University of Alabama
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PH 102-1 / LeClair — Summer II 2008

Problem Set 1: Relativity
Solutions

1. 5 points. Neutrons have an average lifetime of 15 minutes when at rest in the laboratory. What is the average lifetime (measured in the lab) of a neutron moving at a speed of **(a)** 25% of the speed of light? **(b)** 50%? **(c)** 95%?

The neutron's lifetime is always the same in its own reference frame, a constant 15 minutes which we will call the 'proper' time interval Δt_p. In the laboratory, we are in motion *relative to the neutron*, and hence we measure a dilated (longer) time interval $\Delta t'$. For each speed, then, we just need to calculate the dilated time interval, and that is the observed lifetime in the laboratory frame. The dilated interval is:

$$\Delta t' = \gamma \Delta t_p = \gamma\,(15\,\text{min})$$

For each part, then, we just need to calculate γ for the given speed v.

part a: $$\Delta t'_a = \gamma\,(15\,\text{min}) = \frac{1}{\sqrt{1-(0.25c)^2/c^2}}\,(15\,\text{min}) = \frac{15\,\text{min}}{\sqrt{1-0.25^2}} = 1.03\,(15\,\text{min}) = 15.5\,\text{min}$$

part b: $$\Delta t'_b = \frac{1}{\sqrt{1-(0.5c)^2/c^2}}\,(15\,\text{min}) = \frac{15\,\text{min}}{\sqrt{1-0.5^2}} = 1.15\,(15\,\text{min}) = 17.3\,\text{min}$$

part c: $$\Delta t'_c = \frac{1}{\sqrt{1-(0.95c)^2/c^2}}\,(15\,\text{min}) = \frac{15\,\text{min}}{\sqrt{1-0.95^2}} = 3.20\,(15\,\text{min}) = 48.0\,\text{min}$$

2. 5 points. If a moving clock has a time dilation factor of 10, what is its speed?

The 'time dilation factor' is just the ratio of the time that passes for the moving clock compared to that of an observer in motion relative to it. It is implied that the observer is keeping the 'proper' time. We know how to relate these times:

$$\Delta t'_{\text{clock}} = \gamma \Delta t_{\text{observer}}$$

The dilation factor is then:

$$[\text{dilation factor}] = \frac{\Delta t'_{\text{clock}}}{\Delta t_{\text{observer}}} = \gamma = 10$$

We now just need the definition of γ, and we can solve for the velocity of the clock v

$$\begin{aligned}
\gamma = \frac{1}{\sqrt{1-v^2/c^2}} &= 10 \\
\sqrt{1-\frac{v^2}{c^2}} &= \frac{1}{10} \\
1-\frac{v^2}{c^2} &= \frac{1}{10^2} = \frac{1}{100} \\
\frac{v^2}{c^2} &= 1-\frac{1}{100} \\
v^2 &= c^2\left(1-\frac{1}{100}\right) \\
v &= \pm c\sqrt{1-\frac{1}{100}} \\
v &\approx \pm 0.995c
\end{aligned}$$

One minor point: remember that when we take a square root, we have a positive and a negative answer, hence the $\pm$. Physically, this represents the fact that we can't tell from the information given whether the clock is coming or going - the answer is the same no matter what direction the clock is moving relative to the observer, it is only important that it *is* moving.

3. 10 points. A bassist taps the lowest E on his bass at 140 beats per minute during one portion of a song. What tempo would an observer on a ship moving *toward* the bassist at $0.70c$ hear?

What we are really interested in is the time interval between the taps. That time interval will be dilated (longer) for the moving observer, and hence the taps will sound farther apart (the tapping will be slower).

The 'proper time' interval Δt_p is that measured by the bassist, which is 1/140 min/beat (so that there are 140 beats/min).[i] The time interval *between taps* measured by the moving observer $\Delta t'$ is longer by a factor gamma:

$$\Delta t' = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{0.7^2c^2}{c^2}}} \cdot \left(\frac{1\,\text{min}}{140\,\text{beats}}\right) = \frac{1}{\sqrt{1-0.7^2}} \cdot \left(\frac{1\,\text{min}}{140\,\text{beats}}\right) \approx \frac{0.1\,\text{min}}{\text{beat}}$$

The beats per minute heard by the moving observer is just $1/\Delta t'$:

$$[\text{beats per minute, heard by observer}] = \frac{1}{\Delta t'} = \frac{1}{0.1} = 100$$

So, the bass line seems to be moving about 40% slower.

