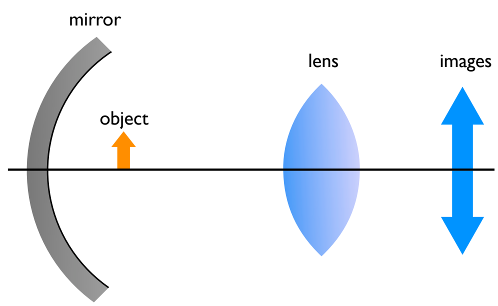
University of Alabama
Department of Physics and Astronomy

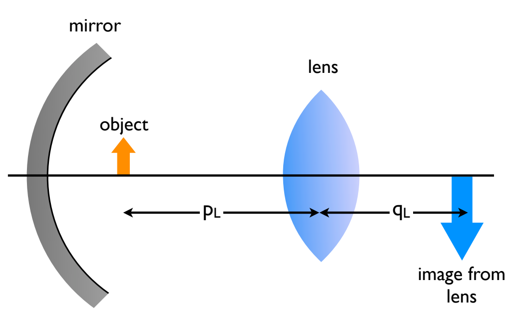
PH 102-2 / LeClair | Spring 2008

Problem Set 9: SOLUTIONS

1. 15 points. An observer to the right of the mirror-lens combination shown in the figure sees two real images that are the same size and in the same location. One image is upright and the other is inverted. Both images are 1.70 times larger than the object. The lens has a focal length of 11.2 cm. The lens and mirror are separated by 40.0 cm. Determine the focal length of the mirror. (Don't assume that the figure is drawn to scale.)



What we first need to do here is sort out which image is which. One image must be that formed from light going directly from the object through the lens. If this is the case, since it is a biconvex lens any *real* image formed will be inverted. One can only have an upright *virtual* image from this type of lens. Since the object and image are on opposite sides of the lens, a real image is formed, and the upside-down image must be the one formed directly by the lens, as shown below:



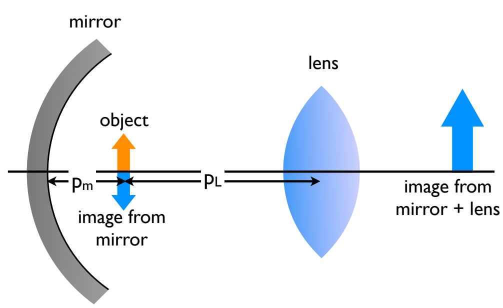
Now that we know which one is the image from the lens, we can use the known focal length and magnification factor to find the location of the image and object. The magnification of both mirror and lens is given as $M = -1.7$, thus:

$$M = -\frac{q_l}{p_l} = -1.7$$
$$\implies \quad q_l = 1.7 p_l$$

Given a relationship between the image and object distances for the lens, and the known focal length ($f_l = 11.2\,\mathrm{cm}$), we can find the image and object distances:

$$\begin{aligned}
\frac{1}{p_l} + \frac{1}{q_l} &= \frac{1}{f_l} \\
\frac{1}{p_l} + \frac{1}{1.7p_l} &= \frac{1}{0.112\,\mathrm{m}} \\
\frac{1.7}{1.7p_l} + \frac{1}{1.7p_l} &= \frac{2.7}{1.7p_l} = \frac{1}{0.112\,\mathrm{m}} \\
1.7p_l &= 2.7\,(0.112\,\mathrm{m}) \\
\implies \quad p_l &= 0.178\,\mathrm{m} \\
\implies \quad q_l &= 1.7p_l = 0.303\,\mathrm{m}
\end{aligned}$$

Great! But what about the second image, and how do we now involve the mirror? If the first image is light coming directly off of the object and focused by the lens, the second image must be light coming off of the object, then reflected back through the lens and focused. This makes some sense - a concave spherical mirror like the one shown always makes an *inverted* real image. The image formed by the mirror is an object for the lens, so it must be a real image. The inverted real image from the mirror is *again* inverted by the lens, so that the image formed by the combination of the mirror and lens is real and upright, as shown below:



Thus, light from the object moving to the left is reflected by the mirror, forming a real inverted image between the mirror and lens, which is then focused by the lens to create a real *upright* image to the right of the mirror. Now we know that the second image is at the same horizontal position as that formed by the lens alone. If this is the case, *the image formed by the mirror must be at the same position as the original object, but inverted.* If the image from the mirror and the object itself both form an image at the same position on the other side of the lens, they must be at the same position themselves - for a given image position relative to the lens, there is only a single possible object position. Thus, if the two images are at the same place, they both result from objects at the same place.

