UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 102 / LeClair Summer II 2009

Problem Set 2: Solutions

Notes:

1. My solutions tend to be longer and more pedantic than yours need to be.
2. Problem 4 in particular is far too difficult for an exam.
3. Problems 1 and 5 are *probably* not too difficult for an exam, but it is close.

1. At each corner of a square is a particle with charge q. Fixed at the center of the square is a point charge with opposite sign, of magnitude Q. What value must Q have to make the total force on each of the four particles zero? With Q set at that value, the system, in the absence of other forces, is in equilibrium. Do you think the equilibrium is stable?

First, we need a sketch of situation described in the problem, see Fig. 1 below.

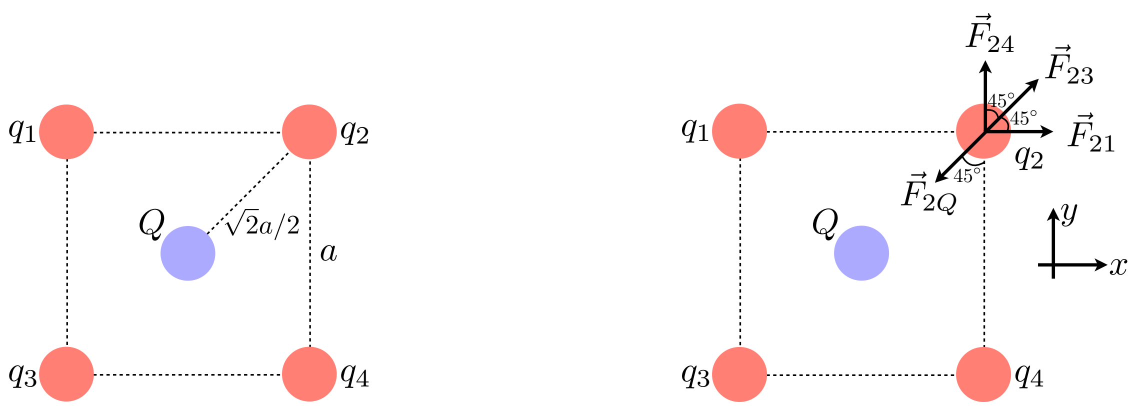


Figure 1: Left: *Situation described in problem 1; the charges q_1-q_4 are equivalent, while charge Q has the opposite sign. We assume the square has side a, though we won't need this distance in the final answer.* **Right:** *Forces on charge q_2 due to the other charges.*

We have four identical charges, labeled q_1 through q_4 just to keep track of things, on the corners of a square whose side we will assume has length a. An opposite charge Q sits in the middle of the square. Now, we are not given a, and we will not need it in the final answer. Our final answer should not depend on any precise dimensions, it should only depend on the fact that we have a square, and a square of any dimension should give the same result by symmetry. Whether you believe this or not is irrelevant, it something that should be proven: we'll give the square a side a for now, and see if it cancels in the end ...

If this structure is to be in equilibrium, then the net force on any given corner charge should be zero. All corner charges are the same, so it should not matter which one we choose to analyze. By considering the symmetry of the structure, since we know $q_1 = q_2 = q_3 = q_4$ we can see that if all the forces sum to zero for charge q_1, then it must be true for q_2, q_3, and q_4 as well. Thus, we only need to worry about one of the four charges.

It gets better. Since this structure has four-fold symmetry, the horizontal and vertical directions are equivalent. If we define x and y axes to be horizontal and vertical, respectively (Fig. 1) this means that the net force along x must be the same as the net force along y. It *must* be so, since there is nothing special to distinguish x from y in the problem. That means we need only to consider either the net x or net y component rather than the whole resultant force on a single charge. Thus, the problem is reduced to the following: pick one corner charge, and sum the forces along one axis.

