University of Alabama
Department of Physics and Astronomy

PH 102-2 / LeClair Spring 2008

Problem Set 6: Magnetism

1. 10 points. A wire with a weight per unit length of 0.10 N/m is suspended directly above a second wire. The top wire carries a current of 30 A and the bottom wire carries a current of 60 A. Find the distance of separation between the wires so that the top wire will be held in place by magnetic repulsion.

What we want to do here is balance the force of gravity pulling the top wire downward, F_g, with a magnetic force between the two wires that pushes the top wire up, $F_{B,12}$. This means that the wires should *repel* each other, so the currents must be running antiparallel to each other (*i.e.,* in opposite directions). We know that the magnetic force *per unit length* between the two current-carrying wires, $F_{B,12}$ can be written as:

$$\frac{F_{B,12}}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

This tells us that if we have a length l of one wire, the force on that length due to the other wire is $F_{B,12}$. What is that length? We aren't given a length at all, which leads one to suspect it isn't necessary. In fact, it isn't, as we will find out shortly.

Next we need the gravitational force on the top wire - its weight. We are in fact given its weight per unit length already, which we will call $w = 0.10\,\mathrm{N/m}$. Actually, this is highly convenient: this is the gravitational force (what weight is, really) per unit length, F_g/l. If we think about it: we have an expression for the magnetic force per unit length already, and now we have the gravitational force per unit length. All we have to do is equate them:

$$\begin{aligned} F_g &= F_{B,12} \\ \frac{F_g}{l} &= \frac{F_{B,12}}{l} \\ w &= \frac{\mu_0 I_1 I_2}{2\pi d} \\ \Longrightarrow \qquad d &= \frac{\mu_0 I_1 I_2}{2\pi w} = \frac{\left[4\pi \times 10^{-7}\,\mathrm{N/A^2}\right][30\,\mathrm{A}][60\,\mathrm{A}]}{2\pi\,[0.1\,\mathrm{N/m}]} \approx 3.6 \times 10^{-3}\,\mathrm{m} = 3.6\,\mathrm{mm} \end{aligned}$$

Note that the constant μ_0 can be expressed in either $\mathrm{T \cdot m/A}$ or $\mathrm{N/A^2}$ – the two sets of units are equivalent, though the latter is much more convenient in this particular case.

2. 5 points. An electron moving along the positive x axis perpendicular to a magnetic field experiences a magnetic deflection in the negative y direction. What is the direction of the magnetic field?

We need to remember two things here: the right hand rule, and the fact that positive and negative charges move oppositely in response to electromagnetic fields. First, let us pretend that the charge in question is positive, just for clarity.

If a positive charge is moving along the x axis, and is deflected in the negative y direction, that means it is being acted on by a force in the negative y direction. Using the right hand rule, your fingers should point along the particle's velocity, and the force should come out of the back of your hand. Your thumb then points along the required direction of the magnetic field, namely, out of the page, or in the $+z$ direction.

But wait! That is what will happen for a positive charge. Since the electron in question has a *negative* charge, it will simply move in the opposite direction, so to move it in the negative y direction will require a magnetic field in the opposite direction compared to a positive charge. Thus, the field must be pointing *into* the plane of the page, in the $-z$ direction.

3. 10 points. A conductor suspended by two flexible wires as shown in the figure has a mass per unit length of 0.0400 kg/m. What current must exist in the conductor in order for the tension in the supporting wires to be zero when the magnetic field is 3.60 T into the page? What is the required direction for the current?

Each of the wires has in it a tension equal to the weight of the conductor, and this weight must be balanced by the magnetic force on the conductor for the tension to be zero:

$$\begin{aligned} & T = W - F_B = mg - F_B = 0 \\ \Longrightarrow \quad & F_B = mg \end{aligned}$$

The magnetic force on a conductor of length l, carrying a current I, in a magnetic field B is $F_B = BIl$. Once again, do not get nervous about the equation having a length l in it that you don't know. Work out the problem in a way that makes *sense* – if you do that, all the quantities you don't know should disappear. Either that, or they should be something you can determine from other quantities you do know. Of course, this only works on problem sets. Real life problems always have things you don't know in them, so you learn to estimate ...

