UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 102 / LeClair Summer II 2009

Problem Set 4: Solutions

Instructions:

1. Answer all questions below. Show your work for full credit.
2. Due before the end of the day, 17 July 2009
3. Email: leclair.homework@gmail.com; hard copies: Gallalee 110 or Bevill 228.
4. You may collaborate, but everyone must turn in their own work

1. If the voltage at the terminals of an automobile battery drops from 12.3 to 9.8 V when a 0.5 Ω resistor is connected across the battery, what is the internal resistance of the battery?

Before the 0.5 Ω resistor is connected, there is no current flowing in the battery, and what we measure at the terminals is its ideal rated voltage, $\Delta V = 12.3\,\text{V}$. Once the resistor is connected, a current I flows, and the output voltage of the battery is reduced by its internal resistance. What we measure at the battery terminals is the rated voltage *minus the voltage drop across the internal resistance.* Thus, we have two circuits to consider, and between the two we can find the internal resistance.

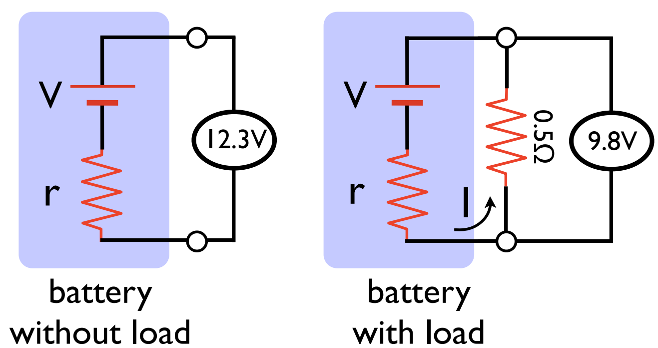


Figure 1: Left: *Before connecting the external resistor to the battery, we measure the full rated voltage of the battery, since no current flows.* Right: *After connecting an external resistor, we measure a lower voltage – once there is a current in the circuit, there is a voltage drop across the battery's internal resistance, which reduces the voltage available for the external resistance.*

Call the internal resistance r, and the external 0.5 Ω resistor R. With the resistor connected, conservation of energy says that the sum of voltage sources and drops around the entire circuit must be zero:

$$\Delta V - Ir - IR = 0$$

What we measure at the terminals of the battery is $\Delta V - Ir = 9.8\,\text{V}$ - the ideal battery voltage minus what is lost due to internal resistance. Using this relationship in the equation above, we have:

$$
\begin{aligned}
\Delta V - Ir - IR &= 0 \\
9.8\,\mathrm{V} - IR &= 0 \\
\implies \quad I &= \frac{9.8\,\mathrm{V}}{0.5\,\Omega} = 19.6\,\mathrm{A}
\end{aligned}
$$

Now that we know what I is, we can use the measured voltage with the resistor connected to find r, the internal resistance:

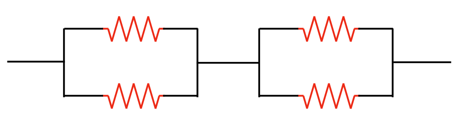
$$
\begin{aligned}
\Delta V - Ir &= 9.8\,\mathrm{V} \\
12.3\,\mathrm{V} - (19.6\,\mathrm{A})\,r &= 9.8\,\mathrm{V} \\
\implies \quad r &= \frac{2.5\,\mathrm{V}}{19.6\,\mathrm{A}} \approx 0.13\,\Omega
\end{aligned}
$$

2. Are the two headlights of a car wired in series or in parallel? How can you tell?

Have you ever seen cars driving down the road with only one working headlight? If headlights were wired in series, when one light goes out, both would go out. Wiring headlights in parallel means that when one bulb goes out, the other stays lit.

3. What advantage might there be in using two identical resistors in parallel connected in series with another identical parallel pair, rather than just using a single resistor?

The combination we are talking about is this one:

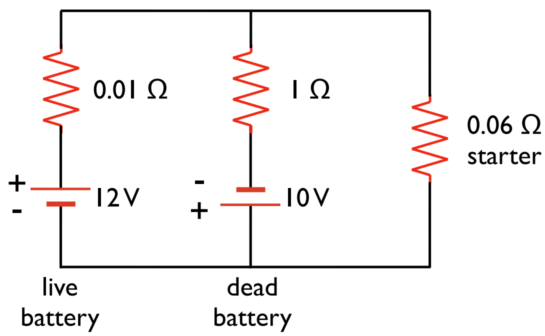


You can verify for yourself that if each individual resistor has a value R, the equivalent resistance of the arrangement above is also R. The advantage in this situation compared to using a single resistor of value R is that while the *total* power dissipation is the same, it is now divided between four resistors. This arrangement lets one use several physically smaller low power components instead of one bulky high-power component. For instance, if you only had resistors rated at 15 W, but your circuit required 30 W, one could use the arrangement above safely.

