UNIVERSITY OF ALABAMA
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PH 102 / LeClair Summer II 2009

Problem Set 7: Solutions

1. A cell membrane typically has a capacitance around $1\,\mu\text{F/cm}^2$. It is believed the membrane consists of material having a dielectric constant of $\kappa \sim 3$. Find the thickness this implies. Other electrical measurements have indicated that the resistance of $1\,\text{cm}^2$ of cell membrane is around $1000\,\Omega$. Show that the time constant of such a leaky capacitor is independent of the area of the capacitor. How large is it in this case? What is the resistivity?

Our first key assumption is that we can model our cell membrane as a parallel plate capacitor. This seems outlandishly stupid on first sight, but in fact it is a reasonable starting point for modeling the phospholipid bilayer in a cell membrane. The polar head groups on the lipid molecules in the lipid bilayer do form an extended structure that resembles a capacitor, if we think about the charge transfer involved. Anyway: if you are not up on your microbiology (and it has been a while for me), just take this starting point for granted for now.

This problem is a lot easier than it looks if we just write down the expression for capacitance per unit area. That is what we are given, and sometimes when you aren't quite sure what to do, just write down what you know. Let the interior of the cell membrane have a thickness d and a relative dielectric constant κ. Then,

$$\frac{C}{A} = \frac{\kappa\epsilon_o}{d} = 1\,\mu\text{F/cm}^2$$

As it turns out, capacitance per unit area depends only on the spacing between charges, in this case the thickness of the cell membrane, so we can find d easily:

$$d = \frac{\kappa\epsilon_o}{C/A} = \approx 2.7\times 10^{-9}\,\text{m}$$

Be careful with the units! This is about 3 nm, which seems about the right order of magnitude for the length of a small lipid molecule. What about our leaky capacitor? All this means is that the capacitor lets some dc current through it, as if it had a resistor in *parallel* that let some current "leak" around it. In other words, a "leaky" capacitor is just a parallel RC circuit.

The resistive part of the circuit we can model as a cylinder of area A and the same thickness d, which has then a resistance of $R=\rho d/A$ (here we imagine charge transfer across the cell membrane, along the direction of the lipid molecules). Notice that the *resistance area product* is just $RA=\rho d$. Now, the time constant of a parallel RC circuit is just $\tau=RC$. This is the same thing as multiplying the capacitance per unit area and the resistance area product: $\tau = RC = (RA)(C/A)$. Basically: we can calculate the time constant of our leaky capacitor without knowing its area.

$$\tau = RC = (RA)\left(\frac{C}{A}\right) = \left(\frac{\rho d}{A}\right)\left(\frac{\kappa\epsilon_o A}{d}\right) = \rho\kappa\epsilon_o$$

Independent of area, as desired, and also independent of thickness! Given the value of the resistance R for a specific area A, we can find the area product RA and calculate the resistivity:

$$\rho = \frac{RA}{d} \approx 4 \times 10^7\,\Omega \cdot \mathrm{m}$$

Given a resistivity, we can get a number for the time constant, $\tau \approx 10^{-3}\,\mathrm{s}$. This implies that charge (ion) transfer across a cell membrane should have an intrinsic time scale on the order of milliseconds.

2. The rate of flow of a conducting liquid can be measured with an electromagnetic flowmeter that detects the voltage induced by the motion of the liquid in a magnetic field. Suppose that a plastic pipe of diameter $0.10\,\mathrm{m}$ carries beer with a speed of $1.5\,\mathrm{m/s}$. The pipe is in a transverse magnetic field (*i.e.*, perpendicular to the pipe axis) of about $1.5 \times 10^{-2}\,\mathrm{T}$. **(a)** Presume the beer is an ideal conductor. What voltage will be induced between the opposite sides of the column of liquid? **(b)** Does it matter whether the conductivity of the beer is due to mobile positive or negative charges?

This is in the end just a motion-induced voltage problem. The presence of a magnetic field perpendicular to the movement of ions in the beer means that they experience a magnetic force.

$$F_m = qvB$$

If the flow of beer is from left to right, and the magnetic field is pointing into the page, then the force for positive ions is pointing up, and for negative ions it is pointing down. This serves to separate spatially the positive and negative charges along the diameter of the pipe, the same way that positive and negative charges were separated in a conducting rod moving perpendicularly to a magnetic field. This continues until the ions reach the surface of the pipe, at which point they are separated by its diameter d.

If the positive and negative charges are separated spatially by a distance d, then this gives rise to a (uniform) electric field E, and a potential difference $\Delta V = Ed$. At equilibrium, the electric and magnetic forces are balanced - the magnetic force pulls the charges apart, and it is perfectly counterbalanced by the resulting electric force. Set the magnetic and electric fields equal to one another, use the expression for ΔV, and take care with units:

$$\begin{aligned} F_m &= qvB = F_e = qE = q\left(\frac{\Delta V}{d}\right) \\ \Rightarrow \Delta V &= Bvd \approx 2.25\,\mathrm{mV} \end{aligned}$$

(b) Does it matter whether the conductivity of the beer is due to mobile positive or negative charges?

