University of Alabama
Department of Physics and Astronomy

PH 102-2 / LeClair   Spring 2008

Problem Set 3 SOLUTIONS

1. 10 points. Remember #7 on last week's homework? Calculate the potential energy of that system of three charges, for a circle of radius r. Take the zero of potential energy to be infinitely far away from all charges. Express your answer in terms of the energy of charges q_1 and q_2 separated by r – *e.g.,* a constant times $k_e q_1 q_2/r$.

From last week, we had an arrangement of three charges on a circular track of radius r. Two of the charges (q_1 and q_2, $q_1 = q_2$) were separated by 90° on the track, and hence a distance $r_{12} = r\sqrt{2}$ apart. The third charge (q_3) was a distance d from both of those. We found that d could be expressed in terms of r:

$$d = r\sqrt{2+\sqrt{2}}$$

We also found that the magnitude of q_3 was fixed by the geometry given in the problem, and could express the magnitude of q_3 relative to q_1 or q_2:

$$q_3 = \frac{1}{2}\left[2+\sqrt{2}\right]^{\frac{3}{2}} q_1 = \frac{1}{2}\sqrt{2+\sqrt{2}}\left[2+\sqrt{2}\right] \approx 3.15 q_1$$

How do we find the potential energy of this system of charges? We just have to add together the potential energy of each unique pair of charges. In this case, there are three pairings:

$$PE_{\text{system}} = PE_{1\&2} + PE_{1\&3} + PE_{2\&3}$$

The potential energy of a pair of charges q_1 and q_2 separated by a distance r_{12} is straightforward:

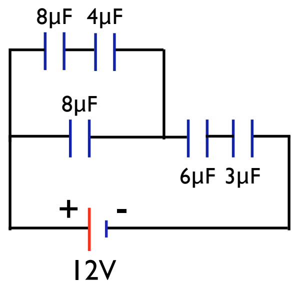
$$PE_{\text{pair}} = \frac{k_e q_1 q_2}{r_{12}}$$

All we need to do is evaluate the sum above, plugging in the values we know:

$$\begin{aligned}
PE_{\text{system}} &= \frac{k_e q_1 q_2}{r_{12}} + \frac{k_e q_1 q_3}{r_{13}} + \frac{k_e q_2 q_3}{r_{23}} \\
&= \frac{k_e q_1 q_2}{r\sqrt{2}} + \frac{k_e q_1 q_3}{r\sqrt{2+\sqrt{2}}} + \frac{k_e q_1 q_3}{r\sqrt{2+\sqrt{2}}} \\
&= \frac{k_e q_1 q_2}{r\sqrt{2}} + \frac{k_e q_1\left[\frac{1}{2}\left(2+\sqrt{2}\right)^{\frac{3}{2}} q_1\right]}{r\sqrt{2+\sqrt{2}}} + \frac{k_e q_1\left[\frac{1}{2}\left(2+\sqrt{2}\right)^{\frac{3}{2}} q_1\right]}{r\sqrt{2+\sqrt{2}}} \\
&= \frac{k_e q_1 q_2}{r}\left[\frac{1}{\sqrt{2}} + \frac{\frac{1}{2}\left[2+\sqrt{2}\right]^{\frac{3}{2}}}{\sqrt{2+\sqrt{2}}} + \frac{\frac{1}{2}\left[2+\sqrt{2}\right]^{\frac{3}{2}}}{\sqrt{2+\sqrt{2}}}\right] \\
&= \frac{k_e q_1 q_2}{r}\left[\frac{1}{\sqrt{2}} + \frac{1}{2}\left(2+\sqrt{2}\right) + \frac{1}{2}\left(2+\sqrt{2}\right)\right] \\
&= \frac{k_e q_1 q_2}{r}\left[\frac{1}{\sqrt{2}} + 2 + \sqrt{2}\right] = \frac{k_e q_1 q_2}{r}\left[\frac{3\sqrt{2}+4}{2}\right] \approx 4.12\frac{k_e q_1 q_2}{r}
\end{aligned}$$

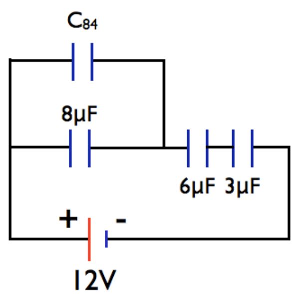
So the energy of this system of charges is about 4.12 times higher than if we had 2 charges alone separated by r, or about 2.9 times higher than if we removed q_3 and just had q_1 and q_2 separated by $r\sqrt{2}$.

2. 15 points. **(a)** Find the equivalent capacitance of the capacitors in the figure below. **(b)** Find the charge on each capacitor. **(c)** Find the potential difference across each capacitor.



