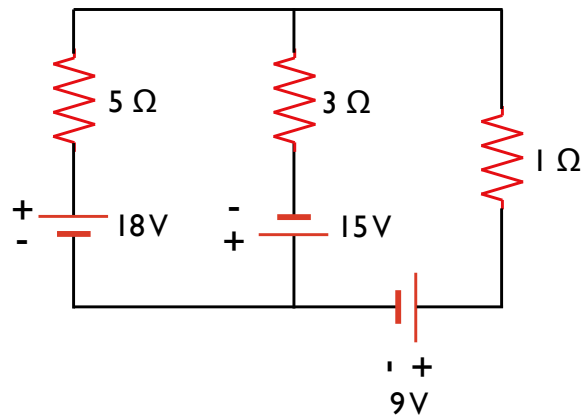
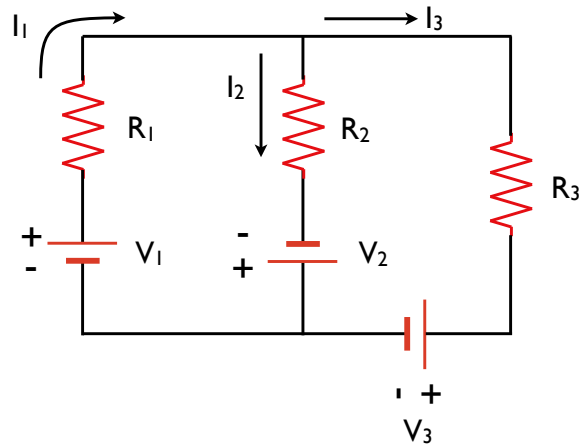


Problem Set 5: Solutions

1. Find the current in the $1\ \Omega$ resistor in the circuit below.



Solution: Since this circuit has several branches and multiple batteries, we cannot reduce it by using our rules of series and parallel resistors - we have to use our general circuit rules (Kirchhoff's rules). In order to do that, we first need to assign currents in each branch of the circuit. It doesn't matter what directions we choose at all, assigning directions is just to define what is, relatively speaking, positive and negative. If we choose the direction for one current incorrectly, we will get a negative number for that current to let us know. Below, we choose initial currents I_1 , I_2 , and I_3 in each branch of the circuit.



Here we have also labeled each component symbolically to make the algebra a bit easier to sort out - we don't want to just plug in numbers right away, or we'll make a mess of things. Note that since we have

three unknowns – the three currents – so we will need three equations to solve this problem completely. We have three possible loops (left, right, outer), which gives us two equations, and two junctions, which gives us one more. If we have N loops or N junctions, we get $N - 1$ equations from either.

Now we are ready to apply the rules. First, conservation of charge (the “junction rule”). We have only two junctions in this circuit, in the center at the top and bottom where three wires meet. The junction rule basically states that the current into a junction (or node) must equal the current out. In the case of the upper node, this means:

$$I_1 = I_2 + I_3 \quad (1)$$

You can easily verify that the lower node gives you the same equation. Next, we can apply conservation of energy (the “loop rule”). There are three possible loops we can take: the rightmost one, the leftmost one, and the outer perimeter. We only need to work through two of them - we have already one equation above, and we only need two more. Given that we want the current in R_3 , we would do well to choose two loops that contain R_3 , namely, the rightmost one and the outer perimeter.

First, the right-hand loop. Starting below the V_2 battery and ‘walking’ clockwise, we first cross V_2 from + to -. This is the direction a positive charge would like to go, which means a loss of potential. We next cross R_2 against the current, which is undesirable and means we gain potential. After that, we cross R_3 *with* the current, which is favorable and means a loss of potential. Finally, we cross V_3 from + to -, again losing potential. Overall:

$$-V_2 + I_2 R_2 - I_3 R_3 - V_3 = 0 \quad (2)$$

For the outer loop, we’ll start below V_1 and walk clockwise around the loop. Following the same rules, we have

$$V_1 - I_1 R_1 - I_3 R_3 - V_3 = 0 \quad (3)$$

Now we have three equations and three unknowns, and we are left with the pesky problem of solving them for the three currents. There are many ways to do this, we will illustrate two of them. Before we get started, let us repeat the three questions in a more ordered fashion:

$$I_1 - I_2 - I_3 = 0 \quad (4)$$

$$R_2 I_2 - R_3 I_3 = V_2 + V_3 \quad (5)$$

$$R_1 I_1 + R_3 I_3 = V_1 - V_3 \quad (6)$$

The first way we can proceed is by substituting the first equation into the second:

$$R_2 I_2 - R_3 I_3 = R_2 (I_1 - I_3) - R_3 I_3 = R_2 I_1 - (R_2 + R_3) I_3 = V_2 + V_3 \quad (7)$$