What this means then, is two things: first, since the image from the mirror forms at the same position as the object, $p_m = q_m$. Recall that for a mirror, both p and q are positive if they are on the same side as the mirror, when the image is real. Second, given that the spacing between the mirror and lens is known, we can relate p_m and p_l - together, they must make up the total distance between mirror and lens (see figure above):

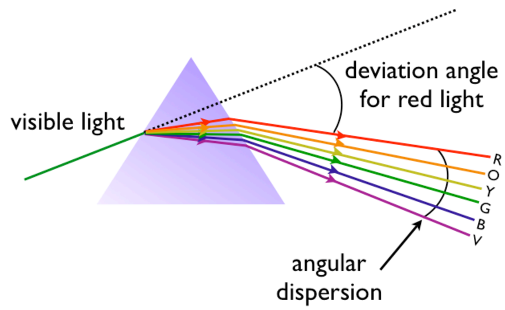
$$\begin{aligned}
p_m + p_l &= 0.400\,\mathrm{m} \\
p_m &= 0.400 - p_l = 0.222\,\mathrm{m}
\end{aligned}$$

Now we have p_m in terms of p_l, and we know that $p_m = q_m$, so we can use the mirror equation to find the focal length.

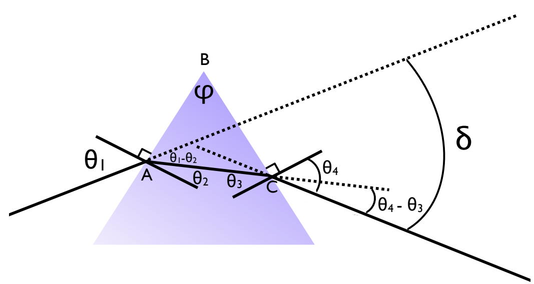
$$
\begin{aligned}
\frac{1}{p_m} + \frac{1}{q_m} &= \frac{1}{f_m} \\
\frac{1}{p_m} + \frac{1}{p_m} &= \frac{2}{p_m} = \frac{1}{f_m} \\
\frac{2}{0.400\,\mathrm{m} - p_l} &= \frac{1}{f_m} \\
f_m &= \frac{0.400\,\mathrm{m} - p_l}{2} = \frac{0.222\,\mathrm{m}}{2} \\
\Longrightarrow \quad f_m &= 0.111\,\mathrm{m}
\end{aligned}
$$

2. 15 points. The index of refraction for **violet** light in silica flint glass is $n_{\text{violet}} = 1.66$, and for **red** light it is $n_{\text{red}} = 1.62$. In air, $n = 1$ for both colors of light.

What is the **angular dispersion** of visible light (the angle between red and violet) passing through an equilateral triangle prism of silica flint glass, if the angle of incidence is $50°$? The angle of incidence is that between the ray and a line *perpendicular* to the surface of the prism. Recall that all angles in an equilateral triangle are $60°$.



What we need to do is find the deviation angle for both red and violet light in terms of the incident angle and refractive index of the prism. The angular dispersion is just the difference between the deviation angles for the two colors. First, let us define some of the geometry a bit better, referring to the figure below.



Let the angle of incidence be θ_1, and the refracted angle θ_2 at point A. The incident and refracted angles are defined with respect to a line *perpendicular* to the prism's surface. Similarly, when the light rays exit the prism, we will call the incident angle within the prism θ_3, and the refracted angle exiting the prism θ_4 at point C. If we call index of refraction of the prism n, and presume the surrounding material is just air with index of refraction 1.00, we can apply Snell's law at both interfaces:

$$
\begin{aligned}
n \sin\theta_2 &= \sin\theta_1 \\
n \sin\theta_3 &= \sin\theta_4
\end{aligned}
$$