We digress. The magnetic force on the conductor was easy, now we just need to find the weight of the conductor. We can find that from its mass, which in turn must just be its mass per unit length ($\lambda = 0.04\,\mathrm{kg/m}$) times the length l - the kilograms per meter given, times meters, gives mass in kilograms:

$$\begin{aligned} m &= \lambda l \\ W &= mg = \lambda l g \end{aligned}$$

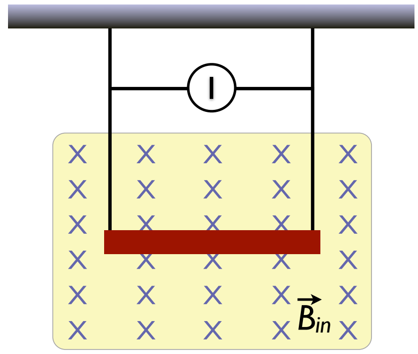
Now we just put it together, and balance the weight and magnetic force:

$$\begin{aligned} & F_B = mg \\ & BIl = \lambda l g \\ \Longrightarrow \quad & I = \frac{\lambda g}{B} = \frac{[0.04\,\mathrm{kg/m}]\,[9.81\,\mathrm{m/s^2}]}{3.60\,\mathrm{T}} \approx 0.109\,\frac{[\mathrm{kg/m}]\cdot[\mathrm{m/s^2}]}{[\mathrm{N/A}]\cdot\mathrm{m}} = 0.109\,\frac{\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{A}}{\mathrm{N}\cdot\mathrm{s^2}} = 0.109\,\mathrm{A} \end{aligned}$$

In order to make the units come out correctly on this one, note that a Tesla (T) is the same as a Newton per Amp per meter: $1\,\mathrm{T} = \mathrm{N/A}\cdot\mathrm{m}$. What about the direction of the current? We want the magnetic force to point upward, balancing the weight of the conductor. Using the right-hand rule: with force pointing upward, and magnetic field pointing into the page, the current must flow to the right side. Try it: put your thumb along B, and make the back of your hand face upward in the direction of the force. This puts your fingers pointing to the right, which must be the direction of the current.

4. 10 points. A 40.0 cm length of wire carries a current of 20.0 A. It is bent into a loop and placed with its normal perpendicular to a magnetic field with a magnitude of 0.520 T. What is the torque on the loop if it is bent into a **(a)** equilateral triangle? What is the torque if the loop is **(b)** a square, or **(c)** a circle?

The torque on a current loop, provided the current is constant, is just



$$\tau = BIA \sin\theta$$

We already know the current in each loop I, and the external magnetic field B. We also know that the area normal is perpendicular to B, so $\theta = 90$ and we have simply $\tau = BIA$. What about the area?

First, the square. With 0.4 m of wire in total, we can afford to make each side 0.1 m long. In that case, the area of the square is just $(0.1\,\mathrm{m})^2$, or $0.01\,\mathrm{m}^2$. Thus:

$$\tau_{\text{square}} = BIA = [0.52\,\mathrm{T}]\,[20\,\mathrm{A}]\,\left[0.01\,\mathrm{m}^2\right] \approx 0.104\,\mathrm{N} \cdot \mathrm{m}$$

The units are easier to see if you note, as in the last problem, that $1\,\mathrm{T} = \mathrm{N}/\mathrm{A} \cdot \mathrm{m}$, which makes the torque come out in units of $\mathrm{N} \cdot \mathrm{m}$.

How about the triangle? With 0.4 m of wire, each (equal) side can be $0.4/3 \approx 0.133$ m long. The area of a triangle is half the base times the height. Since each side of the triangle has an equal length of $0.4/3$ m, you should be able to verify that its height is $\sqrt{3}/2$ times the length a side, and its area is then (approximately) $7.7 \times 10^{-3}\,\mathrm{m}^2$. The torque is then:

$$\tau_{\text{triangle}} = BIA \approx [0.52\,\mathrm{T}]\,[20\,\mathrm{A}]\,\left[0.0077\,\mathrm{m}^2\right] = 0.081\,\mathrm{N} \cdot \mathrm{m}$$