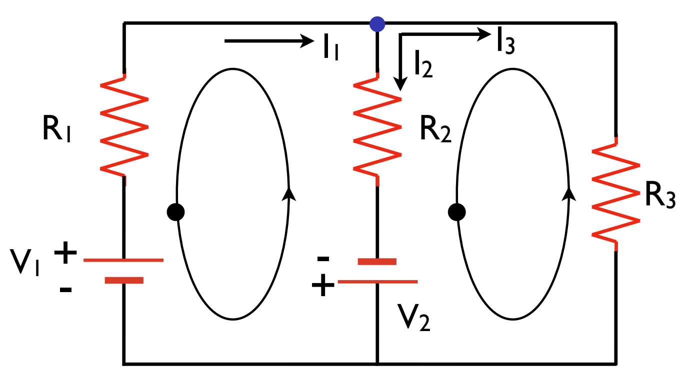
Each in this case would each have half of the total voltage (due to the series combination) and half the total current (due to the parallel combination). Since power is current times voltage, the total power in any given resistor is one quarter what it would be for a single resistor connected to the same power source.

Another advantage would be redundancy, as with the headlights in question 1 - in this arrangement, one single failure will still allow the circuit to operate. For instance, the four resistors might be electric heaters connected to a constant voltage source (like a wall socket). A single heater could fail, and if the rest of the circuit were properly designed, the remaining three would still provide 2/3 of the original power.

4. A dead battery is charged by connecting it to the live battery of another car with jumper cables (see below). Determine the current in the starter and in the dead battery.



Since this circuit has several branches and multiple batteries, we cannot reduce it by using our rules of series and parallel resistors - we have to use our general circuit rules (Kirchhoff's rules). In order to do that, we first need to assign currents in each branch of the circuit. It doesn't matter what directions we choose at all, assigning directions is just to define what is, relatively speaking, positive and negative. If we choose the direction for one current incorrectly, we will get a negative number for that current to let us know. Below, we choose initial currents I_1, I_2, and I_3 in each branch of the circuit.



Here we have also labeled each component symbolically to make the algebra a bit easier to sort out – we don't want to just plug in numbers right away, or we'll make a mess of things. Note that since we have three unknowns

– the three currents – so we will need three equations to solve this problem completely. We have three possible loops (left, right, outer), which gives us two equations, and two junctions, which gives us one more. If we have N loops or N junctions, we get $N - 1$ equations from either.

Now we are ready to apply the rules. First, conservation of charge (the "junction rule"). We have only two junctions in this circuit, in the center at the top and bottom where three wires meet. The junction rule basically states that the current into a junction (or node) must equal the current out. In the case of the upper node, this means:

$$I_1 = I_2 + I_3 \tag{1}$$

You can easily verify that the lower node gives you the same equation. Next, we can apply conservation of energy (the "loop rule"). There are three possible loops we can take: the rightmost one containing R_3 and R_2, the leftmost one containing R_1 and R_2, and the outer perimeter (containing R_1 and R_3). We only need to work through two of them - we have already one equation above, and we only need two more. Somewhat arbitrarily, we will pick the right and left side loops.

First, the left-hand side loop. Start just above the live battery V_1, and walk *clockwise* around the loop. We cross the battery from negative to positive for a *gain* in potential energy, and we cross R_1 and R_3 in the direction of current flow for a *loss* of potential energy. These three have to sum to zero for a closed loop:

$$-I_1R_1 - I_2R_2 + V_2 + V_1 = 0 \tag{2}$$

Next, the right-hand side loop. Again, start just above the battery (V_2 this time), and walk *clockwise* around the loop. Now we cross R_2 against the current and R_3 with the current for a gain and loss of voltage, respectively, and then cross the battery in the wrong direction for a potential drop:

$$I_2R_2 - I_3R_3 - V_2 = 0 \tag{3}$$

Now we have three equations and three unknowns, and we are left with the pesky problem of solving them for the three currents. There are many ways to do this, we will illustrate two of them. Before we get started, let us repeat the three questions in a more consistent form.

