Problem Set 3: Solutions

1. A sphere the size of a basketball (radius 0.12 m) is charged to a potential of -500 V. About how many extra electrons are on it, per cm^2 of surface?

If the charge is spread evenly over the surface of a spherical object, like a basketball, then Gauss' law says we may treat the charge distribution as a point charge. Thus, we may consider the uniformly charged basketball equivalent to a single point charge at the center of the basketball. A men's regulation basketball has a diameter of 9.39 in, and thus a radius of 0.12 m. We know then that the potential at the surface of the basketball is -1000 V, and this potential results from an effective point charge q at the center of the basketball, 0.12 m from the surface. Using the expression for the potential from a point charge:

$$-500\,\text{V} = \frac{k_e q}{r} = \frac{\left[9 \times 10^9 \,\text{N} \cdot \text{m}^2/\text{C}^2\right] q}{0.12\,\text{m}}$$
$$\implies \quad q = -6.67 \times 10^{-9}\,\text{C}$$

This is the effective charge on the basketball. Given that one electron has -1.6×10^{-19} C of charge, this must correspond to:

$$(\text{number of electrons}) = \frac{-6.67 \times 10^{-9}\,\text{C}}{-1.6 \times 10^{-19}\,\text{C/electron}} = 4.17 \times 10^{10}\,\text{electrons}$$

If these electrons are spread out evenly over the surface, the electron density can be calculated from the surface area of the basketball (remembering that the surface area of a sphere is $4\pi r^2$, and we were asked to use cm^2):

$$(\text{density of electrons}) = \frac{4.17 \times 10^{10}\,\text{electrons}}{4\pi \left(12\,\text{cm}\right)^2} \approx 2.3 \times 10^7\,\text{electrons/cm}^2$$

2. A positive charge of q and a negative charge of $-4q$ are placed a distance d apart. Taking the zero for electric potential to be an infinite distance from both charges, find another point where the electric potential is zero.

First, we need to sketch the problem at hand and choose coordinates. Place the $-4q$ and q charge on the x axis, with the q charge at the origin, and the $+x$ direction running to the right of the q charge, as shown below.
We can already guess the possible positions where $V = 0$. The potential from the $-4q$ charge will be negative everywhere, and the potential from the q charge will be positive everywhere. The $4q$ charge will provide a much larger potential than the q charge, so for the potential to be zero, we'll need to find a spot closer to the q charge than the $-4q$ charge. This could be either between the two charges, closer to the q charge, or to the right of both
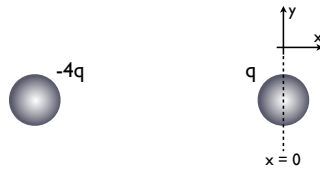
charges. For simplicity, we will seek a solution on the x axis.

If we are a distance x from the q charge, we must be a distance $d + x$ from the $-4q$ charge. For $x > 0$, that means we are to the right of both charges, for $x < 0$, we are between the two charges. The total potential for any x is the sum of the potential from the $-4q$ charge and the potential from the q charge:

$$V_{\text{tot}} = V_{-4q} + V_q = \frac{k_e\,(-4q)}{d + x} + \frac{k_e q}{x} = 0 \tag{1}$$

Now, we just need to solve for x:

$$\frac{4k_e q}{d + x} = \frac{k_e q}{x}$$
$$4k_e q x = k_e q\,(d + x)$$
$$4x = d + x$$
$$x = \frac{d}{3} \tag{2}$$

Thus, one point where the potential is zero is along the x axis, $d/3$ to the right of the q charge.

3. A single atomic layer of singly-charged ions (charge $+q$) can be arranged on a neutral insulating surface in one of two ways: either as a square or a triangular lattice. For four ions in the former configuration, and three in the latter, calculate the potential energy *per unit charge*. Which lattice is more stable?
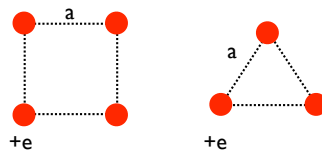
Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just:

$$PE_{\text{pair}} = \frac{k_e q^2}{d}$$

Since all of the charges are the same in both configurations, all we need to do is figure out how many pairs there

are in each situation, and for each pair, how far apart the charges are.

How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force - which we will do as a check - but we can also calculate how many there are. In both configurations, we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair $(2, 1)$ is the same as $(1, 2)$. Pairings like this are known as combinations, as opposed to *permutations* where $(1, 2)$ and $(2, 1)$ are *not* the same. Calculating the number of possible combinations is done like this:[i]

$$\text{ways of choosing pairs from four charges} = \binom{4}{2} = {}^4C_2 = \frac{4!}{2! \, (4 - 2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

So there are 6 unique ways to choose 2 charges out of 4, and similarly we find only 3 unique ways to choose 2 charges out of 3. First, let's consider the square lattice. In order to enumerate the possible pairings, we should label the charges to keep them straight. Label the corner charges $1-4$. Then our possible pairings are:

$(1, 2)$ $(1, 3)$ $(1, 4)$

$(2, 3)$ $(2, 4)$

$(3, 4)$

And there are six, just as we expect. In this configuration, there are only two different distances that can separate a pair of charges: pairs on adjacent corners are a distance a apart, and pairs on opposite corners are a distance $a\sqrt{2}$ apart. We can take our list above, and sort it into pairs that have the same separation:

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Table 1: *Charge pairings in the square lattice*

#, pairing type	separation			pairs	
4, adjacent corners	a	$(1, 2)$	$(2, 3)$	$(3, 4)$	$(1, 4)$
2, opposite corners	$a\sqrt{2}$			$(1, 3)$	$(2, 4)$