You can verify that the sign of the potential difference does *not* depend on whether the ions are mostly positive or negative, since the force is in different directions for each ion. No matter what, positive ions go up, negative ions go down, and the same potential difference results.

3. A technician wearing a conducting bracelet enclosing an area $0.005\,\mathrm{m}^2$ places her hand in a solenoid whose magnetic field is $5.0\,\mathrm{T}$ directed perpendicular to the plane of the bracelet. The resistance around the circumference of the bracelet is $0.02\,\Omega$. A power failure causes the field to drop to $1.50\,\mathrm{T}$ in a time of $20.0\,\mathrm{ms}$. Find **(a)** the current in the bracelet, and **(b)** the power delivered to the bracelet..

Once the power goes out, the magnetic field drops by an amount $\Delta B = 5 - 1.5 = 3.5\,\mathrm{T}$ in a time $\Delta t - 0.020\,\mathrm{s}$. This means, given the fixed area of the bracelet, that the magnetic flux is changing through the bracelet:

$$\frac{\Delta\Phi_B}{\Delta t} = A\frac{\Delta B}{\Delta t} = \left(0.005\,\mathrm{m}^2\right)\left(\frac{3.5\,\mathrm{T}}{0.020\,\mathrm{s}}\right) \approx 0.875\,\mathrm{Tm}^2/\mathrm{s} = 0.875\,\mathrm{V}$$

The change in flux leads to an induced voltage across the bracelet. Since the bracelet has a single loop of wire, the induced voltage is just

$$\Delta V = -\frac{\Delta\Phi_B}{\Delta t} = -0.875\,\mathrm{V}$$

The minus sign here is not important, since the ultimate direction of current flow is not important. Given the bracelet's resistance, we can find the induced current:

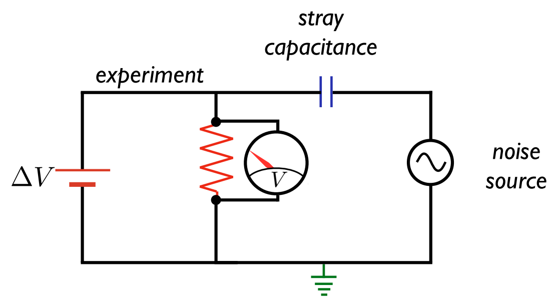
$$I = \frac{\Delta V}{R} \approx 43.75\,\mathrm{A}$$

Finally, we can find the power from the current and voltage, or either one and the resistance:

$$\mathscr{P} = I^2R = I\Delta V = \frac{\Delta V^2}{R} \approx 38.28\,\mathrm{W}$$

This is an unreasonably large amount of power to be dumped into a bracelet on one's wrist, leading to an unreasonable generation of heat …for this reason, you will not see personnel working regularly with MRI machines wearing much jewelry. At least, they shouldn't be.

4. Any two adjacent conductors can be considered as a capacitor, although the capacitance will be small unless the conductors are close together or long. This (unwanted) effect is termed "stray" or "parasitic" capacitance. Stray capacitance can allow signals to leak between otherwise isolated circuits (an effect called crosstalk), and it can be a limiting factor for proper functioning of circuits at high frequency. A stray capacitance can result when you touch or come close the wires in a circuit – your body provides a capacitive path between the circuit of interest and an adjacent noise source. **(a)** Explain, referencing the figure at right, why the stray capacitance allows unwanted ac signals to couple into the circuit, but does not allow dc signals. **(b)** Suggest a method for minimizing this effect.



In the circuit shown, the noise source and experiment share a common ground point, which means the circuits are connected at one point. Connecting the circuits at a second point will allow signals to pass from one circuit to the other. If this connection is made by a stray capacitance, constant dc currents can pass through the ground connection, but not through the capacitance. Thus, dc signals still have only one path into the experimental circuit, there is no closed loop for them to couple into the experiment. Ac signals, on the other hand, *can* travel through the capacitor, and now have a closed loop to travel between the experiment and noise source. The higher the frequency, the easier the signals can couple into the experiment.

The solution to this problem is simply to use coaxial shielded cable. The stray capacitance arises when wires from the experiment and noise source come too close together, and the intervening region makes a capacitor out of the adjacent wires. Time-varying electric fields from one circuit can be coupled across the intervening region into the other circuit, just like ac currents can pass through a capacitor. If the experimental circuit's wires are encased in a conducting shell, the electric fields from nearby circuits outside the shell are shielded out, a result of the fact that the field inside a conductor must be zero. Of course, another solution is just to put the experiment far, far away from any potential noise sources, but this is not always practical.