Pick charge q_2, and let us sum the forces along the x axis. There are four forces acting, corresponding to the other 3 corner charges and the central charge. Charge q_1 sits a distance a away, and gives a repulsive force in the $+x$ direction. We need not worry about components, the entire force acts along the x axis:

$$F_{21} = \frac{k_e q_1 q_2}{a^2} \tag{1}$$

Charge q_3 sits a distance $a\sqrt{2}$ away, and gives a repulsive force upward and to the right at $45°$ (since we have a square). The x component is then found by multiplying the magnitude of the force times $\cos 45° = \sqrt{2}/2$:

$$\begin{aligned} F_{23} &= \frac{k_e q_2 q_3}{\left(a\sqrt{2}\right)^2} = \frac{k_e q_2 q_3}{2a^2} \\ F_{23,x} &= F_{23} \cos 45° = \frac{k_e q_2 q_3}{2a^2} \frac{\sqrt{2}}{2} = \frac{\sqrt{2} k_e q_2 q_3}{4a^2} \end{aligned} \tag{2}$$

Charge q_4 gives a repulsive force purely along the y axis, and does not contribute to our force balance at all.[i] That leaves only the charge Q, a distance $\sqrt{2}a/2$ away, which provides an attractive force on q_2. We find its x component just like last time.

$$\begin{aligned} F_{2Q} &= -\frac{k_e q_2 Q}{\left(\sqrt{2}a/2\right)^2} = -\frac{2k_e q_2 Q}{a^2} \\ F_{2Q,x} &= -\frac{2k_e q_2 Q}{a^2} \cos 45° = -\frac{2k_e q_2 Q}{a^2} \frac{\sqrt{2}}{2} = -\frac{\sqrt{2} k_e q_2 Q}{a^2} \end{aligned} \tag{3}$$

Now we just need to add together all the x components and set the result equal to zero to impose

[i] If we had chosen to sum the forces along the y axis, q_4 would give us the a contribution identical to F_{21} above, but q_1 would give no contribution, so the result would be the same in the end.

equilibrium:

$$F_{\text{net,x}} = F_{21} + F_{23,x} + F_{2Q,x} = \frac{k_e q_1 q_2}{a^2} + \frac{\sqrt{2} k_e q_2 q_3}{4a^2} - \frac{\sqrt{2} k_e q_2 Q}{a^2} = 0 \tag{4}$$

We no longer need to separately keep track of the q_1-q_4, so we can replace them all by just plain q. The remaining task is simply to find a relationship between Q and q:

$$0 = \frac{k_e qq}{a^2} + \frac{\sqrt{2} k_e qq}{4a^2} - \frac{\sqrt{2} k_e qQ}{a^2} \tag{5}$$

$$0 = \frac{q}{a^2} + \frac{\sqrt{2} q}{4a^2} - \frac{\sqrt{2} Q}{a^2} \qquad \text{(cancel } k_e,\, q \text{ from all)} \tag{6}$$

$$0 = q + \frac{\sqrt{2}}{4} q - \sqrt{2} Q \qquad \text{(cancel } a^2 \text{ from all)} \tag{7}$$

$$\sqrt{2} Q = q \left(1 + \frac{\sqrt{2}}{4}\right) \tag{8}$$

$$\Longrightarrow Q = q \left[\frac{1 + \frac{\sqrt{2}}{4}}{\sqrt{2}}\right] = q \left[\frac{\sqrt{2}}{2} + \frac{1}{4}\right] \approx 0.957q \tag{9}$$

Is the equilibrium stable? Qualitatively, we can answer this in a straightforward way; mathematically it is harder (requiring calculus). If the equilibrium is stable, that means that if we displace Q by a tiny amount, it should feel a restoring force pushing it back to its original position. If it is unstable, any tiny push would send it to a completely new position. What happens in this case?

If we push Q (for example) toward q_2, it will feel a stronger attractive force from q_2 than any other charge, and it will move toward q_2. The same is true for any direction we push Q – any displacement would put it closer to one of the q charges than the other three. The forces keeping it in place would no longer be balanced, and even a tiny perturbation would send it flying toward the closest q charges. This means the equilibrium is unstable.