$$\begin{aligned}
I_1 - I_2 - I_3 &= 0 \\
R_1I_1 + R_2I_2 &= V_1 + V_2 \\
R_2I_2 - R_3I_3 &= V_2
\end{aligned}$$

The first way we can proceed is by substituting the first equation into the second:

$$V_1 + V_2 = R_1 I_1 + R_2 I_2 = R_1 \left(I_2 + I_3\right) + R_2 I_2 = \left(R_1 + R_2\right) I_2 + R_1 I_3$$

Now our three equations look like this:

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ \left(R_1 + R_2\right) I_2 + R_1 I_3 &= V_1 + V_2 \\ R_2 I_2 - R_3 I_3 &= V_2 \end{aligned}$$

The last two equations now contain only I_1 and I_2, so we can solve the third equation for I_2:

$$I_2 = \frac{I_3 R_3 + V_2}{R_2}$$

…and plug it in to the second one:

$$\begin{aligned} V_1 + V_2 = \left(R_1 + R_2\right) I_2 + R_1 I_3 &= \left(R_1 + R_2\right) \left(\frac{I_3 R_3 + V_2}{R_2}\right) + R_1 I_3 \\ R_2 \left(V_1 + V_2\right) &= R_3 \left(R_1 + R_2\right) I_3 + V_2 \left(R_1 + R_2\right) + R_1 R_2 I_3 \\ V_1 R_2 - V_2 R_1 &= \left(R_1 R_3 + R_2 R_3 + R_1 R_2\right) I_3 \\ \implies \quad I_3 &= \frac{R_2 V_1 - R_1 V_2}{R_1 R_2 + R_2 R_3 + R_1 R_3} \approx 169\,\text{A} \end{aligned}$$

There is a sort of pleasing symmetry to the analytical answer. Now that you know I_3, you can plug it in the expression for I_2 above, you should find $I_2 \approx 20\,\text{A}$, and similarly you can find $I_1 \approx 189\,\text{A}$

Optional: There is one more way to solve this set of equations using matrices and Cramer's rule,[i] if you are familiar with this technique. If you are not familiar with matrices, you can skip to the next problem - you are not required or necessarily expected to know how to do this. First, write the three equations in matrix form:

$$\begin{bmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -R_3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 + V_2 \\ V_2 \\ 0 \end{bmatrix}$$

$$aI = V$$

The matrix a times the column vector I gives the column vector V, with the matrices defined thusly:

[i] See 'Cramer's rule' in the Wikipedia to see how this works.

$$a = \begin{bmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -R_3 \\ 1 & -1 & -1 \end{bmatrix} \qquad I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \qquad V = \begin{bmatrix} V_1 + V_2 \\ V_2 \\ 0 \end{bmatrix}$$

Now we can use the determinant of the matrix a with Cramer's rule to find the currents. For each current, we construct a new matrix, which is the same as the matrix a except that the column of a corresponding to that current is replaced the column vector V. Thus, for I_1, we replace column 1 in a with V, and for I_2, we replace column 2 in a with V. We find the current then by calculating the determinant of the new matrix and dividing it by $\det a$. Below, we have highlighted the columns in a which have been replaced to make this more clear:

$$I_1 = \frac{\begin{vmatrix} V_1 + V_2 & R_2 & 0 \\ V_2 & R_2 & -R_3 \\ 0 & -1 & -1 \end{vmatrix}}{\det a} \qquad I_2 = \frac{\begin{vmatrix} R_1 & V_1 + V_2 & 0 \\ 0 & V_2 & -R_3 \\ 1 & 0 & -1 \end{vmatrix}}{\det a} \qquad I_3 = \frac{\begin{vmatrix} R_1 & R_2 & V_1 + V_2 \\ 0 & R_2 & V_2 \\ 1 & -1 & 0 \end{vmatrix}}{\det a}$$

Now we need to calculate the determinant of each new matrix, and divide that by $\det a$.[ii] First, the determinant of a.

$$\det a = -R_1R_2 - R_1R_3 - R_2R_3 = -\left(R_1R_2 + R_2R_3 + R_1R_3\right)$$

We can now find the currents readily from the determinants of the modified matrices and $\det a$ we just found. Thankfully, the matrices have enough zeros that it is relatively easy. In case your memory is rusty, here is the determinant of an arbitrary 3×3 matrix:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left(a_1b_2c_3 - a_1b_3c_2\right) + \left(a_2b_3c_1 - a_2b_1c_3\right) + \left(a_3b_1c_2 - a_3b_2c_1\right)$$