This, combined with the third equation, gives us two equations in terms of only I_2 and I_3 :

$$R_2 I_1 - (R_2 + R_3) I_3 = V_2 + V_3 \quad (8)$$

$$R_1 I_1 + R_3 I_3 = V_1 - V_3 \quad (9)$$

If we multiply the second equation above by R_2/R_1 :

$$R_2 I_1 - (R_2 + R_3) I_3 = V_2 + V_3 \quad (10)$$

$$R_2 I_1 + \frac{R_2 R_3}{R_1} I_3 = (V_1 - V_3) \frac{R_2}{R_1} \quad (11)$$

Now we can subtract the equations we just found from each other, leaving

$$-(R_2 + R_3) I_3 - \frac{R_2 R_3}{R_1} I_3 = V_2 + V_3 - (V_1 - V_3) \frac{R_2}{R_1} \quad (12)$$

$$I_3 = \frac{V_2 + V_3 - (V_1 - V_3) \frac{R_2}{R_1}}{-(R_2 + R_3) - \frac{R_2 R_3}{R_1}} = \frac{R_1 V_2 + R_1 V_3 - R_2 V_1 + R_2 V_3}{-R_2 R_1 - R_1 R_3 - R_2 R_3} = \frac{R_2 (V_1 - V_3) - R_1 (V_2 + V_3)}{R_2 R_1 + R_1 R_3 + R_2 R_3} \quad (13)$$

Plugging in $V_1 = 18 \text{ V}$, $V_2 = 15 \text{ V}$, $V_3 = 9 \text{ V}$ and $R_1 = 5 \Omega$, $R_2 = 3 \Omega$, $R_3 = 1 \Omega$, we find $I_3 = -4.04 \text{ A}$. The minus sign just means that we guessed wrong, and I_3 goes up through the right-hand branch instead of down, but the magnitude is just fine.

Once you know I_3 , you can plug that back in to one of the equations above to get I_2 and I_1 in short order.

If you're into the math, you can also see an example from last year (clickable link: http://faculty.mint.ua.edu/~pleclair/ph102/Homework/Sum09/HW4_17Jul09/HW4_circuits_17Jul09_SOLN.pdf) where I demonstrate how you can use matrices and Cramer's rule to solve the whole thing pretty quickly. Of course, these days one can just use a trusty TI-89 ...

2. The electric field of a long, straight line of charge with λ coulombs per meter is

$$E = \frac{2k_e \lambda}{r}$$

where r is the distance from the wire. Suppose we move this line of charge parallel to itself at speed v .

(a) The moving line of charge constitutes an electric current. What is the magnitude of this current?
 (b) What is the magnitude of the magnetic field produced by this current? (c) Show that the magnitude of the magnetic field is proportional to the magnitude of the electric field, and find the constant of proportionality.

Solution: The current can be found by thinking about how much charge passes through a given region of space per unit time. If we were standing next to the wire, in a time Δt , the length of wire that passes by us would be $v\Delta t$. The corresponding charge is then $\Delta q = \lambda v\Delta t$, and thus the current is

$$I = \frac{\Delta q}{\Delta t} = \frac{\lambda v\Delta t}{\Delta t} = \lambda v$$

From the current, we can easily find the magnetic field a distance r from the wire.

$$B = \frac{\mu_o I}{2\pi r} = \frac{\mu_o \lambda v}{2\pi r}$$

If the wire were sitting still (or we were traveling parallel to it at the same velocity v), it would produce the electric field given above. Rearranging the given expression, we can relate λ and E , $\lambda = Er/2k_e$. Substituting this in our expression for the magnetic field,

$$B = \frac{\mu_o \lambda v}{2\pi r} = \frac{\mu_o Erv}{4\pi k_e r} = \mu_o \epsilon_o vE = vE/c^2$$

For the last steps, we noted that $\epsilon_o = 1/4\pi k_e$ and $c^2 = 1/\mu_o \epsilon_o$.

3. A 50 kV direct-current power line consists of two conductors 2 m apart. When this line is transmitting a power of 10 MW, how strong is the magnetic field halfway between the conductors?

Solution: Since the currents are running antiparallel to each other, they will reinforce each other (as opposed to canceling each other). You should be able to verify this by using the right hand rule and drawing the field lines around each wire - at the midpoint between the wires, the field from both wires is in the same direction, perpendicular to a line connecting the two wires. Since the currents are of the same magnitude, if we can find the field from one, the total field is just twice that. In order to find the field, we need the current, which we can get from the given power and voltage:

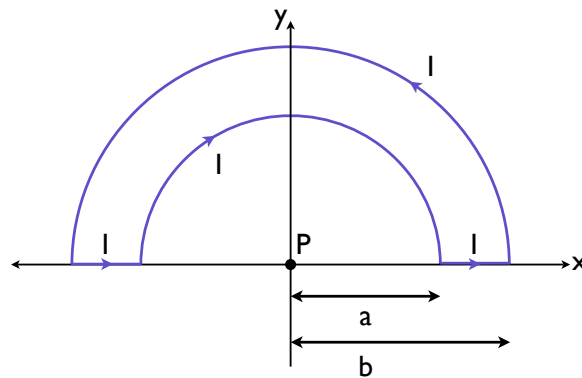
$$I = \frac{\mathcal{P}}{\Delta V} = \frac{10 \times 10^6 \text{ W}}{50 \times 10^3 \text{ V}} = 200 \text{ A}$$