2. A charge of $100\,\mu$C is at the center of a cube of side 0.8 m. (a) Find the total flux through each face of the cube. (b) Find the flux through the whole surface of the cube. (c) Would your answers to the first two parts change if the charge were not at the center of the cube?

It is easier to answer part (b) first. If the cube encloses the point charge, then the flux through the entire cube must be found in accordance with Gauss' law:[ii]

$$\Phi_{E,\text{tot}} = 4\pi k_e q = 4\pi \left(9 \times 10^9 \left[\frac{\text{N m}^2}{\text{C}^2}\right]\right) \left(100 \times 10^{-6}\,[\text{C}]\right) = 1.13 \times 10^7 \left[\frac{\text{N m}^2}{\text{C}}\right] \tag{10}$$

[ii]Don't forget that μ means 10^{-6}.

Back to **(a)**. If the charge sits *symmetrically* at the center of the cube, then the flux must be the same over every face of the cube. Thus, each side takes $1/6$ of the total flux, since there are six sides:

$$\Phi_{E,\text{side}} = \frac{1}{6}\Phi_{E,\text{tot}} = 1.88 \times 10^{6} \left[\frac{\text{N m}^2}{\text{C}}\right] \tag{11}$$

If the charge were not in the center of the cube, the flux through the *whole* cube in **(b)** would be the same, provided that the charge were still fully enclosed by the cube. If the charge were partially penetrated by one side of cube (say, half in and half out) then this would not be the case, and we would need to figure out how much of the charge was inside the cube and how much was outside. In either case, we could not say anything very clever about the flux through any given side – figuring that out relied crucially on the charge being symmetrically placed with respect to the six sides of the cube, so we could say they all had equal flux. If the charge were off-center, closer to one side than the others, this is no longer true, and our answer to **(a)** no longer holds.

3. A pyramid has a square base of side a, and four faces which are equilateral triangles. A charge Q is placed on the center of the base of the pyramid. What is the net flux of electric field emerging from one of the triangular faces of the pyramid?

This is another one of those cases where we have to rely on the symmetry of the situation in order to have any hope. First, we know that the *total* flux must just be $4\pi k_e Q$ or Q/ϵ_o. If the charge Q is embedded in the base of our pyramid, half in and half out, the pyramid must be intercepting half of the total flux. Even if it is just above or just below the base, it does not matter, it only matters that the charge Q is actually on the base. The field from the charge Q is then parallel to the base of the pyramid, and thus the base itself intercepts no field lines and has zero flux.

Thus, the pyramid intercepts half of the total flux, but does so only with its four triangular sides. The field lines from the charge are parallel to the base of the pyramid, so there can be no net flux through the base. Since the charge is exactly at the center of the pyramid, each of the other four sides of the pyramid must receive the same fraction of the flux through the pyramid. Thus, the pyramid receives half of the total flux, and each triangular side must have one quarter of *that*, or one eighth of the total per side. The flux per triagular face must then be $\frac{1}{2}\pi k_e Q$, or $Q/8\epsilon_o$.

4. Suppose three positively charged particles are constrained to move on a fixed circular track. If all the charges were equal, an equilibrium arrangement would obviously be a symmetrical one with the particles spaced $120°$ apart around the circle. Suppose two of the charges have equal charge q, and the equilibrium arrangement is such that these two charges are $60°$ apart rather than $120°$. What must be the relative magnitude of the third charge?

The first thing we need to do is figure out the geometry and draw a picture. First, all three charges are confined to a circular track, which we will say has radius r. Two of the charges are the same, which we will call q_1 and q_2, and they sit $60°$ apart on the circle. Where will the third, unequal charge (q_3) sit? In order for the forces on it due to charges 1 and 2 to be balanced, it must be equidistant from both on the circle. If charges 1 and 2 are $60°$ apart, then there are $300°$ left in the circle, and the third charge must sit halfway around that - the third charge must be $150°$ from both of the other charges.