Before we start, it is useful to remember that one farad times one volt gives one coulomb: $1\,[\mathrm{F}]\cdot 1\,[\mathrm{V}] = 1\,[\mathrm{C}]$, and that capacitance times voltage gives stored charge: $Q = CV$. Knowing this now will save some confusion on units later on. For that matter, it is also good to remember that the prefix μ means 10^{-6}.

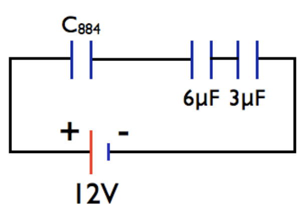
In order to find a single equivalent capacitor that could replace all five in the diagram above, we need to look for purely series and parallel combinations that can be replaced by a single capacitor. The uppermost $8\,\mu\mathrm{F}$ and $4\,\mu\mathrm{F}$ capacitors are purely in series, so they can be replaced by a single equivalent we will call C_{84}, as shown below:



Using our rule for combining series capacitors, we can find the value of C_{84} easily:

$$\begin{aligned} \frac{1}{C_{84}} &= \frac{1}{8\,\mu\mathrm{F}} + \frac{1}{8\,\mu\mathrm{F}} \\ \Longrightarrow \quad C_{84} &= \frac{8}{3}\,\mu\mathrm{F} \approx 2.67\,\mu\mathrm{F} \end{aligned}$$

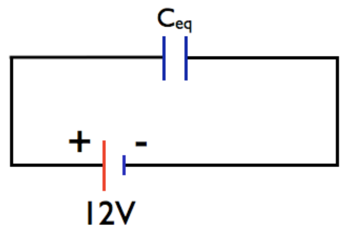
Now we have this equivalent capacitance purely in *parallel* with the second $8\,\mu\mathrm{F}$ capacitor. We can replace C_{84} and the second $8\,\mu\mathrm{F}$ capacitors with a single equivalent, which we will call C_{884}:



Using our addition rule for parallel capacitors, we can find its value:

$$C_{884} = C_{84} + 8\,\mu\mathrm{F} \approx 10.67\,\mu\mathrm{F}$$

This leaves us with three capacitors in series, as shown below:



Adding together these three in series, we have the overall equivalent capacitance, C_{eq}:

$$\begin{aligned} \frac{1}{C_{eq}} &= \frac{1}{C_{884}} + \frac{1}{3\,\mu\text{F}} + \frac{1}{6\,\mu\text{F}} \\ \implies \quad C_{eq} &\approx 1.68\,\mu\text{F} \end{aligned}$$

Since we now have one single capacitor connected to a single voltage source, we can find the total charge stored in the equivalent capacitor, Q_{eq}:

$$Q_{eq} = C_{eq}V = (1.68\,\mu\text{F})\,(12\,\text{V}) \approx 20.16\,\mu\text{C}$$

Now in order to get the charge and voltage on each single capacitor, we have to work backwards and rebuild our original circuit. We know that the C_{eq} capacitor is really three capacitors in series - the $6\,\mu$F, the $3\,\mu$F, and C_{884}. Series capacitors always have the same charge, and one must have the same charge as the equivalent capacitor: $Q_{6\mu\text{F}} = Q_{3\mu\text{F}} = Q_{884}$ Since we know the charge and capacitance for all three of these capacitors, we can now find the voltage on each, since $V = Q/C$:

$$\begin{aligned} V_{6\mu\text{F}} &= \frac{Q_{6\mu\text{F}}}{6\mu\text{F}} = \frac{20.16\,\mu\text{C}}{6\mu\text{F}} \approx 3.4\,\text{V} \\ V_{3\mu\text{F}} &= \frac{Q_{3\mu\text{F}}}{3\mu\text{F}} = \frac{20.16\,\mu\text{C}}{3\mu\text{F}} \approx 6.7\,\text{V} \\ V_{884} &= \frac{Q_{884}}{C_{884}} = \frac{20.16\,\mu\text{C}}{10.67\,\mu\text{F}} \approx 1.9\,\text{V} \end{aligned}$$

Notice that the voltage on all three of these series capacitors adds up to the total battery voltage - it must be so, based on conservation of energy. Next, we know that C_{884} is really two capacitors in parallel - the lower $8\,\mu$F capacitor and C_{84}. Parallel capacitors have the same voltage, so we know that both of these have to have $V_{\text{lower }8\mu\text{F}} = V_{84} = V_{884} = 1.9\,\text{V}$ across them. We know the voltage and the capacitance for C_{84} and the lower $8\,\mu$F capacitors now, so we can find the stored charge, $Q = CV$:

$$\begin{aligned} Q_{\text{lower }8\mu\text{F}} &= 8\,\mu\text{F} \cdot V_{884} = 8\,\mu\text{F} \cdot 1.9\,\text{V} \approx 15.2\,\mu\text{C} \\ Q_{84} &= C_{84} \cdot V_{884} = 2.67\,\mu\text{F} \cdot 1.9\,\text{V} \approx 5.1\,\mu\text{C} \end{aligned}$$