We want to find the field at the midpoint between the two wires. If the distance between the wires is $d = 2.0 \text{ m}$, then the midpoint is a distance of $2.0 \text{ m}/2 = 1.0 \text{ m}$ from either wire. The field due to one current-carrying wire at a distance of $d/2 = 1.0 \text{ m}$ is found as above:

$$B_{\text{wire}} = \frac{\mu_0 I}{2\pi (d/2)} \approx 40 \mu\text{T}$$

The total field due to both wires is just twice that, or $80 \mu\text{T}$.

4. Find the magnetic field at point P due to the current distribution shown below. *Hint: Break the loop into segments, and use superposition.*



Solution: The easiest way to do solve this is by superposition – our odd current loop is just the same as two semicircles plus two small straight segments. We know that the magnetic field at the center of a *full* circular loop of radius r carrying a current I is

$$B = \frac{\mu_0 I}{2r} \quad (\text{loop radius } r)$$

Since the magnetic field obeys superposition, we could just as well say that our full circle is built out of two equivalent half circles! The field from each half circle, by symmetry, must be half of the total field, so the field at the center of a semicircle must simply be

$$B = \frac{\mu_0 I}{4r} \quad (\text{semicircle, radius } r)$$

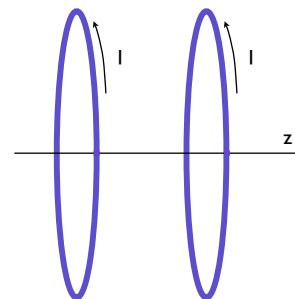
In other words: a half circle gives you half the field of a full circle. Here we have two semicircular current segments contributing to the magnetic field at P: one of radius b , and one of radius a . The currents are in the opposite directions for the two loops, so their fields are in opposing directions. Based on the axes given, it is the outer loop of radius b that has its field pointing out of the page in the \hat{z} direction, and the inner loop of radius a in the $-\hat{z}$ direction.

What about the straight bits of wire? For those segments, the direction field is zero. Since the magnetic

field “circulates” around the wire, along the wire axis it must be zero. Even if it were not, by symmetry the two straight bits would have to give equal and opposite contributions and cancel each other anyway. There is no field contribution at P from the straight segments! Thus, the total field is just that due to the semicircular bits,

$$\vec{B} = \frac{\mu_0 I}{4b} \hat{z} - \frac{\mu_0 I}{4a} \hat{z} = \frac{\mu_0 I}{4} \left(\frac{1}{b} - \frac{1}{a} \right) \hat{z}$$

5. A Helmholtz coil consists of two identical circular coils separated by a distance equal to their radius R , as shown at right. Each carries current I in the same direction. Find the field at any point along the axis between the two coils (the z axis in the figure). *Hint: The field from a single loop of radius R a distance z along the axis is:*



Helmholtz Coil

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \quad (\text{single loop})$$

Solution: We want to find the field at an *arbitrary* point z somewhere between the two coils. That being said, given that it is a bit messy, I told a good number of you that it was enough to just find the field in the middle, and that was ok.

Let $z=0$ at the intersection of the plane of the left coil and the z axis (i.e., in the middle of the left coil). The field from the left coil at an arbitrary point a distance z along the axis is just the quantity given above. At a position z , since the separation of the coils is R , we are a distance $R - z$ from the right coil. We need only replace z with $R - z$ in the expression above to find the field from the right coil at a distance $z < R$ from the bottom coil. Since the currents are in the same directions for both coils, the magnetic fields are in the same direction, and we may just add them together:

$$B_{\text{tot}} = B_{\text{left}} + B_{\text{right}} = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} + \frac{\mu_0 I}{2} \frac{R^2}{[(R - z)^2 + R^2]^{3/2}}$$

The field midway between the two coils, with $z = R/2$ is $B = 8\mu_0 I/5^{3/2}R$, about 1.4 times the field one would get directly at the center of a single coil. Not only is the field more uniform, it is larger. How convenient!

6. Microphones transduce sound waves into voltages which vary in time (with the frequency of the voltage corresponding to the sound wave frequency, or pitch) and intensity (with the magnitude of the

voltage corresponding to sound intensity). Describe how you could, using permanent magnets, wire, and other items, turn this voltage signal back into a sound wave (i.e., how can you construct a simple speaker?).

Solution: See, for example:

<http://en.wikipedia.org/wiki/Loudspeaker>

<http://electronics.howstuffworks.com/speaker5.htm>

(page 5 has the most relevant bit, but the whole article is good)