And the circle? With 0.4 m of wire, the *circumference* of the wire can be 0.4 m, so $C = 2\pi r = 0.4\,\mathrm{m}$. Thus, $r \approx 0.0647\,\mathrm{m}$, and the circle's area is approximately $0.0127\,\mathrm{m}^2$. The torque is then:

$$\tau_{\text{circle}} = BIA = [0.52\,\mathrm{T}]\,[20\,\mathrm{A}]\,\left[0.0127\,\mathrm{m}^2\right] \approx 0.132\,\mathrm{N} \cdot \mathrm{m}$$

Thus, the torque is largest on the circle, next largest on the square, and smallest on the circle:

$$\tau_{\text{circle}} > \tau_{\text{square}} > \tau_{\text{triangle}}$$

A circle is in fact the two-dimensional surface with the maximal area for a given perimeter, so it makes sense that its torque is the largest for the same current and field. If all sides are the same length, the more sides an N sided polygon has, the larger the area.

5. 5 points. A proton moving in a circular path perpendicular to a constant magnetic field takes $1.00\,\mu$s to complete one revolution. Determine the magnitude of the magnetic field.

If the proton moves in a circular path of radius r, its period of motion T can be found by equating the magnetic force to the net centripetal force it must feel, noting that the period is the distance covered in one revolution ($2\pi r$) divided by the velocity v. First, a force balance gives the velocity:

$$\begin{aligned} F_B &= F_c \\ qvB &= \frac{mv^2}{r} \\ v &= \frac{qrB}{m} \end{aligned}$$

Next, the velocity gives the period:

$$\begin{aligned} T &= \frac{2\pi r}{v} \\ &= \frac{2\pi mr}{qrB} \\ &= \frac{2\pi m}{qB} \end{aligned}$$

This is a result we already derived in class. Now all we have to do is solve this for B, and plug in the given period of $1\,\mu$s:

$$\begin{aligned} B &= \frac{2\pi m}{qT} = \frac{2\pi\left[1.67\times10^{-27}\,\text{kg}\right]}{\left[1.6\times10^{-19}\,\text{C}\right]\left[1.00\,\mu\text{s}\right]} \\ &\approx 6.55\times10^{-2}\,\frac{\text{kg}}{\text{C}\cdot\text{s}} = 6.55\times10^{-2}\,\frac{\text{kg}\cdot\text{s}}{\text{C}\cdot\text{s}^2} \\ &= 6.55\times10^{-2}\,\frac{\text{kg}\cdot\text{m}\cdot\text{s}}{\text{C}\cdot\text{m}\cdot\text{s}^2} = 6.55\times10^{-2}\,\frac{\text{N}}{\text{A}\cdot\text{m}} = 6.55\times10^{-2}\,\text{T} \end{aligned}$$

As it turns out, one Tesla is also a kilogram per Coulomb per second! ($1\,\text{T}=1\,\text{kg}/\text{C}\cdot\text{s}$).

6. 10 points. What current is required in the windings of a long solenoid that has 1000 turns uniformly distributed over a length of 0.400 m to produce at the center of the solenoid a magnetic field of magnitude of 1.00×10^{-4} T.

The magnetic field produced by a solenoid of N turns over length L carrying a current I is given by:

$$B = \mu_0 \frac{N}{L} I$$

All we need to do is plug in the given numbers, and solve this for I:

$$\begin{aligned} I &= \frac{BL}{\mu_0 N} \\ &= \frac{\left[10^{-4}\,\text{T}\right]\left[0.4\,\text{m}\right]}{\left[4\pi\times10^{-7}\,\text{N}/\text{A}^2\right]\left[1000\right]} \\ &= 0.0318\,\frac{\text{T}\cdot\text{m}}{\text{N}/\text{A}^2} = 0.0318\,\text{A}^2\cdot\frac{\text{T}\cdot\text{m}}{\text{N}} \\ &= 0.0318\,\text{A}^2\cdot\frac{1}{\text{A}} = 0.0318\,\text{A} \end{aligned}$$

Based on the problems above, you should now know that $1\,\mathrm{T} = 1\,\mathrm{N/A \cdot m}$, so $1\,\mathrm{T \cdot m/N} = 1/\mathrm{A}$.

7. 15 points. One electron collides elastically with a second electron initially at rest. After the collision, the radii of their trajectories are 1.00 cm and 2.40 cm. The trajectories are perpendicular to a magnetic field of magnitude 0.0440 T. Determine the energy (in keV) of the incident electron.