Finally, the capacitor C_{84} is really two capacitors in series, which must both have the same charge: $Q_{4\mu\text{F}} = Q_{\text{upper }8\mu\text{F}} = Q_{84}$. Given the charge on both of the remaining capacitors and their capacitances, we can find the voltages:

$$\begin{aligned} V_{4\mu\text{F}} &= \frac{Q_{4\mu\text{F}}}{4\mu\text{F}} = \frac{5.1\,\mu\text{C}}{4\mu\text{F}} \approx 1.26\,\text{V} \\ V_{\text{upper }8\mu\text{F}} &= \frac{Q_{\text{upper }8\mu\text{F}}}{8\mu\text{F}} = \frac{5.1\,\mu\text{C}}{8\mu\text{F}} \approx 0.63\,\text{V} \end{aligned}$$

Now we know the charge and voltage on every single capacitance, as well as the overall charge (Q_{eq}) and effective capacitance (C_{eq}). Your numbers may be very slightly different than those above due to different choices in rounding, this is normal. The results are summarized in the table below:

Table 1: *Equivalent capacitances, charges, and voltages*

Capacitor [μF]	**Charge [μC]**	**Voltage [V]**
top 8 μF	5.1	0.63
4 μF	5.1	1.26
lower 8 μF	15.2	1.9
6 μF	20.16	3.4
3 μF	20.16	6.7
C_{eq}	20.16	12

3. 10 points. A parallel-plate capacitor has $4.00\,\text{cm}^2$ plates separated by 6.00 mm of air. If a 12.0 V battery is connected to this capacitor, how much energy does it store in Joules? In electron volts?

If we want to find the energy stored in the capacitor, we need to know two of three things, minimally: the amount of charge stored, the voltage applied, and the capacitance. Any two of these three are sufficient, based on our formula for the potential energy stored in a capacitor:

$$\Delta PE = \frac{1}{2}Q\Delta V = \frac{1}{2}C\left(\Delta V\right)^2 = \frac{Q^2}{2C}$$

We already know the applied voltage, $\Delta V = 12.0\,\text{Volts}$. Since this is a parallel plate capacitor and we know its area A and plate spacing d we can easily calculate the capacitance ... if we are very careful with units. Recall that the dielectric constant of air is essentially one ($\kappa \approx 1$).

$$\begin{aligned}
C &= \frac{\kappa\epsilon_0 A}{d} \\
&= \frac{1\cdot\epsilon_0\left(4.00\,\text{cm}^2\right)\cdot\left(\frac{1\,\text{m}}{100\,\text{cm}}\right)^2}{6.00\times10^{-3}\,\text{m}} \\
&= \frac{\left(8.85\times10^{-12}\,\text{F/m}\right)\cdot\left(4\times10^{-4}\,\text{m}^2\right)}{6.00\times10^{-3}\,\text{m}} \\
&= \frac{8.85\times4.00}{6.00}\cdot10^{-13}\,\text{F} \\
&\approx 5.90\times10^{-13}\,\text{F} = 0.590\,\text{pF}
\end{aligned}$$

Now we know the capacitance and the voltage, we can find the energy readily:

$$\begin{aligned}
\Delta PE &= \frac{1}{2}\left(5.90\times10^{-13}\,\text{F}\right)\left(12.0\,\text{V}\right)^2 = 4.25\times10^{-11}\,\text{F}\cdot\text{V}^2 = 4.25\times10^{-11}\,\text{J} \\
&= \left(4.25\times10^{-11}\,\text{J}\right)\left(\frac{1\,\text{eV}}{1.60\times10^{-19}\,\text{J}}\right) = 2.66\times10^{8}\,\text{eV} = 266\,\text{MeV}
\end{aligned}$$

This problem brings to mind a few hand SI unit conversions, which you should be able to verify: $1\,\text{J} = 1\,\text{F}\cdot1\,\text{V}^2 = 1\,\text{C}\cdot1\,\text{V}$, $1\,\text{C} = 1\,\text{F}\cdot1\,\text{V}$.

4. 5 points. A capacitor with air between its plates is charged to 150 V and then disconnected from the battery. When a piece of glass is placed between the plates, the voltage across the capacitor drops to 25 V. What is the dielectric constant of the glass? (Assume the glass completely fills the space between the plates.)