Fair enough, but now we need to use some geometry to relate these four angles to each other, the deviation angle δ, and the prism's apex angle φ. Have a look at the triangle formed by points A, B, and C. All three angles in this triangle must add up to $180°$. At point A, the angle between the prism face and the line $\overline{AC}$ is $\angle BAC = 90-\theta_2$ - the line we drew to define θ_1 and θ_2 is by construction perpendicular to the prism's face, and thus makes a $90°$ angle with respect to the face. The angle $\angle BAC$ is all of that $90°$ angle, *minus* the refracted angle θ_2. Similarly, we can find $\angle BCA$ at point C. We know the apex angle of the prism is φ, and for an equilateral triangle, we must have $\varphi = 60°$

$$\begin{aligned}(90° - \theta_2) + (90° - \theta_3) + \varphi &= 180° \\ \implies \varphi = \theta_2 + \theta_3 &= 60°\end{aligned}$$

How do we find the deviation angle? Physically, the deviation angle is just how much in total the exit ray is "bent" relative to the incident ray. At the first interface, point A, the incident ray and reflected ray differ by an angle $\theta_1 - \theta_2$. At the second interface, point C, the ray inside the prism and the exit ray differ by an angle $\theta_4 - \theta_3$. These two differences *together* make up the total deviation - the deviation is nothing more than adding together the differences in angles at each interface due to refraction. Thus:

$$\delta = (\theta_1 - \theta_2) + (\theta_4 - \theta_3) = \theta_1 + \theta_4 - (\theta_2 + \theta_3)$$

Of course, one can prove this rigorously with quite a bit more geometry, but there is no need: we know physically what the deviation angle is, and can translate that to a nice mathematical formula. Now we can use the expression for φ in our last equation:

$$\delta = \theta_1 + \theta_4 - \varphi$$

We were given $\theta_1 = 50°$, so now we really just need to find θ_4 and we are done. From Snell's law above, we can relate θ_4 to θ_3 easily. We can also relate θ_3 to θ_2 and the apex angle of the prism, φ. Finally, we can relate θ_2 back to θ_1 with Snell's law. First, let us write down all the separate relations:

$$\begin{aligned}\sin\theta_4 &= n\sin\theta_3 \\ \theta_3 &= \varphi - \theta_2 \\ n\sin\theta_2 &= \sin\theta_1 \\ \text{or}\quad \theta_2 &= \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\end{aligned}$$

If we put all these together (in the right order) we have θ_4 in terms of known quantities:

$$\begin{aligned}\sin\theta_4 &= n\sin\theta_3 \\ &= n\sin(\varphi - \theta_2) \\ &= n\sin\left[\varphi - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]\end{aligned}$$

With that, we can write the full expression for the deviation angle:

$$\delta = \theta_1 + \theta_4 - \varphi = \theta_1 + n\sin\left[\varphi - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \varphi$$

Now we just need to calculate the deviation separately for red and violet light, using their different indices of refraction. You should find:

$$\begin{aligned}\delta_{\text{red}} &= 48.56° \\ \delta_{\text{blue}} &= 53.17°\end{aligned}$$

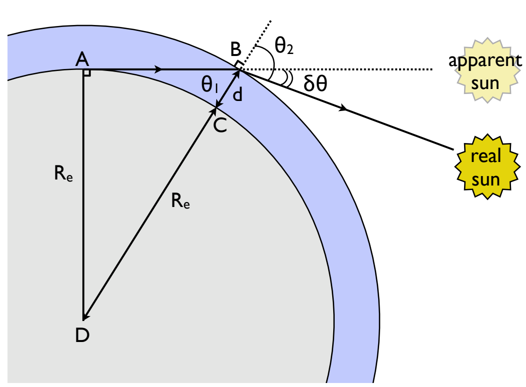
The angular dispersion is just the difference between these two:

$$\text{angular dispersion} = \delta_{\text{blue}} - \delta_{\text{red}} = 4.62^\circ$$

3. 15 points. As light from the Sun enters the atmosphere, it refracts due to the small difference between the speeds of light in air and in vacuum. The optical length of the day is defined as the time interval between the instant when the top of the Sun is just visibly observed above the horizon, to the instant at which the top of the Sun just disappears below the horizon. The geometric length of the day is defined as the time interval between the instant when a geometric straight line drawn from the observer to the top of the Sun just clears the horizon, to the instant at which this line just dips below the horizon. The day's optical length is slightly larger than its geometric length.

By how much does the duration of an optical day exceed that of a geometric day? Model the Earth's atmosphere as uniform, with index of refraction $n = 1.000293$, a sharply defined upper surface, and depth 8767 m. Assume that the observer is at the Earth's equator so that the apparent path of the rising and setting Sun is perpendicular to the horizon. You may take the radius of the earth to be 6.378×10^6 m. Express your answer to the nearest hundredth of a second.