4. 10 points. Two identical spaceships are traveling in the same direction. An observer on earth measures the first to have a speed of $0.80c$, and observes the second to be 1.50 times as long as the first one. What is the speed of the second spaceship, relative to the earth?

[i] The time interval is just the inverse of the *rate* of tapping in beats per unit time.

The main trick to this problem is keeping everything straight - beyond that, it is just an application of length contraction. We don't even really need to worry about reference frames or transformations - everything is measured by the earthbound observer. This earthbound observer will see the ships as appearing shorter than their rest lengths, since they are in relative motion.

Both ships have a proper length L_p when measured in their own rest frame. The observer on earth measures the first ship to have some contracted length L_1 and the second L_2, and we are told $L_2 = 1.5L_1$. Already, from that last fact we know that ship 2 must be going slower than ship 1: if both ships are identical, and ship 1 appears shorter, then its length contraction is more severe, and it is traveling at a higher relative velocity. Before we get started, make the following definitions:

$$\begin{aligned}
v_1 &= \text{velocity of ship 1, relative to earth} \\
\gamma_1 &= \text{ship 1, relative to earth} \\
v_2 &= \text{velocity of ship 2, relative to earth} \\
\gamma_2 &= \text{ship 2, relative to earth}
\end{aligned}$$

For the first ship, its length contraction must be

$$L_1 = \frac{L_p}{\gamma_1}$$

For ship 2, it is:

$$L_2 = \frac{L_p}{\gamma_2} = 1.5\frac{L_p}{\gamma_1}$$

From the last equation, we can relate γ_1 and γ_2, and solve for v_2:

$$\begin{aligned}
\frac{L_p}{\gamma_2} = \frac{1.5L_p}{\gamma_1} \qquad \Longrightarrow \qquad \gamma_1 &= 1.5\gamma_2 \\
\frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} &= \frac{1.5}{\sqrt{1-\frac{v_2^2}{c^2}}} && \text{cross multiply \& square} \\
1.5^2\left(1-\frac{v_1^2}{c^2}\right) &= \left(1-\frac{v_2^2}{c^2}\right) && \text{solve for } v_2^2 \\
v_2^2 &= 1-1.5^2\left(1-\frac{v_1^2}{c^2}\right) && \text{note } v_1 = 0.8c \\
v_2 &\approx 0.44c
\end{aligned}$$

In this case, we are told the space ships are traveling in the same direction, so we know we should take the positive root in the last step.

5. 10 points. A radioactive atom in a beam produced by an accelerator has a speed of $0.80c$ relative to the laboratory. The atom decays and ejects an electron of speed $0.50c$ relative to itself. What is the speed of the electron relative to the laboratory if ejected in **(a)** the forward direction? **(a)** The backward direction?

The essence of this problem is that we have one object, a radioactive atom, going at $0.80c$ relative to the laboratory, and it ejects a second object, an electron, at $0.50c$ relative to itself. How fast is the second object going with respect to the lab? This is just velocity addition, the same way we would find out without relativity - we add the first object's velocity to the second.

First, let's be clear on our definitions. Let the laboratory reference frame be an unprimed system, with the the atom moving along the $+x$ direction. Thus, if the electron is ejected in the forward direction, its velocity is positive, if it is ejected in the backward direction, its velocity is positive. Let the reference frame of the first object (the radioactive atom) be the primed system. Thus,

$$\begin{aligned}
v_1 &= \text{velocity of the atom, relative to the lab} \\
v_2 &= \text{velocity of the electron, relative to the lab} \\
v_2' &= \text{velocity of the electron, relative to the atom}
\end{aligned}$$

The velocity of the electron relative to the lab frame is just the velocity of the atom relative to the lab, plus the velocity of the electron relative to the atom, corrected by our relativistic factor. For a forward-ejected electron, the latter velocity is positive, $v_2'=0.5c$, and we need just apply the velocity addition formula:

$$\text{forward ejection:} \quad v_2 = \frac{v_1 + v_2'}{1 + \frac{v_1 v_2'}{c^2}} = \frac{0.8c + 0.5c}{1 + \frac{(0.5c)(0.8c)}{c^2}} \approx 0.93c$$

