University of Alabama
Department of Physics and Astronomy

PH 102-2 / LeClair Spring 2008

Problem Set 4 SOLUTION

1. 10 points. An 11.0 W compact fluorescent bulb is designed to produce the same illumination as a conventional 40.0 W incandescent bulb. Assuming a cost of \$0.0800/kWh for electrical power, how much money does the user of the fluorescent bulb save over 100 hr of use?

Since the cost of electricity is given in \$/kWh, we just need to multiply kW for each bulb by the 100 hr time period:

$$\begin{aligned}
\text{cost of CF bulb} &= 11\,\text{W} \cdot \left(\frac{1\,\text{kW}}{1000\,\text{W}}\right) \cdot (100\,\text{hr}) \cdot (\$0.0800/\text{kWh}) \approx \$0.088 \\
\text{cost of conventional bulb} &= 40\,\text{W} \cdot \left(\frac{1\,\text{kW}}{1000\,\text{W}}\right) \cdot (100\,\text{hr}) \cdot (\$0.0800/\text{kWh}) \approx \$0.32 \\
\text{difference over 100 hours} &= \$0.23 \\
\text{difference per hour} &\approx \$0.0023
\end{aligned}$$

Over 100 hours of continuous use, one saves about 23 cents.

2. 5 points. If the voltage at the terminals of an automobile battery drops from 12.3 to 9.8 V when a 0.5 Ω resistor is connected across the battery, what is the internal resistance?

Before the 0.5 Ω resistor is connected, there is no current flowing in the battery, and what we measure at the terminals is its ideal rated voltage, $\Delta V = 12.3\,\text{V}$. Once the resistor is connected, a current I flows, and the output voltage of the battery is reduced by its internal resistance. What we measure at the battery terminals is the rated voltage *minus the voltage drop across the internal resistance.* Call the internal resistance r, and the external 0.5 Ω resistor R. With the resistor connected, conservation of energy says that the sum of voltage sources and drops around the entire circuit must be zero:

$$\Delta V - Ir - IR = 0$$

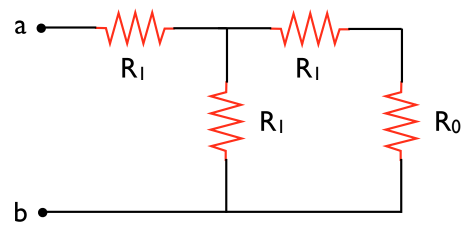
What we measure at the terminals of the battery is $\Delta V - Ir = 9.8\,\text{V}$ - the ideal battery voltage minus what is lost due to internal resistance. Using this relationship in the equation above, we have:

$$\begin{aligned}
\Delta V - Ir - IR &= 0 \\
9.8\,\text{V} - IR &= 0 \\
\Longrightarrow \quad I &= \frac{9.8\,\text{V}}{0.5\,\Omega} = 19.6\,\text{A}
\end{aligned}$$

Now that we know what I is, we can use the measured voltage with the resistor connected to find r, the internal resistance:

$$\begin{aligned}
\Delta V - Ir &= 9.8\,\text{V} \\
12.3\,\text{V} - (19.6\,\text{A})\,r &= 9.8\,\text{V} \\
\Longrightarrow \quad r &= \frac{2.5\,\text{V}}{19.6\,\text{A}} \approx 0.13\,\Omega
\end{aligned}$$

3. 15 points. In the circuit below, if R_0 is given, what value must the R_1 have for the equivalent resistance between the two terminals a and b to be R_0?



This one is, admittedly, a bit messy. The end result does have a certain elegance though ...

With any complicated resistor problem, we first try to find sets of two resistors purely in parallel or purely in series. Combine any such pairs, lather, rinse, repeat. The first pair we can spot - and the only one which is purely in series or parallel - is resistor R_0 in series with the rightmost R_1. We cannot combine any other resistors, since no other pairs are purely in series or parallel. Putting together R_1 and R_0 makes an equivalent resistor R_2, whose value we can calculate easily:

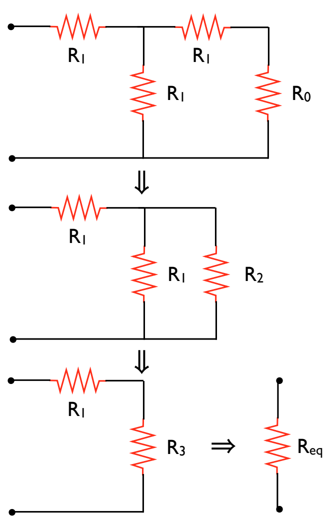
$$R_2 = R_1 + R_0$$

This will leave the new resistor purely in *parallel* with the middle R_1, which means we can combine R_2 and R_1 into a new resistor R_3:

$$\frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_1} = \frac{1}{R_1 + R_0} + \frac{1}{R_1} = \frac{R_1 + R_0 + R_1}{R_1\left(R_1 + R_0\right)} = \frac{2R_1 + R_0}{R_1^2 + R_1 R_0}$$

$$\Longrightarrow \quad R_3 = \frac{R_1 R_0 + R_1^2}{2R_1 + R_0}$$

Our progress so far is shown below.



Now we only have R_3 and one R_1 left, purely in series. Combining them will give us one single equivalent resistor R_{eq}:

$$
\begin{aligned}
R_{eq} = R_1 + R_3 = \frac{R_1R_0 + R_1^2}{2R_1 + R_0} + R_1 &= \frac{R_1R_0 + R_1^2}{2R_1 + R_0} + \frac{R_1\left(2R_1 + R_0\right)}{2R_1 + R_0} \\
&= \frac{R_1R_0 + R_1^2 + 2R_1^2 + R_1R_0}{2R_1 + R_0} \\
&= \frac{3R_1^2 + 2R_1R_0}{2R_1 + R_0}
\end{aligned}
$$

The final bit of the problem says that we want the equivalent resistance to be exactly R_0. We just need to set the above equal to R_0, and solve for R_1 in terms of R_0.

$$
\begin{aligned}
R_0 &= \frac{3R_1^2 + 2R_1R_0}{2R_1 + R_0} \\
R_0\left(2R_1 + R_0\right) &= 3R_1^2 + 2R_1R_0 \\
2R_0R_1 + R_0^2 &= 3R_1^2 + 2R_1R_0 \\
R_0^2 &= 3R_1^2 \\
\Longrightarrow \qquad R_1 &= \frac{R_0}{\sqrt{3}}
\end{aligned}
$$

4. 10 points. An aluminum wire with a cross-sectional area of $4.00 \times 10^{-6}\,\mathrm{m}^2$ carries a current of 5.00 A. Find the drift speed of the electrons in the wire. The density of aluminum is $2.70\,\mathrm{g/cm^3}$; assume each Al atom provides a single electron for conduction. *Hint: how many atoms per unit volume are there? How many charges per unit volume does this imply?*

Each aluminum atom donates one mobile electron for conducting electricity. Therefore, finding the number of carriers per unit volume n is equivalent to finding the number of aluminum atoms per unit volume - easily found from the density and molar mass of aluminum and Avogadro's number:

$$
\begin{aligned}
n &= \left[2.70\,\frac{\mathrm{g}}{\mathrm{cm}^3}\right]\left[\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right]^3\left[\frac{1\,\mathrm{mol}}{27\,\mathrm{g}}\right]\left[6.02\times10^{23}\,\frac{\mathrm{atoms}}{\mathrm{mol}}\right]\left[1\,\frac{\mathrm{electron}}{\mathrm{atom}}\right] \\
&= 6.02\times10^{28}\,\frac{\mathrm{electrons}}{\mathrm{m}^3}
\end{aligned}
$$

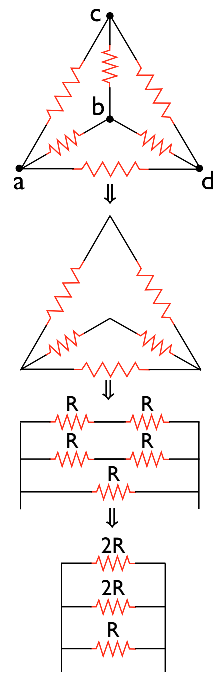
Now we have the carrier density n, and we further know the current I and conductor area A. Since the conduction is stated to be due to electrons, we also know the charge per carrier $q{=}e$. This is as much as we need to find the drift velocity:

$$
\begin{aligned}
v_d = \frac{I}{nqA} &= \frac{5.00\,\mathrm{A}}{\left(6.02\times10^{28}\,\frac{\mathrm{electrons}}{\mathrm{m}^3}\right)\left(1.60\times10^{-19}\,\frac{\mathrm{C}}{\mathrm{electron}}\right)\left(4.00\times10^{-6}\,\mathrm{m}^2\right)} \\
&= 1.3\times10^{-4}\,\frac{\mathrm{A}\cdot\mathrm{m}}{\mathrm{C}} = 1.3\times10^{-4}\,\mathrm{m/s}
\end{aligned}
$$

If we remember that one Ampere is one Coulomb per second and the units come out just fine.

5. 15 points. A regular tetrahedron is a pyramid with a triangular base. Six $14.0\,\Omega$ resistors are placed along its six edges, with junctions at its four vertices. A 9.0 V battery is connected to any two of the vertices. **(a)** Find the equivalent resistance of the tetrahedron between these vertices. **(b)** Find the current in the battery.

Just by inspection, it is clear that we can't use our usual rules for parallel and series resistors to reduce this circuit - those simple rules cannot handle junctions with three branches. We can figure out how to reduce it by symmetry, however. Consider a current flowing out of point a below. The current must split up into three equal portions, since all three branches from point a are connected to the same resistance. Thus, the currents in branches ab, ac, and ad must be equal. If this is true, then the potential difference at points b, c, and d must be the same - at all three points, the same current has flowed from point a through the same resistance, so the voltage drops from a to c, a to b, and a to d are the same. This means that points b and c are at the same potential (as is point d), and there is no voltage across the resistor between points b and c. If there is no potential difference across the resistor bc, then there is no current by Ohm's law.



If there is no voltage difference and no current across the resistor between b and c, then it may be removed from the circuit - it isn't doing anything! If we take out that resistor, we have the second diagram above. If we rearrange this circuit - which we can always do, so long as no wires are broken in the process - then we see it is equivalent to the third diagram above. This circuit *does* allow a simple analysis based on series and parallel combinations.

First, combine the series resistors in the upper to branches. These add together to give two equivalent resistors of $2R$, which results in the simple parallel resistor diagram in the last panel above. These three parallel resistors give one equivalent resistance:

$$\frac{1}{R_{eq}} = \frac{1}{2R} + \frac{1}{2R} + \frac{1}{R} = \frac{4}{2R}$$

$$\Longrightarrow \quad R_{eq} = \frac{1}{2}R$$

Thus, a regular tetrahedron of resistors can be replaced by a single resistor of half the value of the individual components. In this case, that means the equivalent resistance is $7\,\Omega$. If this $7\,\Omega$ equivalent resistance is connected to a $9\,\mathrm{V}$ battery, the total current flowing out of the battery must be:

$$I = \frac{\Delta V}{R_{eq}} = \frac{9\,\mathrm{V}}{7\,\Omega} \approx 1.29\,\mathrm{A}$$

Incidentally, there is another way to analyze this circuit. Once we realize there is no potential difference between points b and c, rather than removing the resistor we could just connect those two points together. Since they are at the same voltage, this does not affect the operation of the circuit. You can verify for yourself that if you simply connect points b and c in the second diagram above, the same equivalent resistance results.

6. 5 points. A conductor of uniform radius $1.2\,\mathrm{cm}$ carries a current of $3.0\,\mathrm{A}$ produced by an electric field of $120\,\mathrm{V/m}$. What is the resistivity of the material?

We can use the current density J to relate both current I and area A and electric field E and resistivity ϱ. Keep in mind that by area we mean the cross-sectional area of the conductor, $A = \pi r^2$.

$$J = \frac{I}{A} = \frac{E}{\varrho}$$
$$\implies \quad \varrho = \frac{EA}{I} = \frac{(120\,\mathrm{V/m}) \cdot \left[\pi\,(0.012\,\mathrm{m})^2\right]}{3.0\,\mathrm{A}} = 0.018\,\Omega \cdot \mathrm{m}$$

7. 5 points. A common 1.5 V "D" cell battery can supply about 0.100 A of current for about 100 h, hence its capacity rating of 10000 mA·h. How high could you lift yourself with one "D" cell battery powering a 50% efficient winch? Note that your mass in kg can be found by dividing your weight in pounds by 2.2.