When we insert a dielectric into a parallel plate capacitor, for the same amount of charge stored the voltage is *reduced* by a factor κ, the dielectric constant:

$$\Delta V_{\text{empty}} = \kappa \Delta V_{\text{filled}}$$

If the voltage is 150 V while the capacitor is empty, and 25 V when filled, then we must have $\kappa = 6$.

Another way to see this is to think about the charge stored. If our (ideal) capacitor is fully charged at 150 V and disconnected, it keeps a total amount of charge Q. We can relate this charge to the voltage applied when the capacitor is empty:

$$Q = C_{\text{empty}} \Delta V_{\text{empty}} = 150 C_{\text{empty}}$$

When we insert the dielectric between the plates, the capacitor remains disconnected *but is still fully charged.* The charges can't go anywhere while the capacitor is disconnected, so we have the same Q. With the dielectric, however, we know that the capacitance *increases* by a factor κ: $C_{\text{full}} = \kappa C_{\text{empty}}$. We can relate the stored charge Q to the new capacitance and voltage, and combine that with the expression above to find κ:

$$\begin{aligned}
Q &= C_{\text{full}} \Delta V_{\text{full}} = \kappa C_{\text{empty}} V_{\text{full}} = 25 \kappa C_{\text{empty}} \\
Q &= 150 C_{\text{empty}} \\
25 \kappa C_{\text{empty}} &= 150 C_{\text{empty}} \\
\implies \kappa &= 6
\end{aligned}$$

Remember, dielectrics increase the total charge stored and the capacitance, but decrease the voltage required to store the same amount of charge.

5. 10 points. A potential difference of 100 mV exists between the outer and inner surfaces of a cell membrane. The inner surface is negative relative to the outer. How much work is required to move a sodium ion Na^+ outside the cell from the interior? Answer in electron volts and Joules. A singly-charged ion has a charge of $1e$.

The work done in moving a charge q across a potential difference ΔV is readily calculated: $W = -\Delta PE = -q\Delta V$. In this case the charge is $e = 1.6 \times 10^{-19}$ C, and $\Delta V = 0.1$ V. Watch how easy it is to find the answer in electron volts:

$$\begin{aligned}
W = -q\Delta V &= \left(1.6 \times 10^{-19}\,\text{C}\right)(0.1\,\text{V}) \cdot \left(\frac{1\,\text{eV}}{1.6 \times 10^{-19}\,\text{J}}\right) \\
&= -\left(\cancel{1.6 \times 10^{-19}}\,\text{C}\right)(0.1\,\text{V}) \cdot \left(\frac{1\,\text{eV}}{\cancel{1.6 \times 10^{-19}}\,\text{J}}\right) \\
&= -0.1\,\frac{\cancel{[\text{C} \cdot \text{V}]} \cdot \text{eV}}{\cancel{\text{J}}} \qquad \textit{remember: } 1\,\text{J} = 1\,\text{C} \cdot 1\,\text{V} \\
&= -0.1\,\text{eV}
\end{aligned}$$

In fact, we didn't even need to go through all that. An electron volt is *defined* as the energy required to move one electron's equivalent of charge - $1e$ - through a potential difference of 1 Volt. Our ion has the same magnitude of charge as an electron, and we move it through 0.1 V. Following the definition of an electron volt, we must have $\Delta PE = 0.1$ eV. This is one reason why electron volts are such a handy unit for many areas of physics.

Anyway: how about the answer in Joules? Well, if $1\,\text{eV} = 1.6 \times 10^{-19}$ J, then $0.1\,\text{eV} = 1.6 \times 10^{-20}$ J.

6. 5 points. A proton and an electron are accelerated from rest through a potential difference of 120 V. Calculate the speed and kinetic energy of each.

This one is just conservation of energy: electric potential energy is converted into kinetic energy. Let's work it through all the way though, but for a generic charge q of mass m. This solves the general problem, and at the end we can just plug in the appropriate numbers for a proton and electron. This saves us the work of solving the same problem twice.

Initially, the charge is at rest, and thus has no kinetic energy. Our potential difference really only tells us the *difference* in potential energy from the initial to final state, we still need to define a zero point for potential energy. We will define the *final* potential energy to be zero, for convenience - this means the charge starts out at a potential energy of $e\Delta V$ and ends at 0, moving from high potential to low. With all of this, we can write down conservation of energy, and solve for kinetic energy and velocity.

$$\begin{aligned} PE_i + KE_i &= PE_f + KE_f \\ q\Delta V + 0 &= 0 + KE_f \end{aligned}$$

Already, we have the kinetic energy. Since the proton and electron have the same magnitude of charge e, and the voltage is the same in both cases, the kinetic energy is the same:

$$KE_f = e\Delta V = \left(1.6 \times 10^{-19}\,\mathrm{C}\right)(120\,\mathrm{V}) \cdot \frac{1\,\mathrm{eV}}{1.6 \times 10^{-19}\,\mathrm{J}} = 120\,\mathrm{eV}$$

We could have just written this down without calculating anything at all. Recall that an electron volt is *defined* as the energy required to move one electron's equivalent of charge - $1e$ - through a potential difference of 1 Volt. Both the proton and electron have a charge of e, and both move through a potential difference of 120 V. Therefore, the potential energy change is 120 eV in both cases, so the kinetic energy is also 120 eV in both cases (or 1.92×10^{-17} J).