Next, we should pick a coordinate system and origin. For reasons I hope will be clear soon, we will choose the origin to be on charge q_1, with the $+y$ direction pointing toward the center of the circle and the x axis tangential to the circle, as shown below. We could have equally chosen q_2 as the origin, since it is identical to q_1, it makes no difference.[iii] For convenience, we label the center of the circle as point C so we can easily refer to it later.

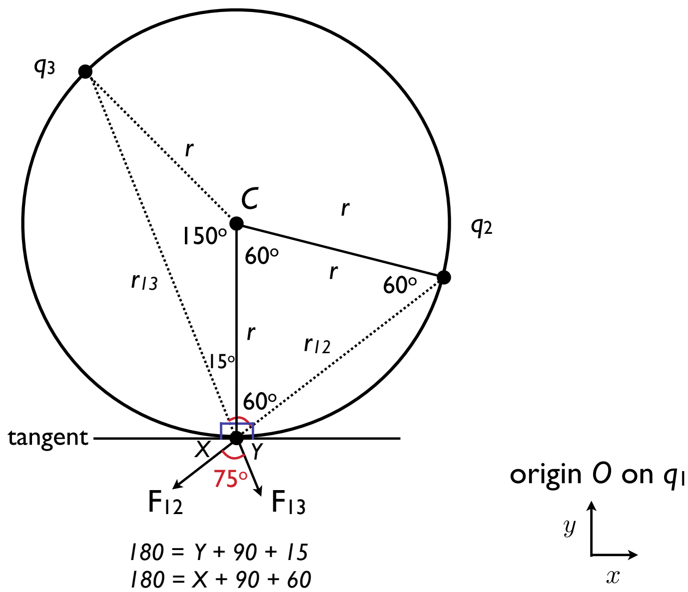


Figure 2: *Geometry implied by problem 4.*

Since charges q_1 and q_2 are $60°$ apart on the circle, we can form an equilateral triangle with point C as one corner. Based on this, we can find the distance between q_1 and q_2 in terms of the radius of the circle r: $r_{12} = r$. Charges q_1 and q_2 are identical, and therefore experience a repulsive force of magnitude F_{12} directed along the line connecting them. This force must be at a $30°$ angle to the x and y axes, based on the geometry above. Charge q_3 has a different magnitude, but the same sign as q_1, and thus the force between them F_{13} is also repulsive.

In order for the charges to stay in the positions above, what must be true? For charge q_1, the forces in the y direction are irrelevant, since q_1 is constrained to stay on the circle anyway. Only net forces along the x direction will force it to move around the circle one way or the other. Thus, in order for this situation to

[iii]One could choose any point as the origin and get the same result, but in my opinion the geometry is more transparent in the present case.

be the equilibrium configuration, the forces on q_1 in the x direction tangential to the track must cancel. Since q_1 and q_2 are identical, the forces along the tangential direction of the circle will also vanish for q_2 automatically. Finally, since the system is symmetric, q_3 must also have no net force along the direction of the circle if neither of the other charges do. Thus, it is sufficient to find the forces in the x direction for q_1 and equate them. This means we need to find the x components of F_{12} and F_{13}, set them equal to one another, and solve for q_3.