What we want to do is find the total energy content of the battery, and relate this to a change in gravitational potential energy, less the 50% (in)efficiency. Power is energy per unit time, so the energy content of the battery is the power it can deliver multiplied by the length of time it will last:

$$E = \mathscr{P}\Delta t = (I\Delta V)\,\Delta t = (1.5\,\mathrm{V}) \cdot (0.1\,\mathrm{A}) \cdot \left(100\,\mathrm{h} \cdot \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \cdot \frac{60\,\mathrm{s}}{1\,\mathrm{min}}\right) = 54000\,\mathrm{J}$$

We are told our winch is 50% efficient - what this means is that half of this total energy content can be converted into mechanical energy, 27000 J. In order to lift oneself with this winch, mechanical energy must be converted into gravitational potential energy. Using my mass:

$$\begin{aligned}
\text{mechanical energy} &= \text{gravitational potential energy} \\
0.5E &= mg\Delta h \\
27000\,\mathrm{J} &= mg\Delta h \\
\implies \quad \Delta h &= \frac{27000\,\mathrm{J}}{(63\,\mathrm{kg}) \cdot \left(9.81\,\mathrm{m/s^2}\right)} \approx 44\,\mathrm{m}
\end{aligned}$$

Recall that a Joule is equivalent to $1\,\mathrm{kg} \cdot \mathrm{m^2/s^2}$.

By the way, Mythbusters did demonstrate that a handheld winch running off of battery power could lift a person as high as 20 m. They didn't use D-cells though - the problem is not the energy content, but that it can only be withdrawn fairly slowly. The D-cell above likes to deliver $\mathscr{P} = I\Delta V = 0.15\,\mathrm{W}$, which means 0.15 J/s. At that rate, raising yourself one meter would take about 4000 s, or a bit over an hour.

`http://mythbustersresults.com/episode86`

8. 10 points. If the current carried by a conductor is doubled, what happens to the **(a)** charge carrier density? **(b)** Current density? **(c)** Electron drift velocity? **(d)** Average time between collisions?

As noted in problem 4 above, the carrier density in typical metallic conductors is mainly determined by the number and type of atoms present. It is a fixed property of a conductor, and does not change when current changes. Current is the rate at which charge flows, a larger current does not imply more charges, only that they move faster.[i]

The current density is just current per unit area. For a fixed conductor, if current doubles, then current density must also double. Drift velocity is related to current in a simple way, we found:

$$I = nqv_dA$$

[i] This is not true in a semiconductor, where the number of carriers is a sensitive function of temperature, purity, doping, *etc.*

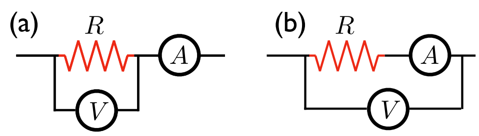
The number of carriers remains constant, as does the area A. The charge per carrier q is obviously fixed as well, which means that if I doubles, v_d must also double. This makes sense - larger current means on the whole the charges are moving faster along the wire, not that there are more of them.

Finally, the collision time τ can be related to resistivity:

$$\varrho = \frac{m}{ne^2\tau}$$

In this equation, all parameters are fixed - ϱ is a material constant, m and e are fixed properties of the charge carriers, and n we have already decided is constant. Thus, the time between collisions does not depend on the current value. This makes sense too - our model of conduction assumes that random thermal motion of the carriers is responsible for collisions, and far outweighs the relatively tiny drift velocity.

9. 15 points. The value of an unknown resistor is to be determined with an ammeter and voltmeter, as shown below. The ammeter has an internal resistance of $0.500\,\Omega$, and the voltmeter has an internal resistance of $20,\mathrm{k}\Omega$. Within what range of actual values of R will the measured values be correct to within 5% if the measurement is made using the circuit shown in **(a)** and **(b)**?



In circuit **(a)**, at least a little current goes through the voltmeter, which reduces the current through the resistor slightly. Since it is in parallel with the resistor, it does at least measure the proper voltage. The ammeter measures the total current, however, since it is outside of the parallel circuit formed by the voltmeter and resistor. This means that the apparent resistance - the measured voltage divided by the measured current - will be a little to *low*, since the measured current is higher than the actual value.

