on the x axis such that the resulting electric force on it is zero?

We have two positive charges and one negative charge along a straight line. If we want there to be no net force on the negative charge, the electric forces from both of the positive charges on it must cancel. For that to happen, there is only one possibility: the negative charge has to be between the two positive charges. Outside that middle region, both positive charges will exert an attractive force on the negative charge in the same direction, and there is no way they can cancel each other. Only in the middle region do the forces from both positive charges act in opposite directions on a negative charge, and only there can they cancel each other.

For convenience, let $q_1 = 15\,\mu\text{C}$ and $q_2 = 10\,\mu\text{C}$. Our negative charge will be q_3.We will choose the origin of our x axis to be on the larger charge q_1, with our smaller charge q_2 placed 1.0 m in the positive direction. Rather than solve the problem for just one specific separation of $1\,m$, we will just call that distance d and plug in 1 m in the end. This solves a more general problem for us, and makes the result much more useful. Physicists are economical this way - never solve a problem twice if you can avoid it.

Intuitively, we know that the negative charge q_3 must be closer to the smaller of the positive charges. Since electric forces get larger as separation decreases, the only way the force due to the larger charge can be the same as that due to the smaller charge is if *if the negative charge is farther away from the larger charge.*

Let us just specify the position of q_3 as x, so that q_3 will be a distance x from q_1 and a distance $d-x$ from q_2. Let F_{32} be the force on the negative charge q_3 due to q_2, and F_{31} be the force on q_3 due to q_1. In the region between the two positive charges (*only!*), we know that they act in opposite directions, so if we want the net force on the negative force to be zero, these two forces must be equal:

$$\begin{aligned}
F_{32} &= F_{31} \\
\frac{k_e q_3 q_2}{(d-x)^2} &= \frac{k_e q_3 q_1}{x^2} \\
\frac{q_2}{(d-x)^2} &= \frac{q_1}{x^2}
\end{aligned}$$

Notice how there are no terms with q_3 left! It doesn't matter *how* big the negative charge is at all, or even that it is negative. Equivalently, we could have asked at what points in space do the electric *fields* from q_1 and q_2 cancel, since it is at those points that a third charge of any kind would experience no force. The only difference mathematically would be to divide both sides of the first and second equations above by q_2, since $E = F/q$. Anyway: now cross multiply and collect terms ...

$$\begin{aligned}
q_2 x^2 &= q_1 (d-x)^2 \\
q_2 x^2 &= q_1 \left(x^2 - 2xdr + d^2\right) \\
\Longrightarrow \quad 0 &= (q_1 - q_2)\, x^2 - 2q_1 dx + q_1 d^2
\end{aligned}$$

If we had identical positive charges, $q_1 = q_2$, the equation above reduces to $2q_1 dx - q_1 d^2 = 0$, or $x = \frac{1}{2}d$. This is exactly what we expect, the third charge should sit halfway between two identical charges to feel no force. For the general case: solve the quadratic, plug in the numbers given, and we are done.

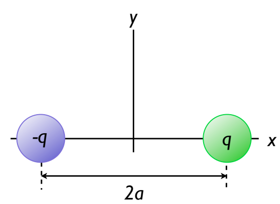
$$\begin{aligned}
x &= \frac{-(-2q_1 d) \pm \sqrt{4q_1^2 d^2 - 4q_1 d^2 (q_1 - q_2)}}{2(q_1 - q_2)} = \frac{2q_1 d \pm \sqrt{4d^2 q_1 q_2}}{2(q_1 - q_2)} = \left[\frac{q_1 \pm \sqrt{q_1 q_2}}{(q_1 - q_2)}\right] d & (q_1 \neq q_2) \\
x &= \left[\frac{15 \pm \sqrt{150}}{5}\right] \cdot 1\,\text{m} = \frac{15 \pm 5\sqrt{6}}{5}\,\text{m} = \left(3 \pm \sqrt{6}\right)\,\text{m} & (q_1 \neq q_2) \\
\Longrightarrow \quad x &\approx (5.45, 0.551)\,\text{m} & (q_1 \neq q_2)
\end{aligned}$$

Just as we expected: one solution ($x \approx 0.55$ m) is between the two charges, a little bit closer to the smaller charge. What about the positive solution? This corresponds to a position far away from both charges 5.45 m to the right of q_1. As stated above, the forces act in the same direction outside of the middle region, and cannot cancel! This solution is physically impossible, just an artifact of the mathematics. We specified originally that the equations were only good for the middle region, so if we get an answer that falls outside we must discard it as outside the scope of our equations.