Well. We still need the speed. Just plug in the non-relativistic kinetic energy formula, and solve for velocity:

$$\begin{aligned} KE_f &= \frac{1}{2}mv_f^2 = q\Delta V \\ v_f^2 &= \frac{2q\Delta V}{m} \\ v_f &= \sqrt{\frac{2q\Delta V}{m}} \end{aligned}$$

Now we have something of a problem. For electrons, the charge is $q = -e$, doesn't that lead to an imaginary number based on the formula above? Yes, it would ... if we don't think a bit first. The problem stated that the electron was *accelerated through a potential difference* of 120 V, it did not state whether the potential was higher at the start or the end! From the phrasing of the problem, we have to assume that the electron starts out at a lower potential and moves to a higher potential, since a negative charge will only be accelerated toward a higher potential. If this is the case, the final potential is higher, then ΔV must be negative. In the formula above, both q and ΔV will be negative, and there is no problem. The proton, on the other hand, will be attracted to regions of lower potential, so we have to assume that it moves from high to low potential, meaning ΔV must be positive.

The key is to remember that the change in kinetic energy must be equal and opposite the change in potential energy. Both charges gain kinetic energy and lose potential energy, so in both cases the initial potential energy must be positive (since the final potential energy is set to zero). For protons, this means starting at a high potential and moving to a lower one. For electrons, this means starting at a low potential and moving to a higher one, owing to its negative charge.

Another way to go about it is to use the now known value of kinetic energy, which sneaks around this problem all together:

$$\begin{aligned} KE_f &= \frac{1}{2}mv_f^2 \\ v_f^2 &= \frac{2KE}{m} \\ v_f &= \sqrt{\frac{2KE}{m}} \end{aligned}$$

This formula is perhaps easier to use, since we already know KE. Plug in the known masses and charge, and you should find:

$$
\begin{aligned}
v_f &= 1.5 \times 10^5 \,\text{m/s} \qquad \text{proton} \\
v_f &= 6.5 \times 10^6 \,\text{m/s} \qquad \text{electron}
\end{aligned}
$$

7. 5 points A parallel plate capacitor is held at constant voltage. Initially there is only air between the plates. If a dielectric with a dielectric constant of 2 is inserted into the capacitor, what happens to the energy stored in the capacitor?

This is a lot like question 4. If we insert a dielectric into the capacitor with a fixed voltage, the capacitance goes *up* by a factor κ, as does the total amount of stored charge. Without a dielectric, we can write the stored energy in three different ways:

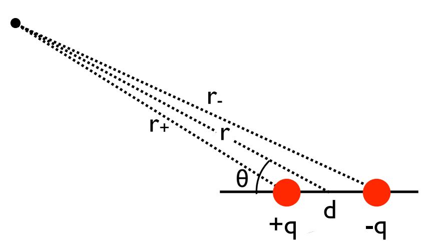
$$
\begin{aligned}
\Delta PE_{\text{empty}} &= \frac{1}{2} C_{\text{empty}} \left(\Delta V\right)^2 \\
\Delta PE_{\text{empty}} &= \frac{Q_{\text{empty}}^2}{2C_{\text{empty}}} \\
\Delta PE_{\text{empty}} &= \frac{1}{2} Q_{\text{empty}} \Delta V
\end{aligned}
$$

We will verify that all three formulas give the same result, just for "fun." Once we fill the capacitor with a dielectric, the voltage is held constant at ΔV, but the capacitance and charge stored increase: $C_{\text{full}} = \kappa C_{\text{empty}}$ and $Q_{\text{full}} = \kappa Q_{\text{empty}}$. Using this and the relations above, we can find the energy stored with dielectric present in terms of the original stored energy without:

$$
\begin{aligned}
\Delta PE_{\text{filled}} &= \frac{1}{2} C_{\text{full}} \left(\Delta V\right)^2 = \frac{1}{2} \kappa C_{\text{empty}} \left(\Delta V\right)^2 = \kappa \Delta PE_{\text{empty}} \\
\Delta PE_{\text{filled}} &= \frac{Q_{\text{full}}^2}{2C_{\text{filled}}} = \frac{\kappa^2 Q_{\text{empty}}^2}{2\kappa C_{\text{empty}}} = \kappa \frac{Q_{\text{empty}}^2}{2C_{\text{empty}}} = \kappa \Delta PE_{\text{empty}} \\
\Delta PE_{\text{filled}} &= \frac{1}{2} Q_{\text{full}} \Delta V = \frac{1}{2} \kappa Q_{\text{empty}} \Delta V = \kappa \Delta PE_{\text{empty}}
\end{aligned}
$$

No matter which formula we choose to solve the problem, the answer is the same: the stored energy also increases by a factor κ, or 2 times. Naturally you didn't need to solve it all three ways, one was enough - I just wanted to show you that it made no difference which formula you started with.