The initial kinetic energy of the first electron, K_i must be split up between both electrons, based on conservation of energy – after the collision, the first electron gives some of its kinetic energy to the second. If we call the final velocity of the electrons v_{1f} and v_{2f}, we have:

$$K_i = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 = K_f$$

Since both particles are electrons, we do note need to label the masses, they are all the same. After the collision, we know that both particles follow a circular path, of radii r_1 and r_2, respectively, due to the B field perpendicular to their plane of motion. This means that the magnetic force gives the centripetal force, and for each particle,

$$\begin{aligned} qvB &= \frac{mv^2}{r} \\ \Longrightarrow \quad v &= \frac{qBr}{m} \end{aligned}$$

Thus, given the radius of the electrons' paths and the magnetic field, we know their velocity, which means we know their kinetic energy. Plugging the equation above into the expression for K_i, and noting that $q = -e$ for an electron,

$$\begin{aligned} K_i &= \frac{1}{2}m\left[\frac{e^2B^2r_1^2}{m^2} + \frac{e^2B^2r_2^2}{m^2}\right] \\ &= \frac{e^2B^2}{2m}\left[r_1^2 + r_2^2\right] \\ &\approx 1.84 \times 10^{-14}\,\frac{\mathrm{C}^2 \cdot \mathrm{T}^2 \cdot \mathrm{m}^2}{\mathrm{kg}} \\ &= 1.84 \times 10^{-14}\,\frac{\mathrm{C}^2 \cdot \mathrm{kg}^2 \cdot \mathrm{m}^2}{\mathrm{kg} \cdot \mathrm{C}^2 \cdot \mathrm{s}^2} \\ &= 1.84 \times 10^{-14}\,\frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}^2} = 1.84 \times 10^{-14}\,\mathrm{J} \\ &= 115\,\mathrm{keV} \end{aligned}$$

Here we used the result from a few problems ago that $1\,\mathrm{T} = 1\,\mathrm{kg/C \cdot s}$. By now, you already know that $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$.

8. 10 points. Electrons are accelerated from rest through a potential difference of 350 V. The electrons travel along a curved path because of the magnetic force exerted on them, and the radius of the path is measured to be 7.5 cm. If the magnetic field is perpendicular to the motion of the electrons, what is the magnitude of the magnetic field?

This one is just conservation of energy to start with, as we did weeks ago. The electrons of charge $-e$ are accelerated from rest by a potential difference ΔV, which means they loose a potential energy $e\Delta V$ and gain kinetic energy. In order for the electrons to be accelerated by the potential difference, it must be negative - staring out low, and ending high, since electrons are negatively charged.

$$
\begin{aligned}
K_i + U_i &= K_f + U_f \\
0 + (-e)(-\Delta V) &= \frac{1}{2} m v_f^2 + 0 \\
\Longrightarrow v &= \sqrt{\frac{2e\Delta V}{m}}
\end{aligned}
$$

Given the electrons velocity v and the radius of its path r, we have already found in several problems above their relationship to the magnetic field B:

$$
\begin{aligned}
B &= \frac{mv}{er} \\
&= \frac{m}{er}\sqrt{\frac{2e\Delta V}{m}} = \sqrt{\frac{2m\Delta V}{er^2}} \\
&= \sqrt{\frac{2\,[9.11\times10^{-31}\,\mathrm{kg}]\,[350\,\mathrm{V}]}{[1.6\times10^{-19}\,\mathrm{C}]\,[0.075\,\mathrm{m}]^2}} \\
&\approx 8.4\times10^{-4}\,\mathrm{T}
\end{aligned}
$$

It takes some doing ... but if you recognize that $1\,\mathrm{V}=1\,\mathrm{N}\cdot\mathrm{m/C}$, $1\,\mathrm{T}=1\,\mathrm{kg/C}\cdot\mathrm{s}$, and $1\,\mathrm{N}=1\,\mathrm{kg}\cdot\mathrm{m/s^2}$, you can make the units work out.

9. 10 points. A coil consists of 200 turns of wire. Each turn is a square of side 18 cm, and a uniform magnetic field directed perpendicular to the plane of the coil is turned on. If the field changes linearly (*i.e.,* uniformly) from 0 to 0.5 T in 0.80 s, what is the magnitude of the induced voltage in the coil while the field is changing?