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And we are nearly done already. We have four pairs of charges a distance a apart, and two that are $a\sqrt{2}$ apart. Write down the energy for each type of pair, multiply by the number of those pairs, and add the results together:

$$PE_{\text{square}} = 4 \, (\text{adjacent-corner pairs}) + 2 \, (\text{opposite-corner pairs})$$

$$= 4 \left[\frac{k_e q^2}{a} \right] + 2 \left[\frac{k_e q^2}{a\sqrt{2}} \right]$$

$$= \frac{k_e q^2}{a} \left[4 + \frac{2}{\sqrt{2}} \right] = \frac{k_e q^2}{a} \left[4 + \sqrt{2} \right] \approx 5.41 \frac{kq^2}{a}$$

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[i]A nice discussion of combinations and permutations is here: http://www.themathpage.com/aPreCalc/permutations-combinations.htm

The energy of this lattice is about 5.4 times larger than just a pair of charges, and since it is positive, we know that the lattice is not stable – the system could lower its overall energy by breaking up the square lattice. The potential energy per unit charge we are asked to find is just $1/4$ this value, since we have four charges:

$$\frac{1}{4}PE_{\text{square}} = \frac{k_e q^2}{a}\left[1 + \frac{\sqrt{2}}{4}\right] \approx 1.35\frac{kq^2}{a} \tag{3}$$

For the triangle lattice, things are even easier. This time, just pick one charge as "1", and label the other two 2 and 3. We only have three possible pairings now:

$$PE_{\text{triangle}} = 3\,(\text{adjacent pairs})$$
$$= 3\frac{k_e q^2}{a}$$

The triangular lattice has a lower overall potential energy, and it is not just because it has fewer charges: the energy per unit charge is also lower:

$$\frac{1}{3}PE_{\text{triangle}} = \frac{k_e q^2}{a} \tag{4}$$

The potential energy per unit charge of the square lattice is higher, meaning it is less favorable than the triangular lattice. Neither one is energetically favored though – since the energy is positive, it means that either configuration of charges is less stable than just having all five charges infinitely far from each other. This makes sense – if all charges have the same sign, they don't want to arrange next to one another, and thus these arrangements cost energy to keep together. If we didn't force the charges together in these patterns, the positive energy tells us that they would fly apart given half a chance. For this reason, neither one is a valid sort of crystal lattice, real crystals have equal numbers of positive and negative charges, and are overall electrically neutral.

4. The gap between the electrodes of a spark plug in an automobile is $0.64\,$mm. In order to produce an electric field of $3 \times 10^6\,$V/m required to initiate a spark, what minimum potential difference must you apply to the plug?

If we assume that the electric field is constant in the gap region, we can relate the potential difference to the gap length and electric field by equating work and potential energy:

$$|W| = |PE| = qE\Delta x = q\Delta V$$
$$E\Delta x = \Delta V$$
$$\Delta V = \left(3 \times 10^6\,\text{V/m}\right)\left(0.64 \times 10^{-3}\,\text{m}\right) \approx 1920\,\text{V}$$

5. An electron in a neon light tube is accelerated from rest through a $2000\,$V potential difference. What velocity

does the electron attain?

If no other forces are acting, we can apply conservation of energy: the electrical potential energy of the electron is converted into kinetic energy. We also know that the potential difference must be negative, since electrons will want to go from low potential to high potential:

$$q\Delta V = -e\Delta V = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{-2e\Delta V}{m}} = \sqrt{\frac{-2\left(1.60 \times 10^{-19}\,\text{C}\right)\left(-2000\,\text{V}\right)}{9.11 \times 10^{-31}\,\text{kg}}} \approx 2.65 \times 10^7\,\text{m/s}$$

6. Find the equivalent capacitance for *both* combinations shown below. Be sure to consider the symmetry involved and the relative electric potential at different points in the circuits.
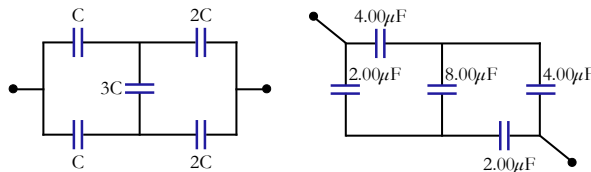
In the first case, by symmetry the $3C$ capacitor has no potential difference across it. Since capacitors in parallel have the same potential difference, both of the C capacitors have the same potential difference, and that means that both ends of the $3C$ capacitor are at the same potential. If that is true, then no charge is stored $Q=C\Delta V=0$, and we can simply replace the $3C$ capacitor with a plain wire. Once we have done that, we have a pair of C capacitors connected in parallel, in series with a pair of $2C$ capacitors which are also connected in parallel. The two C's give an equivalent capacitance of $2C$, and the two $2C$'s give an equivalent capacitance of $4C$, so the whole circuit is equivalent to $2C$ in *series* with $4C$. This gives an equivalent capacitance of $4C/3$.

In the second case, we can apply the same argument to the $8\,\mu\text{F}$ capacitor - it cannot have a potential difference across it, and it can therefore be replaced with a plain old wire. That leaves us with two pairs of $4\,\mu\text{F}$ and $2\,\mu\text{F}$ capacitors in parallel, each of which can be replaced with a single $6\,\mu\text{F}$ equivalent capacitor. The whole circuit is then equivalent to two $6\,\mu\text{F}$ capacitors in series, which is itself equivalent to a single $3\,\mu\text{F}$ capacitor.