First, we need to draw a little picture. This is the situation we have been given:



We presume that some human is standing at point A on the earth's surface, looking straight out toward the horizon. This line of sight intersects the boundary between the atmosphere and space (which we are told to assume is a sharp one) at point B. Light rays from the sun, which is slightly below the horizon, are refracted toward the earth's surface at point B, and continue on along the line of sight from B to A. We know the index of refraction of vacuum is just unity ($n_{\text{vacuum}} = 1$), while that of the atmosphere is $n = 1.000293$. The day appears to be slightly longer because we see the sun even after it has gone through an extra angle of rotation $\delta\theta$ due to atmospheric refraction.

To set up the geometry, we first draw a radial line from point B to the center of the earth. This line, $\overline{BC}$, will intersect the boundary of the atmosphere at point B, and will be normal to the atmospheric boundary. This defines the angle of incidence θ_2 and the angle of refraction θ_1 for light coming from the sun. The difference between these two angles, $\delta\theta$, is how much the light is bent downward upon being refracted from the atmosphere. How do we relate this to the extra length of the day one would observe? We know that the earth revolves on its axis at a constant angular speed - one revolution in 24 hours. Thus, we can easily find the angular speed of the earth:

$$\text{earth's angular speed} = \omega = \frac{\text{one revolution}}{1\text{day}} = \frac{360^\circ}{86400\,\text{s}}$$

Here we used the fact that there are $24 \cdot 60 \cdot 60 = 86400$ seconds in one day. Given the angular velocity of the earth, we know exactly how long it will take for the earth to rotate through the "extra" angle $\delta\theta$ due to refraction:

$$\delta\theta = \omega\delta t$$

We only need one last bit: the atmospheric refraction occurs *twice per day* – once at sun-up and once at sun-down. The total "extra" length of the day is then $2\delta t$. Thus, if we can find $\delta\theta$, we can figure out how much longer the day seems to be due to atmospheric refraction. In order to find it, we need to use the law of refraction and a bit of geometry. First, from the law of refraction and the fact that $\Delta\theta = \theta_2 - \theta_1$, we can state the following:

$$n \sin\theta_1 = \sin\theta_2 = \sin(\theta_1 + \delta\theta)$$

In order to proceed further, we draw a line from point A to the center of the earth, point D. This forms a triangle, $\triangle ABD$. Because line $\overline{AD}$ is a radius of the earth, by construction, it must intersect line $\overline{AB}$ at a right angle, since the latter is by construction a tangent to the earth's surface. Thus, $\triangle ABD$ is a right triangle, and

$$\sin\theta_1 = \frac{\overline{AD}}{\overline{BD}} = \frac{R_e}{R_e + d}$$

Plugging this into the previous equation,

$$\begin{aligned} n \sin\theta_1 &= \sin(\theta_1 + \delta\theta) \\ n\frac{R_e}{R_e + d} &= \sin\theta_2 = \sin(\theta_1 + \delta\theta) \end{aligned}$$

In principle, we are done at this point. The previous expression allows one to calculate θ_1, while the present one allows one to find $\delta\theta$ if θ_1 is known. From that, one only needs the angular speed of the earth.

$$\begin{aligned} \theta_1 &= \sin^{-1}\left[\frac{R_e}{R_e + d}\right] \\ \delta\theta &= \sin^{-1}\left[\frac{nR_e}{R_e + d}\right] - \theta_1 \\ 2\delta t &= \frac{2\delta\theta}{\omega} \approx 163.82\,\mathrm{s} \end{aligned}$$

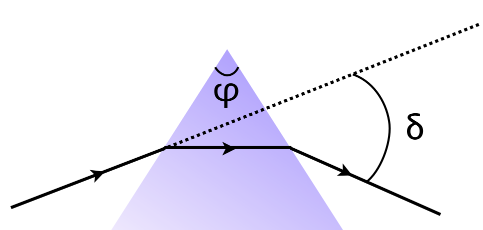
Of course, it is more satisfying to have an analytic approximation. We will leave that as an exercise to the reader for now.

4. 10 points. What is the apparent depth of a swimming pool in which there is water of depth 3 m, **(a)** When viewed from normal incidence? **(b)** When viewed at an angle of 60° with respect to the surface? The refractive index of water is 1.33.