If the square has side $l=0.18\,\mathrm{m}$, the area of one turn of the coil is $A=l^2=0.18^2=0.0324\,\mathrm{m}^2$. The magnetic flux through the coil at $t=0$ is zero because $B=0$ then. At time t, the flux through one turn is

$$
\Phi_B = BA = (0.50\,\mathrm{T})\left(0.0324\,\mathrm{m}^2\right) = 0.0162\,\mathrm{T}\cdot\mathrm{m}^2 \tag{1}
$$

The induced voltage is then the difference in flux for one turn over the elapsed time, multiplied by the number of turns

$$
\Delta V = N\frac{\Delta\Phi_B}{\Delta t} = 200\frac{\left(0.0162\,\mathrm{T}\cdot\mathrm{m}^2 - 0\right)}{0.80\,\mathrm{s}} = 4.1\,\mathrm{Tm^2/s} = 4.1\,\mathrm{V} \tag{2}
$$

10. 10 points. What is the maximum voltage induced across a coil of 4000 turns, average radius 12 cm, rotating at 30 revolutions per second in the earth's magnetic field, where the field is approximately 5×10^{-5} T?

The coil is rotating at constant angular velocity, while the magnetic field is static. As the loop rotates, the magnetic flux oscillates in time, since the area of the facing the magnetic field is oscillating in time. At some point in time, the area normal of the loop will make an angle θ with the magnetic field. Since the loop rotates with constant angular velocity, we know $\theta=wt$. Thus, the flux through the loop must be

$$
\Phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta = BA\cos\omega t
$$

The induced voltage is given by Faraday's law, noting that there are $N=4000$ turns in the loop, and that we know the area of the loop in terms of the given radius r.

$$\Delta V = -N\frac{\Delta\Phi_B}{\Delta t} = -NBA\frac{\Delta(\cos\omega t)}{\Delta t}$$

Though we cannot easily derive what the rate of change is on the right side of the equation above without calculus, your text gives it as $-\omega \sin\omega t$ - the rate of change goes up linearly with ω, which seems logical enough. That gives

$$\Delta V = -NBA(-\sin\omega t)\,\omega = NBA\omega\sin\omega t = NB\omega\pi r^2\sin\omega t$$

We are given the rotation rate in revolutions per second, which is the *frequency* f, not the angular frequency $\omega = 2\pi f$. Making the substitution, and plugging in the values given,

$$\Delta V = 2\pi^2 NBfr^2\sin(2\pi ft) \approx 1.7\sin(188t)\,\mathrm{V}$$

Since we are asked for the maximum voltage, we really want $|\Delta V| = 1.7\,\mathrm{V}$.

11. 5 points. A superconducting solenoid designed for whole-body imaging by nuclear magnetic resonance is 0.9 m in diameter and 2.2 m long. The field at the center is 0.4 T. Estimate roughly the energy stored in the field of this coil, in Joules.

A solenoid is really nothing more than a ginormous inductor, a big coil of wire. The energy stored in an inductor is $U = \frac{1}{2}LI^2$. For the specific case of a single coil - a solenoid - we know the inductance is:

$$L = \mu_0 n^2 V$$

Where n is the number of turns *per unit length* and V the volume of the inductor. Putting this in the energy equation:

$$U = \frac{1}{2}\mu_0 n^2 V I^2$$

Great. But we don't know n or I in this case directly. We do know the *magnetic field* for a solenoid, however, which will give us the product nI, really all we need:

$$\begin{aligned} B &= \mu_0 nI \\ \Longrightarrow nI &= \frac{B}{\mu_0} \end{aligned}$$

Putting it all together, and noting that the volume of a cylinder is $\pi r^2 l$, where r is the radius and l the length:

$$\begin{aligned} U &= \frac{1}{2}\mu_0 n^2 V I^2 \\ &= \frac{1}{2}\mu_0 V (nI)^2 \\ &= \frac{1}{2}\mu_0 V \frac{B^2}{\mu_0^2} \\ &= \frac{\pi r^2 l B^2}{2\mu_0} \approx 89100\,\mathrm{J} \\ &= 89.1\,\mathrm{kJ} \end{aligned}$$