First, we focus on F_{12}, whose x component we will label $F_{12,x}$. We now know the distance between q_1 and q_2, so the magnitude of the *total* force is easily written down with Coulomb's law:

$$F_{12} = \frac{k_e q_1 q_2}{r_{12}^2} = \frac{k_e q_1 q_2}{r^2} \tag{12}$$

In order to find the x component, we need to know the angle that $\vec{\mathbf{F}}_{12}$ makes with the x axis. Based purely on symmetry, we would say it must be $30°$. Explicitly ... the angle we are looking for is the one labeled X in the figure above. We know that the tangent line at charge q_1 forms a $90°$ angle with a (vertical) radial line. We also know that the angle between the line r_{12} and the vertical radial line is $60°$. Together, those three angles make a straight line, for $180°$. Thus, $180 = X + 90 + 60$, or $X = 30°$. Thus,

$$F_{12,x} = F_{12} \cos X = F_{12} \cos 30° = \frac{k_e q_1 q_2}{r^2} \frac{\sqrt{3}}{2} \tag{13}$$

Now, what about the force between charges 1 and 3, F_{31}? We can write down the force between them easily:

$$F_{13} = \frac{k_e q_1 q_3}{r_{13}^2} \tag{14}$$

What is the distance d between q_1 and q_3? For this, we will need the law of cosines (and the fact that $\cos 150° = -\cos 30° = -\sqrt{3}/2$):

$$d^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos 150° = 2r^2 - 2r^2 \left(-\frac{\sqrt{3}}{2}\right) = 2r^2 \left(1 + \frac{\sqrt{3}}{2}\right) \tag{15}$$

In order to find the x component of $\vec{\mathbf{F}}_{13}$, we'll need the angle Y in the figure above, the angle F_{13} makes with the x axis. Again using the fact that angle Y plus a right angle plus $15°$ make a straight line, we deduce $Y = 75°$.[iv] Putting all this together,

$$F_{13,x} = F_{13}\cos Y = \frac{k_e q_1 q_3}{r_{13}^2}\cos 75° = \frac{k_e q_1 q_3}{2r^2\left(1+\frac{\sqrt{3}}{2}\right)}\cos 75° \tag{16}$$

Now, this is probably a little-known fact, but we can find $\cos 75°$ exactly:

$$\cos 75° = \cos\left(45° + 30°\right) = \cos 45°\cos 30° - \sin 45°\sin 30° = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$$

If you didn't know this random factoid, no worries: just leave the $\cos 75°$ in there and continue. What you do *not* want to do is just plug in 0.2588 instead – keep everything symbolic until the last step, and it is much, much easier to check your work. Anyway ...

$$F_{13,x} = \frac{k_e q_1 q_3}{2r^2\left(1+\frac{\sqrt{3}}{2}\right)}\left[\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)\right] \tag{17}$$

Now we just need to set $F_{13,x} = F_{12,x}$ and solve for q_3 in terms of q_1 or q_2:

$$\begin{aligned}
F_{13,x} &= F_{12,x} \\
\frac{k_e q_1 q_3}{2r^2\left(1+\frac{\sqrt{3}}{2}\right)}\left[\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)\right] &= \frac{k_e q_1 q_2}{r^2}\frac{\sqrt{3}}{2} \\
\frac{q_3}{2\left(1+\frac{\sqrt{3}}{2}\right)}\left[\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)\right] &= q_2\frac{\sqrt{3}}{2} \qquad \text{(cancel } q_1\text{; cancel } r^2\text{)} \\
q_3\left[\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4+2\sqrt{3}}\right] &= q_2\sqrt{3} \\
\frac{q_3}{q_1} &= \frac{4\sqrt{3}+6}{\sqrt{2}\left(\sqrt{3}-1\right)} \approx 12.5
\end{aligned}$$

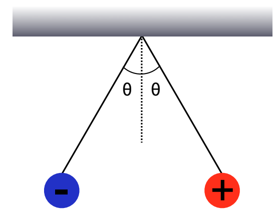
Thus, the charge q_3 must be approximately 12.5 times as big as q_1 and q_2 in order for the latter two charges to be $60°$ apart. Physically, it makes sense that q_3 is bigger - q_1 and q_2 are closer together than they would be if all three charges are equal, so they must be feeling more repulsion from q_3 than from each other, which means q_3 must be bigger.