8. 15 points. Two charges, $+q$ and $-q$, are separated by a distance d. Show that the electric potential far from both charges is approximately $V = \frac{kqd\cos\theta}{r^2}$. The following approximations may be useful (referring to the figure below, with the origin between the two charges): $r_- r_+ \approx r^2$, $r_- - r_+ \approx \frac{xd}{r} = d\cos\theta$.



First, we should pick a coordinate system and an an origin. The easiest choice seems to be putting the origin at the midpoint between the two charges. We will pick an $x - y$ system with the x axis running along a line connecting the two charges, and a y axis running perpendicular. Finally, define the $+x$ and $+y$ directions as the direction from the origin toward the distant point, which we will call P. This puts point P at coordinates $(x, y,)$, and the positive and negative charge at $(x-\frac{d}{2},y)$ and $(x+\frac{d}{2},y)$ respectively. With these choices, we can easily write down the distance from P to the origin (r), to the positive charge (r_+), and to the negative charge (r_-) just using the distance formula:

$$\begin{aligned}
r &= \sqrt{x^2+y^2} \\
r_+ &= \sqrt{\left(x-\frac{d}{2}\right)^2+y^2} \\
r_- &= \sqrt{\left(x+\frac{d}{2}\right)^2+y^2}
\end{aligned}$$

Already, we know enough to write down the potential at point P due to the two charges. The principle of superposition says that the total electric potential at point P can be found by adding together the electric potentials from individual charges. Since potential is a scalar, we don't even need to worry about vectors, we just find the potential due to the positive charge, the potential due to the negative charge, and add them together as numbers. We will choose the zero for electric potential to be infinitely far away, so the potential at point P is just:

$$V_P = \frac{k_e q}{r_+} + \frac{k_e(-q)}{r_-} = \frac{k_e q\,(r_- - r_+)}{r_+ r_-}$$

At this point, it is clear why you are given the approximations above. Plug them in, and we are done:

$$V_P \approx \frac{k_e q d\cos\theta}{r^2}$$

Where did these magical approximations come from? They were given in the problem, so one could just plug them in, but we will quickly derive them just to be safe. First write down r_+ and expand:

$$r_+ = \sqrt{\left(x-\frac{d}{2}\right)^2+y^2} = \sqrt{r^2 - xd + \frac{d^2}{4}} = r\sqrt{1-\frac{xd}{r^2}+\frac{d^2}{4r^2}}$$

One of the conditions of the problem is that the point P is distant, which means $r \gg d$. If this is the case, then the d^2/r^2 term under the square root is negligible:

$$r_+ \approx r\sqrt{1-\frac{xd}{r^2}}$$

Again, since $r \gg d$, the fraction under the radical should be small compared to 1, and we can use the approximation $(1+a)^n \approx 1+ax$. Comparing this with what we have above, $n=\frac{1}{2}$ and $a=-\frac{xd}{r^2}$:

$$r_+ \approx r\left(1-\frac{xd}{2r^2}\right)$$

Now, do the same thing for r_- - you get the same result, except the minus sign becomes a plus. Now we can approximate the difference between the two easily:

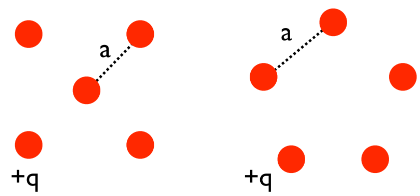
$$r_- - r_+ = r\left(1+\frac{xd}{2r^2}\right) - r\left(1-\frac{xd}{2r^2}\right) = \frac{xd}{r} = d\cos\theta$$

For the last step, we used the fact that $\cos\theta = x/r$, which you should be able to verify. You can also find this approximation geometrically, noting that $r_+ \approx r - \frac{d\cos\theta}{2}$ and $r_+ \approx r - \frac{d\cos\theta}{2}$. Try drawing a line from one charge that meets r perpendicularly to see how this works.