5. 15 points. Light is deviated by a glass prism of index n as shown in the figure below. The ray in the prism is parallel to the base. Show that the refractive index is related to the deviation angle δ and the prism angle φ by the equation

$$n \sin\frac{\varphi}{2} = \sin\left(\frac{\varphi + \delta}{2}\right)$$

for this angle of incidence (*i.e.*,, the angle of incidence such that the ray in the prism is parallel to the base). The deviation angle δ is a minimum for this angle of incidence, and is known as the angle of minimum deviation. *Hint: You can solve this and the first problem together, if you keep things as general as possible - this is just a special case of the first problem.*



6. 10 points. Prove that if two thin lenses are placed in contact, they are equivalent to a single lens of focal length

$$f_{\text{equiv}} = \frac{f_1 f_2}{f_1 + f_2}$$

where f_1 and f_2 are the focal lengths of the two thin lenses. In some sense, lenses in series add like capacitors do.

7. 10 points. A thin lens of focal length f is used to form an enlarged real image of an object on a screen. The separation of the screen and object is s. **(a)** Deduce an expression for the distance between the object and the lens. **(b)** What is the minimum ratio s/f for the formation of a real image?

We have a thin lens forming a real image from an object a distance s from the lens. Based on the fact that it is a real image, it must be a converging lens. Given the focal length f, we need only use the lens equation to find the object distance p. The only trick of sorts is that we are calling the object distance s here instead of the usual p, that is all.

$$\begin{aligned} \frac{1}{s} + \frac{1}{q} &= \frac{1}{f} \\ \frac{1}{s} &= \frac{1}{f} - \frac{1}{q} \\ \frac{1}{s} &= \frac{q-f}{fq} \\ \implies \quad s &= \frac{fq}{q-f} \end{aligned}$$

That is all there is to the first part of the question. The second part asks about the formation of a real image. For a thin converging lens, real images are formed when the object is outside of the focal length, that is, the object distance is greater than the focal length, $s > f$. The minimum ratio of s/f is thus $s/f = 1$, that is all we were asking for.

8. 5 points. A thin convex lens of focal length f produces a real image of magnification m. Show that the object distance p is given by

$$p = \frac{m+1}{m} f$$

The magnification of the lens m can be defined in terms of the image and object distances, q and p, respectively:

$$m = \frac{-q}{p}$$

For a lens, the image is only real if both p and q are positive, meaning they are on opposite sides of the lens. If both p and q are positive, then m is negative, and we could just as well write $mp = !q$ instead of $mp = -q$ - whether the magnification is positive or negative just tells us if the image is inverted or not. For a convex lens forming a real image, we know very well the image is inverted, so we can drop the negative sign without worry. Thus, we just say $mp = q$. Using the lens equation:

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{1}{f} \\ \frac{1}{p} + \frac{1}{mp} &= \frac{1}{f} \\ \frac{m+1}{mp} &= \frac{1}{f} \\ mp &= f\,(m+1) \\ \implies \quad p &= \left(\frac{m+1}{m}\right) f \end{aligned}$$

9. 5 points. A transparent sphere of unknown composition is observed to form an image of the Sun on the surface of the sphere opposite the Sun. What is the refractive index of the sphere material? *Hint: assume the object distance is effectively infinite.* You may take the refractive index of air to be 1.0.

The surface of the sphere forms a lens of radius R. If the image is formed precisely on the back of the sphere, that means that the distance from the front of the sphere, the front of the refracting surface, to the image position is just the diameter of the sphere, or $2R$. If we call the index of refraction of the sphere n and that of the air 1.0, we can use the equation for a spherically refracting surface:

$$\begin{aligned}\frac{1}{p}+\frac{1}{q} &= \frac{n-1}{R}\\ \frac{1}{p}+\frac{1}{2R} &= \frac{n-1}{R}\end{aligned}$$

It is safe to assume that the sun is very, very far away compared to the diameter of the sphere. In that case, if p is very, very large, then $1/p$ is very, very small, and we may neglect it. This leaves us with:

$$\begin{aligned}\frac{n}{2R} &= \frac{n-1}{R}\\ \frac{n}{2} &= n-1\\ \frac{n}{2} &= 1\\ \implies \quad n &= 2\end{aligned}$$