Our equations as we have written them do not take into account the fact that the fields change direction on one side of a charge versus the other. Properly speaking, outside the middle region between the positive charges, we should write $F_{32} = -F_{31}$ since the forces act in the same direction. Try repeating the problem starting there, and you will find that there are no real (non-imaginary) solutions outside the middle region - two positive forces cannot add up to zero.

Remember: in the end, we always need to make sure that the solutions are physically sensible in addition to being mathematically correct.

4. 15 points. Two point charges q and $-q$ are situated along the x axis a distance $2a$ apart as shown below. Show that the electric field at a distant point along $|x| > a$ along the x axis is $E_x = 4k_e qa/x^3$.



Starting this one is not complicated: write down the electric field at a point along the x axis for each charge. The superposition principle says that the total electric field at that point is the sum of the fields from each charge alone. If we are at a point $(x, 0)$, then the $-q$ charge is a distance $x+a$ away, and the $+q$ charge is $x-a$ away. Thus:

$$\begin{aligned}
E_{\text{tot}} &= E_q + E_{-q} \\
&= \frac{k_e q}{(x-a)^2} + \frac{k_e(-q)}{(x+a)^2} = \frac{k_e q (x+a)^2}{(x-a)^2 (x+a)^2} - \frac{k_e q (x-a)^2}{(x-a)^2 (x+a)^2} \\
&= \frac{k_e q \left(x^2 + 2ax + a^2\right) - k_e q \left(x^2 - 2ax + a^2\right)}{(x^2 - a^2)^2} = \frac{4_e kqax}{(x^2 - a^2)^2}
\end{aligned}$$

Now what? The key is that when we specify that we want the field at a "distant" point, we mean the distance x is much, much larger than the spacing a, *i.e.,* $x \gg a$. Large enough that we can use mathematical approximations, basically. First, some rearranging:

$$E_{\text{tot}} = \frac{4k_e qax}{(x^2 - a^2)^2} = \frac{4k_e qax}{x^4 \left(1 - a^2/x^2\right)^2}$$

If we specify that $x \gg a$, then the larger x gets, the smaller a^2/x^2 gets, and for large distances $1 - a^2/x^2 \approx 1$. More directly, before rearranging anything we might have just claimed that since when $x \gg a$, $x^2 - a^2 \approx x^2$ - ignore the a^2 since it is much smaller anyway. Formally, this is considered Not OK, even though it works here. Typically, to make an approximation like this you want to get an expression such that in the limit x tends toward infinity, some term goes to zero and can be ignored - in this case, a^2/x^2 goes to zero, so we drop it. In some sense this is just being pedantic, but this more general trick is very useful for more complicated equations.

In any case, the effect here is the same: the denominator can be approximated as x^4. Using this approximation,

$$E_{\text{tot}} \approx \frac{4kqax}{x^4} = \frac{4kqa}{x^3}$$

A positive and a negative charge like this is a *dipole*, something that comes up a lot – for instance, it is a reasonable approximation of a diatomic molecule (*e.g.,* HCl).

5. 10 points. At what distance below a proton would the upward force on an electron equal the electron's weight?