Given this,

$$I_1 = \frac{-\left(V_1 + V_2\right)R_2 - R_3\left(V_1 + V_2\right) + R_2V_2}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{\left(V_1 + V_2\right)R_3 + R_2V_1}{R_1R_2 + R_2R_3 + R_1R_3} \approx 189\,\mathrm{A}$$

$$I_2 = \frac{-R_1V_2 - \left(V_1 + V_2\right)R_3}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{\left(V_1 + V_2\right)R_3 + R_1V_2}{R_1R_2 + R_2R_3 + R_1R_3} \approx 20\,\mathrm{A}$$

$$I_3 = \frac{R_1V_2 + R_2V_2 - \left(V_1 + V_2\right)R_2}{-\left(R_1R_2 + R_2R_3 + R_1R_3\right)} = \frac{R_2V_1 - R_1V_2}{R_1R_2 + R_2R_3 + R_1R_3} \approx 169\,\mathrm{A}$$

[ii] Again, the Wikipedia entry for 'determinant' is quite instructive.

These are the same results you would get by continuing on with the previous 'plug-n-chug' method. Both numerically and symbolically, we can see from the above that $I_1 = I_2 + I_3$:

$$I_2 + I_3 = \frac{(V_1 + V_2)\, R_3 + R_1 V_2 + R_2 V_1 - R_1 V_2}{R_1 R_2 + R_2 R_3 + R_1 R_3} = \frac{(V_1 + V_2)\, R_3 + R_2 V_1}{R_1 R_2 + R_2 R_3 + R_1 R_3} = I_1$$

5. Two resistors connected in series have an equivalent resistance of $900\,\Omega$. When they are connected in parallel, their equivalent resistance is $180\,\Omega$. Find the resistance of each resistor.

When we combine two resistors in series, they simply add to form a equivalent resistor. In parallel, they add inversely. This implies two equations:

$$\begin{aligned} R_1 + R_2 &= 900 \\ \frac{1}{R_1} + \frac{1}{R_2} &= \frac{1}{180} \end{aligned}$$

It is more convenient if we rearrange the second one (find a common denominator for the left-hand side and invert) to look like this:

$$\frac{R_1 R_2}{R_1 + R_2} = 180$$

Now, plug the first one in to the second and massage it a bit:

$$\begin{aligned} \frac{R_1 R_2}{R_1 + R_2} = \frac{R_1 R_2}{900} &= 180 \\ R_1 R_2 &= 180 \cdot 900 \end{aligned}$$

We can use our first equation a second time, noting that $R_2 = 900 - R_1$:

$$\begin{aligned} R_1 R_2 = R_1 (900 - R_1) = 900R_1 - R_1^2 &= 180 \cdot 900 \\ \Longrightarrow \quad R_1^2 - 900R_1 + 180 \cdot 900 &= 0 \end{aligned}$$

Now we have a quadratic that we can solve for R_1.

$$\begin{aligned} R_1 &= \frac{-(-900) \pm \sqrt{(-900)^2 - 4 \cdot 1 \cdot (180 \cdot 900)}}{2} = \frac{900 \pm \sqrt{900^2 - 720 \cdot 900}}{2} = \frac{900 \pm 900\sqrt{1 - \frac{720}{900}}}{2} \\ &= 450 \left[1 \pm \sqrt{1 - \frac{4}{5}}\right] = 450 \left[1 \pm \frac{1}{\sqrt{5}}\right] = 450 \left[1 \pm \frac{\sqrt{5}}{5}\right] \approx \{249, 651\}\,\Omega \end{aligned}$$

Now we have two solutions for R_1. What is that? No worries. Since we labeled R_1 and R_2 arbitrarily, and our equations are completely symmetric with regard to either, we have actually just found *both* R_1 and R_2. Try plugging them both in to the first equation, and you will see that we really only have one complete solution:

$$R_2 = 900 - R_1 = 900 - 249\,\Omega = 651\,\Omega \qquad \text{1st solution}$$
$$R_2 = 900 - R_1 = 900 - 651\,\Omega = 249\,\Omega \qquad \text{2nd solution}$$

Thus, our two resistors have to be $651\,\Omega$ and $249\,\Omega$.