UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 102 / LeClair Summer II 2009

Problem Set 1: Solutions

1. The orbital speed of the Earth around the Sun is 30 km/s. In one year, how many seconds do the clocks on the Earth lose with respect to the clocks of an inertial reference frame at rest relative to the Sun? *Hint: if v/c is small, the following approximations are valid:*

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1-\frac{1}{2}\frac{v^2}{c^2} \qquad \text{and} \qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1+\frac{1}{2}\frac{v^2}{c^2}$$

If we are to consider an inertial frame at rest relative to the sun, it must be the one keeping the proper time interval Δt_p. Over the course of one year, we have

$$\Delta t_p = 1\,\text{yr} \approx 3.156\times 10^7\,\text{s} \tag{1}$$

The observers on earth are in motion relative to the sun, and therefore they measure a dilated time interval $\Delta t' = \gamma\Delta t_p$. We are asked for the *difference* between the two clocks after one year as measured in the sun's inertial frame, or

$$\text{difference} = \Delta t' - \Delta t_p = \gamma\Delta t_p - \Delta t_p = (\gamma-1)\,\Delta t_p \tag{2}$$

Since in this case the relative velocity of earth is small with respect to c, $v = 30\,\text{km/s} = 3\times 10^4\,\text{s}$ so $v/c = 10^{-4}$, we can use the second approximation given.

$$\gamma - 1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \approx 1+\frac{1}{2}\frac{v^2}{c^2} - 1 = \frac{1}{2}\frac{v^2}{c^2} \tag{3}$$

Thus,

$$\text{difference} \approx \frac{1}{2}\frac{v^2}{c^2}\Delta t_p = \frac{1}{2}\left(10^{-4}\right)^2\left(3.156\times 10^7\,\text{s}\right) = \left(5\times 10^{-9}\right)\left(3.156\times 10^7\,\text{s}\right) \approx 0.16\,\text{s} \tag{4}$$

2. A cannonball flies through our classroom at a speed of $0.30c$. Measurement of the transverse diameter ("width") of the cannonball gives a result of $0.20\,\mathrm{m}$. What can you predict for the measurement of the longitudinal diameter ("length") of the cannonball?

If the cannonball is moving at high speed relative to an observer, along the direction of motion its length will appear *shorter* by a factor γ. Along the transverse direction (perpendicular to the direction of motion), no contraction will occur and the width observed will be the proper one $L_p = 0.2\,\mathrm{m}$. Given the cannonball's speed of $v = 0.3c$, the length will appear to be

$$L' = \frac{L_p}{\gamma} = (0.2\,\mathrm{m})\sqrt{1 - \frac{v^2}{c^2}} \approx 0.19\,\mathrm{m}$$

3. A flexible drive belt runs over two flywheels whose axles are mounted on a rigid base (Fig. 1). In the reference frame of the base, the horizontal portions of the belt have a speed v and therefore are subject to length contraction, which tightens the belt around the flywheels. However, in a reference frame moving to the right with the upper portion of the belt, the *base* is subject to length contraction, which ought to loosen the belt around the flywheels. Resolve this "paradox" with by a qualitative argument. *Hint: consider the lower portion of the belt as seen in the reference frame of the upper portion.*

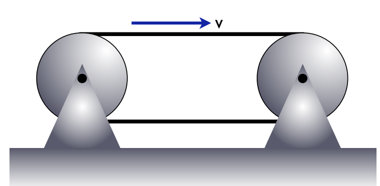

Figure 1: *Question 3*

Viewed from the laboratory frame, both the upper and lower belt should contract, as they are in motion relative to the observer. The fact that the top and bottom move in opposite directions does not matter in this case - both are contracted by the same amount, since length contraction depends on the *square* of the relative velocity. Thus, the belt appears to tighten.

Viewed from a frame traveling with the upper belt, the base appears to contract, since relative to the top portion of the belt, the pulleys on either side are moving away at velocity $|v|$. Why does the belt not loosen? This is because relative to the top of the belt, the bottom of the belt is moving at velocity $|2v|$ (ignoring the proper relativistic addition of velocities for the moment), and is thus length contracted twice as much as the the distance between the pulleys. Thus, the pulleys get closer together, but the bottom of the belt shortens even more, and overall the belt should appear to tighten.

Viewed from the bottom belt, the situation is reversed - both pulleys are moving at velocity v and the base contracts, but the top belt is moving at $|2v|$ and contracts twice as much. Still, the net effect is that the belt appears to tighten, a fact which all three reference frames agree on.

4. A spaceship is moving at a speed of $0.60c$ toward the Earth. A second spaceship, following the first one, is moving at a speed of $0.90c$ toward the Earth. What is the speed of the second spaceship as observed in the reference frame of the first?

Let our primary reference frame O be at rest with respect to the earth, and our second frame O' be at rest with respect to the first ship. The velocities of the two ships with respect to earth are then

$$\begin{aligned} v_1 &= 0.6c \\ v_2 &= 0.9c \end{aligned} \tag{5}$$

The second ship is going in the same direction as the first, but faster. Therefore, observers in the first ship, in the O' frame, will see the second ship as going forward in the same direction as they are. How fast? If we did not worry about relativity, we would say that the speed of the second ship relative to the first is the difference between their velocities - the second ship is going at $0.9c$ relative to earth, but the first ship is already going at $0.6c$ relative to earth. Without relativity, we would simply say that the second ship is going $0.3c$ faster than the first.

Accounting for relativity, we still subtract velocities, but use the proper relativistic formula for the velocity of the second ship viewed from the first:

$$v_2' = \frac{v_2 - v_1}{1 - \frac{v_1 v_2}{c^2}} = \frac{0.3c}{1 - \frac{0.54c^2}{c^2}} \approx 0.65c$$

5. Consider a particle of mass m moving at a speed of $0.10c$. What is its kinetic energy according to the relativistic formula? What is its kinetic energy according to the Newtonian formula? What is the percent deviation between these two results?

The classical formula for kinetic energy gives:

$$K_c = \frac{1}{2}mv^2 = \frac{1}{2}m\left(0.10c\right)^2 = 0.005mc^2 \tag{6}$$

The relativistic formula yields – being *very* careful to carry enough digits – something just a bit larger:

$$K_r = (\gamma - 1)\,mc^2 = \left(\frac{1}{\sqrt{1-0.01}} - 1\right)mc^2 \approx (1.00504 - 1)\,mc^2 = 0.000504mc^2 \tag{7}$$

The percent deviation is then (note that the mc^2 bits cancel)

$$\% \text{ deviation} = 100\% \times \left[\frac{K_r - K_c}{K_r}\right] = 100\% \times \left[\frac{0.00504mc^2 - 0.005mc^2}{0.00504mc^2}\right] \approx 0.8\% \tag{8}$$

Even at a tenth the speed of light, it is not a big difference.

6. Show that the momentum of a particle can be expressed in the concise form $\vec{\mathbf{p}} = \frac{E}{c^2}\vec{\mathbf{v}}$.

Relativistic energy is given by

$$E = \gamma mc^2 \tag{9}$$

meaning we can write

$$\gamma m = \frac{E}{c^2} \tag{10}$$

Relativistic momentum is

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}} \tag{11}$$

Substituting Eq. 10 into Eq. 11 ...

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}} = \frac{E}{c^2}\vec{\mathbf{v}} \tag{12}$$