All we need to do is write down the two forces and balance them. The electron has mass m_e and charge $-e$, the proton has charge e, and the distance between them we'll call d. Our origin will be at the proton's position:

$$
\begin{aligned}
m_e g - \frac{k_e e\,(-e)}{d^2} &= 0 \\
d^2 &= \frac{k_e e^2}{m_e g} \\
d &= \sqrt{\frac{k_e e^2}{m_e g}} = \sqrt{\frac{\left(9\times10^9\,\mathrm{N\cdot m^2/C^2}\right)\left(1.6\times10^{-19}\,\mathrm{C}\right)^2}{\left(9.1\times10^{-31}\,\mathrm{kg}\right)\left(9.8\,\mathrm{m/s^2}\right)}} \\
&\approx \sqrt{26\,\frac{\mathrm{N\cdot m\cdot s^2}}{\mathrm{kg}}} = \sqrt{26\,\frac{\left(\mathrm{kg\cdot m/s^2}\right)\cdot \mathrm{m\cdot s^2}}{\mathrm{kg}}} \\
\Longrightarrow \quad d &\approx 5.1\,\mathrm{m}
\end{aligned}
$$

6. 10 points. A proton accelerates from rest in a uniform electric field of $800\,\mathrm{N/C}$. At some time later, its speed is $1.2\times10^6\,\mathrm{m/s}$. What is the magnitude of the acceleration on the proton?

If the proton, with charge e, is in a uniform electric field E, it experiences a constant force eE. If this is the only force acting on it, then $F = qE = m_p a$, where m_p is the proton mass and a its acceleration. That's it - the speed is superfluous.

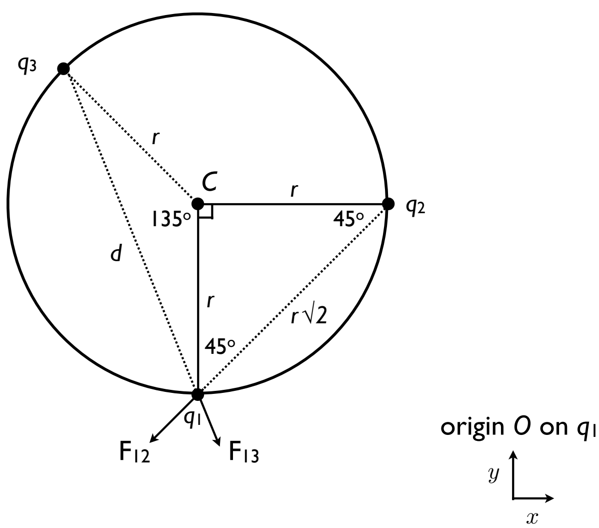
$$
\begin{aligned}
& F = qE = m_p a \\
\Longrightarrow \quad & a = \frac{qE}{m_p} = \frac{\left(1.60\times10^{-19}\,\mathrm{C}\right)\left(800\,\mathrm{N/C}\right)}{1.67\times10^{-27}\,\mathrm{kg}} \approx 7.7\times10^{10}\,\mathrm{N/kg} = 7.7\times10^{10}\,\mathrm{m/s^2}
\end{aligned}
$$

For the last line, we had to use the fact that $1\,\mathrm{N} = 1\,\mathrm{kg\cdot m/s^2}$.

7. 20 points. Suppose three positively charged particles are constrained to move on a fixed circular track. If all the charges were equal, an equilibrium arrangement would obviously be a symmetrical one with the particles spaced 120° apart around the circle. Suppose two of the charges have equal charge q, and the equilibrium arrangement is such that these two charges are 90° apart rather than 120°. What must be the relative magnitude of the third charge?

The first thing we need to do is figure out the geometry and draw a picture. First, all three charges are confined to a circular track, which we will say has radius r. Two of the charges are the same, which we will call q_1 and q_2, and they sit 90° apart on the circle. Where will the third, unequal charge (q_3) sit? In order for the forces on it due to charges 1 and 2 to be balanced, it must be equidistant from both on the circle. If charges 1 and 2 are 90° apart, then there are 270° left in the circle, and the third charge must sit halfway around that - the third charge must be 135° from both of the other charges.

Next, we should pick a coordinate system and origin. For reasons I hope will be clear soon, we will choose the origin to be on charge q_1, with the $+y$ direction pointing toward the center of the circle and the x axis tangential to the circle, as shown below. We could have equally chosen q_2 as the origin, since it is identical to q_1, it makes no difference.[i] For convenience, we label the center of the circle as point C so we can easily refer to it later.