The second approximation we need is more straightforward to find. We just write down r_+r_-, using the approximate forms above, and once again drop terms that have d^2/r^2 in them:

$$\begin{aligned}
r_+r_- &\approx r\left(1-\frac{xd}{2r^2}\right)\cdot r\left(1+\frac{xd}{2r^2}\right)\\
&\approx r^2\left(1-\frac{x^2d^2}{4r^2}\right)\\
&\approx r^2
\end{aligned}$$

9. 15 points. Five identical point charges $+q$ are arranged in two different manners as shown below - in once case as a face-centered square, in the other as a regular pentagon. Find the potential energy of each system of charges, taking the zero of potential energy to be infinitely far away. Express your answer in terms of a constant times the energy of two charges $+q$ separated by a distance a. Bonus (3 points): could one make a two-dimensional repeating crystal with either of these arrangements? Justify your answer.



Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just:

$$PE_{\text{pair}} = \frac{k_e q^2}{d}$$

Since all of the charges are the same in both configurations, all we need to do is figure out how many pairs there are in each situation, and for each pair, how far apart the charges are.

How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force - which we will do as a check - but we can also calculate how many there are. In both configurations, we have 10 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair $(2,1)$ is the same as $(1,2)$. Pairings like this are known as combinations, as opposed to *permutations* where $(1,2)$ and $(2,1)$ are *not* the same. Calculating the number of possible combinations is done like this:[i]

$$\text{ways of choosing pairs from five charges} = \binom{5}{2} = {}^5C_2 = \frac{5!}{2!\,(5-2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = 10$$

So there are 10 unique ways to choose 2 charges out of 5. First, let's consider the face-centered square lattice. In order to enumerate the possible pairings, we should label the charges to keep them straight. Label the corner charges $1-4$, and the center charge 5 (it doesn't matter which way you number the corners, just so long as 5 is the middle charge). Then our possible pairings are:

[i]A nice discussion of combinations and permutations is here: `http://www.themathpage.com/aPreCalc/permutations-combinations.htm`

$$\begin{array}{llll} (1,2) & (1,3) & (1,4) & (1,5) \\ (2,3) & (2,3) & (2,5) & \\ (3,4) & (3,5) & & \\ (4,5) & & & \end{array}$$

And there are ten, just as we expect. In this configuration, there are only three different distances that can separate a pair of charges: pairs on adjacent corners are a distance $a\sqrt{2}$ apart, a center-corner pairing is a distance a apart, and a far corner-far corner pair is $2a$ apart. We can take our list above, and sort it into pairs that have the same separation:

Table 2: *Charge pairings in the square lattice*

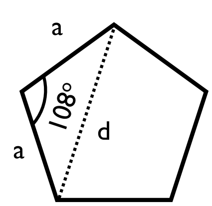
#, pairing type	**separation**				**pairs**
4, center-corner	a	$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$
4, adjacent corners	$a\sqrt{2}$	$(1,4)$	$(3,4)$	$(2,3)$	$(1,2)$
2, far corner	$2a$			$(1,3)$	$(2,4)$

And we are nearly done already. We have four pairs of charges a distance a apart, four that are $a\sqrt{2}$ apart, and two that are $2a$ apart. Write down the energy for each type of pair, multiply by the number of those pairs, and add the results together:

$$\begin{aligned} PE_{\text{square}} &= 4\,(\text{energy of center-corner pair}) + 2\,(\text{energy of far corner pair}) + 4\,(\text{energy of adjacent corner pair}) \\ &= 4\left[\frac{k_e q^2}{a}\right] + 2\left[\frac{k_e q^2}{2a}\right] + 4\left[\frac{k_e q^2}{a\sqrt{2}}\right] \\ &= \frac{k_e q^2}{a}\left[4 + 1 + \frac{4}{\sqrt{2}}\right] \\ &= \frac{k_e q^2}{a}\left[5 + 2\sqrt{2}\right] \approx 7.83\frac{kq^2}{a} \end{aligned}$$

For the pentagon lattice, things are even easier. This time, just pick one charge as "1", and label the others from 2-5 in a clockwise or counter-clockwise fashion. Since we still have 5 charges, there are still 10 pairings, and they are the same as the list above. For the pentagon, however, there are only *two* distinct distances - either charges can be adjacent, and thus a distance a apart, or they can be next-nearest neighbors. What is the next-nearest neighbor distance?

