UNIVERSITY OF ALABAMA
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PH 102 / LeClair Summer II 2010

Problem Set 1: Solution

1. A classic "paradox" involving length contraction and the relativity of simultaneity is as follows: Suppose a runner moving at $0.75c$ carries a horizontal pole 15 m long toward a barn that is 10 m long. The barn has front and rear doors. An observer on the ground can instantly and simultaneously open and close the two doors by remote control. When the runner and the pole are inside the barn, the ground observer closes and then opens both doors so that the runner and pole are momentarily captured inside the barn and then proceed to exit the barn from the back door. Do both the runner and the ground observer agree that the runner makes it safely through the barn?

Solution: This paradox is discussed thoroughly in a number of places online. See, for example:

`http://hyperphysics.phy-astr.gsu.edu/HBASE/Relativ/polebarn.html`
`http://en.wikipedia.org/wiki/Ladder_paradox`

(The links should be clickable.)

2. An astronaut takes a trip to Sirius, which is located a distance of 8 lightyears from the Earth. The astronaut measures the time of the one-way journey to be 6 yr. If the spaceship moves at a constant speed of $0.8c$, how can the 8-ly distance be reconciled with the 6-yr trip time measured by the astronaut?

Solution: The 8 light-year distance is that measured according to the stationary observers, viz., the earthlings. According to the astronaut, who is in motion relative to Earth and Sirius, the distance is shortened by a factor γ:

$$L_{\text{astronaut}} = \frac{1}{\gamma} L_{\text{earthlings}} = (8\,\text{light-years}) \left(\sqrt{1-(0.8c)^2/c^2} \right) = (8\,\text{light-years})\,(0.6) = 4.8\,\text{light-years} \quad (1)$$

The astronaut measures the trip to take 6 yr, which means the astronaut would report a velocity of

$$v = \frac{4.8\,\text{light-years}}{6\,\text{yr}} = 0.8c \quad (2)$$

Thus, there is no paradox: the difference in measured times is due to time dilation/length contraction. More to the point: we can't divide one observers distance by another observer's time and expect to get sensible answers unless they are in the same reference frame!

3. If a duck has a time dilation factor of 10, what is its speed?

Solution: What does this mean? If the duck has a time dilation factor of 10, that means it is moving so fast relative to an external observer that it feels a passage of time which is 10 times slower than the observer. The situation is then that we have an observer on the ground, measuring proper time, who sees a duck fly by at some velocity v. The 'time dilation factor' is just the ratio of the time that passes according to the duck compared to that measured by the observer.

First, we must assume that the duck carries some sort of accurate timepiece. Second, if the duck's time is dilated, the observer on the ground must measure the 'proper' time. We know how to relate these times: the duck is moving, the observer is watching it go by at some velocity v, so

$$\Delta t'_{\text{duck}} = \gamma \Delta t_{\text{observer}}$$

The dilation factor is then:

$$[\text{dilation factor}] = \frac{\Delta t'_{\text{duck}}}{\Delta t_{\text{observer}}} = \gamma = 10$$

We now just need the definition of γ, and we can solve for the velocity of the clock v

$$\begin{aligned}
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} &= 10 \\
\sqrt{1 - \frac{v^2}{c^2}} &= \frac{1}{10} \\
1 - \frac{v^2}{c^2} &= \frac{1}{10^2} = \frac{1}{100} \\
\frac{v^2}{c^2} &= 1 - \frac{1}{100} \\
v^2 &= c^2 \left(1 - \frac{1}{100}\right) \\
v &= \pm c \sqrt{1 - \frac{1}{100}} \\
v &\approx \pm 0.995c
\end{aligned}$$

One minor point: remember that when we take a square root, we have a positive and a negative answer, hence the $\pm$. Physically, this represents the fact that we can't tell from the information given whether the duck is coming or going - the answer is the same no matter what direction the duck is moving relative to the observer, it is only important that it *is* moving.

4. A bassist taps the lowest E on his bass at 140 beats per minute during one portion of a song. What tempo would an observer on a ship moving *toward* the bassist at 0.70c hear?

Solution: What we are really interested in is the time interval between the taps. That time interval will be dilated (longer) for the moving observer, and hence the taps will sound farther apart (the tapping will be slower).

The 'proper time' interval Δt_p is that measured by the bassist, which is $1/140\,\text{min/beat}$ (so that there are $140\,\text{beats/min}$).[i] The time interval *between taps* measured by the moving observer $\Delta t'$ is longer by a factor gamma:

$$\Delta t' = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{0.7^2 c^2}{c^2}}} \cdot \left(\frac{1\,\text{min}}{140\,\text{beats}}\right) = \frac{1}{\sqrt{1-0.7^2}} \cdot \left(\frac{1\,\text{min}}{140\,\text{beats}}\right) \approx \frac{0.01\,\text{min}}{\text{beat}}$$

The beats per minute heard by the moving observer is just $1/\Delta t'$:

$$[\text{beats per minute, heard by observer}] = \frac{1}{\Delta t'} = \frac{1}{0.01} = 100$$

So, the bass line seems to be moving about 40% slower.

5. An atomic clock aboard a spaceship runs slow compared to an Earth-based atomic clock at a rate of 2.0 seconds per day. What is the speed of the spaceship?

Solution: The proper time t_p is that measured on earth, while a dilated time $t = \gamma t_p$ is measured on the ship. If the clock aboard the ship is $2\,\text{s}$ per day slow, then using $1\,\text{day} = 86400\,\text{s}$

$$t - t_p = \gamma t_p - t_p = (\gamma - 1)\,t_p = (\gamma - 1)\,(86400\,\text{s}) = 2\,\text{s} \tag{3}$$

This gives

$$\gamma = \frac{2}{86400} + 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1.00002315 \tag{4}$$

Solving for v, we find $v \approx 0.0068c$.

[i] The time interval is just the inverse of the *rate* of tapping in beats per unit time.