Let R_m be the measured resistance, R the actual value, I_R the current through the resistor, and I the current read by the ammeter. When we use circuit **(a)**, the voltmeter reads a potential difference of $I_R R$ due to the current through the resistor. This voltage difference must also be the same as that due to the current through the voltmeter times its internal resistance. If the total current is I, and I_R flows through the resistor, then conservation of charge says a current $I-I_R$ must go through the voltmeter. Thus:

$$\begin{aligned} I_R R = \Delta V &= 20000\,(I - I_R) \\ \Longrightarrow \quad I &= I_R\left[\frac{R+20000}{20000}\right] \end{aligned}$$

The apparent measured resistance R_m is the voltage measured by the voltmeter divided by the current measured by the ammeter, $R_m\!=\!\Delta V/I$. We want this to be within 5% of the actual resistance R, and we already know that this circuit will always *underestimate* the resistance. Thus, our condition is that $R_m\!\geq\!0.95R$:

$$\begin{aligned} R_m = \frac{\Delta V}{I} &\geq 0.95R \\ \frac{I_R R}{I_R\left[\frac{R+20000}{20000}\right]} &\geq 0.95R \\ \frac{20000}{R+20000} &\geq 0.95 \\ 20000 &\geq 0.95\,(R+20000) \\ \Longrightarrow \quad R &\leq 1050\,\Omega \end{aligned}$$

Thus, this circuit is good for measuring low resistances.

What about circuit **(b)**? In this case, the ammeter reads exactly the current through the resistor, I_R, since they are in the same branch of the circuit. The voltmeter, however, how reads the potential difference across the resistor *and* that across the $0.5\,\Omega$ ammeter. It will always read an anomalously large voltage, which means it will systematically *overestimate* the resistance. If a current I_R flows through the ammeter and resistor, then the voltmeter must read

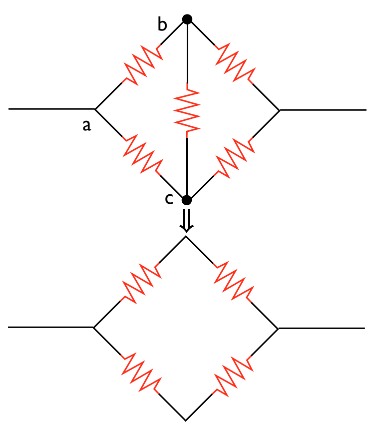
$$\Delta V = I_R\,(R + 0.5\,\Omega)$$

The apparent measured resistance is again just the measured voltage divided by the measured current. Since this circuit overestimates resistance, to stay within 5% our condition is $R_m \leq 1.05R$:

$$\begin{aligned} R_m = \frac{\Delta V}{I} &\leq 1.05R \\ \frac{I_R\,(R + 0.5\,\Omega)}{I_R} &\leq 1.05R \\ R + 0.5\,\Omega &\leq 1.05R \\ \Longrightarrow \quad R &\geq 10\,\Omega \end{aligned}$$

The range of R values seems about right since the ammeters resistance should be less than 5% of the smallest R value we want to measure, and R should be smaller than 5% of the voltmeter's internal resistance. Only for the restricted range between 10 and about $1000\,\Omega$ can we indifferently use either circuit for a reasonably accurate resistance measurement. For low resistances, circuit **(a)** must be used, while only circuit **(b)** can accurately measure a large resistances.

10. 10 points. All of the resistors in the figure below are equal, and have a value R. What is the equivalent resistance between the end points?



For this problem, we proceed much like in problem 5. Current leaving point a from the left must split up into two equal portions in branches ac and ab, since the resistances between a-b and a-c are the same. If the current is the same in ab and ac, then the voltage drop from point a to b is the same as that from point a to c. Thus, there is no potential difference between points b and c, and by Ohm's law, no current.

If the middle resistor between points b and c has no current and no potential difference, it performs no function and can be removed. If we do that, what we are left with is a simple series-parallel combination. There are only two branches now, each with two identical resistors. Applying the rules for series resistors, we have left two parallel resistors, each of value $2R$. These two combine to form a single equivalent resistor of value R.