In a regular pentagon,[ii] each of the angles is $108°$, and in our case, each of the sides has length a, as shown below. We can use the law of cosines to find the distance d between next-nearest neighbors.



$$\begin{aligned} & d^2 = a^2 + a^2 - 2 \cdot a \cdot a \cos 108° = 2a^2\,(1 - \cos 108°) \\ \implies\quad & d = a\sqrt{2 - 2\cos 108°} = a\phi \approx 1.618a \end{aligned}$$

[ii]See `http://www.jimloy.com/geometry/pentagon.htm`

Here the number ϕ is known as the "Golden Ratio."[iii] The distances a and d automatically satisfy the golden ratio in a regular pentagon, $d/a = \phi$. Given the nearest neighbor distance in terms of a, we can complete a table of pairings for the pentagon:

Table 3: *Charge pairings in the pentagonal lattice*

#, pairing type	**separation**					**pairs**
5, next-nearest neighbors	d	$(1,3)$	$(1,4)$	$(2,4)$	$(2,5)$	$(3,5)$
5, adjacent	a	$(1,2)$	$(2,3)$	$(3,4)$	$(4,5)$	$(5,1)$

Now once again we write down the energy for each type of pair, and multiply by the number of pairs:

$$\begin{aligned}
PE_{\text{pentagon}} &= 5\,(\text{energy of adjacent pair}) + 5\,(\text{energy of next-nearest neighbor pair}) \\
&= 5\left[\frac{k_e q^2}{a}\right] + 5\left[\frac{k_e q^2}{d}\right] \\
&= 5\left[\frac{k_e q^2}{a}\right] + 5\left[\frac{k_e q^2}{a\sqrt{2-2\cos 108^\circ}}\right] \\
&= \frac{k_e q^2}{a}\left[5 + \frac{5}{\sqrt{2-2\cos 108^\circ}}\right] \\
&\approx \frac{k_e q^2}{a}\left[5 + \frac{5}{1.618}\right] \approx 8.09\frac{kq^2}{a}
\end{aligned}$$

So the energy of the pentagonal lattice is higher, meaning it is less favorable than the square lattice. Neither one is energetically favored though - since the energy is positive, it means that either configuration of charges is less stable than just having all five charges infinitely far from each other. This makes sense - if all five charges have the same sign, they don't want to arrange next to one another, and thus these arrangements cost energy to keep together. If we didn't force the charges together in these patterns, the positive energy tells us that they would fly apart given half a chance. For this reason, neither one is a valid sort of crystal lattice, real crystals have equal numbers of positive and negative charges, and are overall electrically neutral.

And the bonus? One can tile a floor with square tiles, but never with *regular* pentagons where all five sides are the same length. Try it - it won't work unless you make some of the pentagon's sides different lengths, or squish it in some way. In fact, only three regular polygons can tile a floor without gaps - triangles, squares, and hexagons. A good explanation why can be found here: `http://mathforum.org/sum95/suzanne/whattess.html`.

Also acceptable: since all charges are positive, these arrangements are inherently unstable anyway, and mutual repulsion would prevent one from making a crystal.

10. 10 points. If each of the charges in the pentagon arrangement above are $1\,\mu\text{C}$ and $a = 1\,\text{m}$, what is the electric potential at the center of the pentagon? Again take the zero of potential energy infinitely far away.

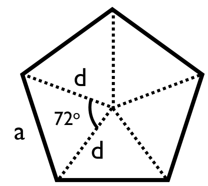
This one is easier than it sounds. The electric potential at the center due to one charge q a distance d away is:

$$V = \frac{k_e q}{d}$$

Since every charge is the same, and the same distance from the center of the pentagon, the principle of superposition says that we just need to find the potential due to one of the charges and multiply it by five. How far is each charge from the center? Have a look at the figure below.

The interior angles of the pentagon, defined by drawing lines from each vertex to the center, are $360^\circ/5 = 72^\circ$. Once again we can use the law of cosines to relate the distance d to a:

[iii] See `http://www.jimloy.com/geometry/golden.htm` and the prior link



$$
\begin{aligned}
& a^2 = d^2 + d^2 + 2 \cdot d \cdot d \cos 72^\circ \\
\implies \quad & d = \frac{a}{\sqrt{2\left(1-\cos 72^\circ\right)}} \approx 0.85\,\mathrm{m}
\end{aligned}
$$

The potential at the center of the pentagon is just five times the potential due to a single charge of $1\,\mu\mathrm{C}$ at a distance of $d \approx 0.5\,\mathrm{m}$:

$$
V_{\text{center}} = 5V_{\text{single}} = 5 \cdot \frac{k_e q}{d} \approx \frac{5\left(9\times10^{9}\,\mathrm{N}\cdot\mathrm{m/C^2}\right)\left(1\times10^{-6}\,\mathrm{C}\right)}{0.85\,\mathrm{m}} \approx 52900\,\mathrm{N}\cdot\mathrm{m/C} = 52.9\,\mathrm{kV}
$$

For the very last part, we note that one newton per coulomb is one volt per meter, $[\mathrm{N/C}]=[\mathrm{V/m}]$, so volts must be newtons times meters per coulomb.