Now, if you left the $\cos 75°$ in there, that is FINE! Knowing how to express $\cos 75°$ in exact form is a rather esoteric bit of knowledge, and fairly useless in our modern electronic age ... it is fine to leave a $\cos 75°$ in your final symbolic answer. Again: the thing you should *resist strongly* is to plug in

[iv] We could also argue on different grounds that it must be half the angle made up by q_3, C, and q_1, $150°/2$.

$\cos 75° \approx 0.2588$ right away. As soon as you start using numbers instead of symbols (or at least values known functions), you have lost one easy way to check your work.

5. Two small spheres, each of mass $2.00\,\mathrm{g}$, are suspended by light strings $10.0\,\mathrm{cm}$ in length. A uniform electric field is applied in the horizontal (x) direction. The spheres have charges equal to $-5 \times 10^{-8}\,\mathrm{C}$ and $+5 \times 10^{-8}\,\mathrm{C}$. Determine the electric field that enables the spheres to be in equilibrium at an angle $\theta = 10.0°$.



Again, we will need a more detailed sketch. Let the distance between the two charges at equilibrium be d. As with the first problem, we need only consider one charge or the other – by symmetry the other charge will have all the same forces. Let us consider the positive charge then. It will have four forces present: those due to the electric field (F_E), the negative charge (F_q), its weight (mg), and the tension in the string (T). Considering all that, here is our sketch and axis definitions:

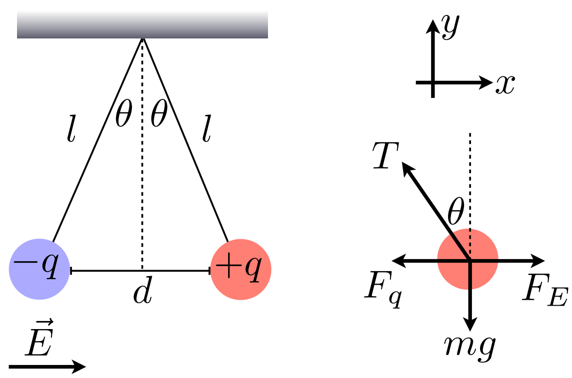


Figure 3: Left: *Situation described in problem 5; we define the distance between the two charges at equilibrium to be d.* **Right:** *Forces on charge $+q$ due to the electric field (F_E), the negative charge (F_q), its weight (mg), and the tension in the string (T).*

The electric forces F_E and F_q are readily calculated. The positive charge $+q$ will feel a force due to the electric field $F_E = qE$, and a force due to the negative charge a distance d away

$$F_q = \frac{-k_e q^2}{d^2} \tag{18}$$

We can find d by trigonometry in terms of the known θ and l:

$$\sin\theta = \frac{d/2}{l} \qquad \Longrightarrow \qquad d = 2l\sin\theta \tag{19}$$

What about the tension in the string? At equilibrium, we know that the y component of the tension must be precisely balanced by the weight of the charge:

$$F_{y,\text{net}} = T\cos\theta - mg = 0 \qquad \Longrightarrow \qquad T = \frac{mg}{\cos\theta} \tag{20}$$

Now we are ready to sum the forces in the x direction, which must also total zero at equilibrium:

$$F_{x,\text{net}} = F_E - F_q - T\sin\theta = 0 \tag{21}$$

$$0 = qE - \frac{k_e q^2}{d^2} - \frac{mg}{\cos\theta}\sin\theta \tag{22}$$

$$qE = \frac{k_e q^2}{4l^2\sin^2\theta} + mg\tan\theta \tag{23}$$

$$\Longrightarrow \quad E = \frac{k_e q}{4l^2\sin^2\theta} + \left(\frac{mg}{q}\right)\tan\theta \approx 4.4\times 10^5 \ [\text{N/C}] \tag{24}$$

When plugging in your numbers, be careful to convert everything to base SI units – m not cm, kg not g.