When the electron is ejected backward, then $v_2'=-0.5c$, but the rest is the same:

$$\text{backward ejection:} \quad v_2 = \frac{v_1 + v_2'}{1 + \frac{v_1 v_2'}{c^2}} = \frac{0.8c + (-0.5c)}{1 + \frac{(-0.5c)(0.8c)}{c^2}} \approx 0.5c$$

A nice numerical coincidence! In the second case, the numbers were 'doctored' to make the velocity come out the same in either reference frame, except for the change of sign. It is nothing more than a coincidence though.

6. 10 points. Show that the velocity of a relativistic particle can be expressed as follows:

$$\vec{\mathbf{v}} = \frac{c\,\vec{\mathbf{p}}}{\sqrt{m^2c^2 + p^2}}$$

The easiest way is to start with the right-hand side and show that it reduces to v. Since there is only one vector on either side, and the rest are only constants, we know that $\vec{\mathbf{v}}$ and $\vec{\mathbf{p}}$ must be in the same direction. Thus, it is sufficient to show that the magnitude of each side is the same, and we can drop the vector notation.
Start by substituting the relativistic expression for γ, and then multiply both numerator and denominator by c. After that, just start grouping terms ...

$$\begin{aligned}
\frac{cp}{\sqrt{m^2c^2 + p^2}} &= \frac{\gamma mvc}{\sqrt{m^2c^2 + p^2}} = \left(\frac{c}{c}\right) \frac{\gamma mvc}{\sqrt{m^2c^2 + p^2}} \\
&= \frac{\gamma mvc^2}{c\sqrt{m^2c^2 + p^2}} = \frac{\left(\gamma mc^2\right) v}{\sqrt{m^2c^4 + p^2c^2}} \qquad \left(\text{note } E = \gamma mc^2 \text{ and } E = \sqrt{m^2c^4 + p^2c^2}\right) \\
&= \frac{Ev}{E} = v
\end{aligned}$$

7. 15 points. Suppose that a spaceship traveling at $0.80c$ through our solar system suffers a totally inelastic collision with a small meteoroid of mass 2.0 kg. **(a)** What is the kinetic energy of the meteoroid in the reference frame of the spaceship? **(b)** In the collision all of this kinetic energy suddenly becomes available for inelastic processes that damage the spaceship. The effect on the spaceship is similar to an explosion. How many tons of TNT will release the same explosive energy? One ton of TNT releases $\approx 4.2\times10^9$ J.

Just remember that it is equivalent to say that in the spaceship's reference frame (in which they are sitting still), the meteoroid is traveling toward them at $0.80c$ when it bombards them. The physics is the same, but it makes the situation more transparent. At that relative velocity, we can readily calculate the meteoroid's kinetic energy:

$$K = (\gamma-1)\,mc^2 = \left[\frac{1}{\sqrt{1-\frac{(0.80c)^2}{c^2}}} - 1\right] mc^2 \qquad = \left[\frac{1}{\sqrt{1-0.80^2}} - 1\right](2\,\mathrm{kg})\left(3\times10^8\,\mathrm{m/s}\right)^2 1.2\times10^{17}\,\mathrm{J}$$

For the units to work out, remember that $1\,\mathrm{J}=1\,\mathrm{N}\cdot\mathrm{m}=1\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$. How many tons of TNT is this? A lot:

$$\frac{1.2\times10^{17}\,\mathrm{J}}{4.2\times10^9\,\frac{\mathrm{J}}{\mathrm{ton\ TNT}}} \approx 2.86\times10^7\,\mathrm{tons\ TNT} = 28.6\,\mathrm{Mtons\ TNT}$$

This corresponds roughly to the yield of the largest hydrogen bombs built by the US; the Soviet "Tsar Bomba," the most powerful weapon ever detonated, reached about 50 Mtons.[ii]

8. 10 points. Given $\vec{\mathbf{p}}=\gamma m\vec{\mathbf{v}}$ and $E=\gamma mc^2$, derive the relationship $E^2=c^2p^2+m^2c^4$.