Since charges q_1 and q_2 are 90° apart on the circle, we can form a 45-45-90 triangle with point C. Based on this, we can find the distance between q_1 and q_2 in terms of the radius r: $r_{12} = r\sqrt{2}$. Charges q_1 and q_2 are identical, and therefore experience a repulsive force of magnitude F_{12} directed along the line connecting them. This force must be at a 45° angle to the x and y axes, based on the geometry above. Charge q_3 has a different magnitude, but the same sign as q_1, and thus the force between them F_{13} is also repulsive.

In order for the charges to stay in the positions above, what must be true? For charge q_1, the forces in the y direction are irrelevant, since q_1 is constrained to stay on the circle anyway. Only net forces along the x direction will force it to move around the circle one way or the other. Thus, in order for this situation to be the equilibrium configuration, the forces in the x direction on q_1 must cancel. Since q_1 and q_2 are identical, the forces along the direction of the circle will also vanish for q_2 automatically. Finally, since the system is symmetric, q_3 must also have no net force along the direction of the circle if neither of the other charges do. Thus, it is sufficient to find the forces in the x direction for q_1 and equate them. This means we need to find the x components of F_{12} and F_{13}, set them equal to one another, and solve for q_3.

First, we focus on F_{12}, whose x component we will label $F_{12,x}$. We now know the distance between q_1 and q_2, so the magnitude of the *total* force is easily written down with Coulomb's law:

$$F_{12} = \frac{k_e q_1 q_2}{r_{12}^2} = \frac{k_e q_1 q_2}{\left(r\sqrt{2}\right)^2} = \frac{k_e q_1 q_2}{2r^2} \tag{1}$$

[i]One could choose any point as the origin and get the same result, but in my opinion the geometry is more transparent in the present case.

In order to find the $x$component, we just need to know the angle that $\vec{\mathbf{F}}_{12}$ makes with the x axis - 45°. You should be able to convince yourself this is true based on the geometry above (the inset to the second figure below may help). The x component is then just $F_{12,x} = F_{12} \sin 45°$. Noting that $\sin 45° = \sqrt{2}2$:

$$F_{12,x} = F_{12} \sin 45^\circ = F_{12} \frac{\sqrt{2}}{2} = \frac{\sqrt{2} k_e q_1 q_2}{4r^2} \tag{2}$$

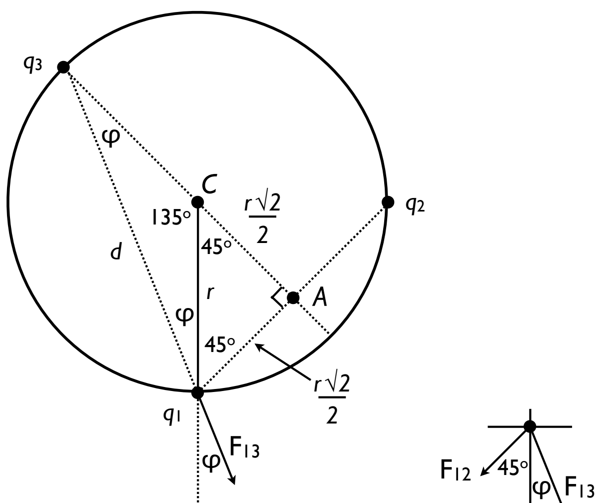
Now, what about the force between charges 1 and 3, F_{31}? We can write down the force between them easily:

$$F_{13} = \frac{k_e q_1 q_3}{r_{13}^2} = \frac{k_e q_1 q_3}{d^2} \tag{3}$$

What is the distance d between q_1 and q_3? For this, we will need the law of cosines (and the fact that $\cos 135° = -\sqrt{2}/2$):

$$d^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos 135^\circ = 2r^2 - 2r^2 \left(-\frac{\sqrt{2}}{2} \right) = 2r^2 \left(1 + \frac{\sqrt{2}}{2} \right) \tag{4}$$

Before we combine that with our expression for F_{13}, let us find the x component, for which we need the angle that $\vec{\mathbf{F}}_{13}$ makes with our axes. The figure below will help us:



The triangle defined by q_1, q_3, and C gives us two equal angles φ. Since the angles of a triangle must add up to 180°, we must have $\varphi = (180° - 135°)/2 = 22.5°$. This is the angle that $\vec{\mathbf{F}}_{13}$ makes with the y axis, and thus $F_{13,x} = F_{13} \sin \varphi$. The inset in the lower right of the figure should help you see this. If we look at the triangle formed by q_1, q_3, and point A, we can find $\sin \varphi$ analytically. Look at the φ nearest q_3: $\sin \varphi = \frac{r\sqrt{2}/2}{d} = \frac{\sqrt{2}r}{2d}$. now we have everything to find $F_{13,x}$:

$$F_{13,x} = F_{13} \sin \varphi = \frac{k_e q_1 q_3}{d^2} \frac{r\sqrt{2}}{2d} = \frac{\sqrt{2} r k_e q_1 q_3}{2d^3} \tag{5}$$

Finally, we have the x components of both forces acting on q_1. All we need to do now is equate them, and solve for q_3:

$$
\begin{aligned}
F_{13,x} &= F_{12,x} && (6)\\
\frac{\sqrt{2}rk_e q_1 q_3}{2d^3} &= \frac{\sqrt{2}k_e q_1 q_2}{4r^2} && (7)\\
\frac{\cancel{\sqrt{2}}r\cancel{k_e}\cancel{q_1}q_3}{\cancel{2}d^3} &= \frac{\cancel{\sqrt{2}}\cancel{k_e}\cancel{q_1}q_2}{\cancel{4}2r^2} && (8)\\
\frac{rq_3}{d^3} &= \frac{q_2}{2r^2} && (9)\\
\implies \quad q_3 &= \frac{q_2 d^3}{2r^3} && (10)
\end{aligned}
$$

Plugging in our expression for d^2 we can find q_3 in terms of only q_2 and numerical factors:

$$
\begin{aligned}
q_3 &= \frac{1}{2}\left(\frac{d}{r}\right)^3 q_2 && (11)\\
&= \frac{1}{2}\left(\frac{d^2}{r^2}\right)^{\frac{3}{2}} q_2 && (12)\\
&= \frac{1}{2}\left(\frac{2r^2\left(1+\frac{\sqrt{2}}{2}\right)}{r^2}\right)^{\frac{3}{2}} q_2 && (13)\\
&= \frac{1}{2}\left(2+\sqrt{2}\right)^{\frac{3}{2}} q_2 \approx 3.15 q_2 && (14)
\end{aligned}
$$

Thus, the charge q_3 must be approximately 3.15 times as big as q_1 and q_2 in order for the latter two charges to be 90° apart. Physically, it makes sense that q_3 is bigger - q_1 and q_2 are closer together than they would be if all three charges are equal, so they must be feeling more repulsion from q_3 than from each other, which means q_3 must be bigger.

8. 20 points. A charge of $100\,\mu\mathrm{C}$ is at the center of a cube of side $0.8\,\mathrm{m}$. **(a)** Find the total flux through each face of the cube. **(b)** Find the flux through the whole surface of the cube. **(c)** Would your answers to the first two parts change if the charge were not at the center of the cube?

From Gauss' law, we know that the *total* flux through any closed surface is just the total charge Q contained by the surface divided by ϵ_0. If the charge is exactly at the center of the cube, the flux through each face of the cube should be the same. Six faces, each with one sixth the flux:

$$\Phi_{E,\text{face}} = \frac{1}{6}\Phi_{E,\text{total}} = \frac{1}{6}\frac{100\,\mu\mathrm{C}}{\epsilon_0} = 1.88\times 10^7\,\mathrm{N\cdot m^2/C}$$

The total flux is just six times this number, clearly. What about if the charge isn't at the center? The total flux remains the same no matter where the charge is, so long as it is inside the cube. However, to find the flux through a given face we assumed that the flux through all faces was equal based on the symmetry of the original problem. If the charge were closer to one face, that face would have higher flux, and the opposite side lower flux. Moving the charge from the center changes the distribution of flux over each face, and they will no longer all be equal, but the total remains the same. In other words, the first answer would change, the second would not.