This time, it is easier to solve the expression above for c^2p^2, and demonstrate the equality. We will need one sneaky trick: at one point, we will both add and subtract the same term (c^2) to come up with a clever grouping of terms. Watch closely ...

$$\begin{aligned}
E^2 &= c^2p^2 + m^2c^4 \\
\Longrightarrow \quad p^2c^2 &= E^2 - m^2c^4 \\
&= \left(\gamma^2m^2v^2\right)c^2 = \left(\gamma^2m^2c^2\right)\left[v^2\right] && \text{(regroup)} \\
&= \left(\gamma^2m^2c^2\right)\left[v^2+c^2-c^2\right] && \text{(add and subtract } c^2\text{)} \\
&= \gamma^2m^2c^2v^2 + \gamma^2m^2c^4 - \gamma^2m^2c^4 && \text{(multiply out and group)} \\
&= \gamma^2m^2c^4 + \gamma^2m^2c^2\left(v^2-c^2\right) && \text{(note } E=\gamma mc\text{)} \\
&= E^2 + \gamma^2m^2c^2\left(v^2-c^2\right)
\end{aligned}$$

Well, we are halfway there, but that last term is a problem. Let's take that separately, and work it through. First plug in the definition of γ:

[ii] http://en.wikipedia.org/wiki/Nuclear_weapon_yield and http://en.wikipedia.org/wiki/Tsar_Bomba

$$
\begin{aligned}
\gamma^2 m^2 c^2 \left(v^2 - c^2\right) &= m^2 c^2 \left[\frac{v^2 - c^2}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^2}\right] = m^2 c^2 \left[\frac{v^2 - c^2}{1 - \frac{v^2}{c^2}}\right] \\
&= m^2 c^2 \left[\frac{c^2 \left(\frac{v^2}{c^2} - 1\right)}{1 - \frac{v^2}{c^2}}\right] \qquad \text{factor out } c^2 \\
&= m^2 c^4 \left[\frac{-\left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2}}\right] \\
&= -m^2 c^4
\end{aligned}
$$

After a messy start, it comes out beautifully simple. Now, plug that back into our work above, and we are done.

$$
\begin{aligned}
& p^2 c^2 = E^2 + \gamma^2 m^2 c^2 \left(v^2 - c^2\right) = E^2 - m^2 c^4 \\
\Longrightarrow \quad & E^2 = c^2 p^2 + m^2 c^4
\end{aligned}
$$

9. 10 points. Combustion of gasoline releases 1.3×10^8 J of energy per gallon. **(a)** How much mass is converted to energy? **(b)** Compare this with 2.8 kg, the mass of one gallon of gasoline.

All we want to find out is what would be the matter equivalent of 1.3×10^8 J worth of rest energy:

$$
\begin{aligned}
& E_R = mc^2 = 1.3 \times 10^8 \text{ J} \\
\Longrightarrow \quad & m = \frac{1.3 \times 10^8 \text{ J}}{c^2} = \frac{1.3 \times 10^8 \text{ J}}{\left(3 \times 10^8 \text{ m/s}\right)^2} = 1.4 \times 10^{-9} \, \frac{\text{J} \cdot \text{s}^2}{\text{m}^2} = 1.4 \times 10^{-9} \text{ kg}
\end{aligned}
$$

For the units to work out, remember that $1\,\text{J} = 1\,\text{N} \cdot \text{m} = 1\,\text{kg} \cdot \text{m}^2/\text{s}^2$. This is a *tiny* amount of mass, especially compared to a gallon of gasoline:

$$
\frac{m}{\text{mass of 1 gal gasoline}} = \frac{1.4 \times 10^{-9} \text{ kg}}{2.8 \text{ kg}} \approx 5.1 \times 10^{-10}
$$

10. 15 points. *Research Problem: one page, double spaced, 1 inch margins.* According to special relativity, the time order of events can be reversed under certain conditions. Does this violate causality? That is, could a ball hit the ground before it had been thrown?

See the notes, and do a little web searching. For instance:

`http://en.wikipedia.org/wiki/Causality_(physics)`

`http://stason.org/TULARC/education-books/startrek-relativity-FTL/8-1-What-is-Meant-Here-by-Causality-and-Unsolvable-Paradoxes.html`