University of Alabama
Department of Physics and Astronomy

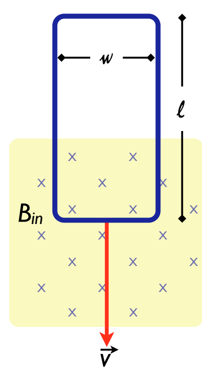
PH 102 / LeClair Summer II 2010

Problem Set 6: Solutions

1. A conducting rectangular loop of mass M, resistance R, and dimensions w by l falls from rest into a magnetic field $\vec{B}$, as shown at right. At some point before the top edge of the loop reaches the magnetic field, the loop attains a constant terminal velocity v_T. Show that the terminal velocity is:

$$v_T = \frac{MgR}{B^2w^2}$$

NB – terminal velocity is reached when the net acceleration is zero. See the schematic figure below.



First, let us analyze the situation qualitatively. As the loop falls into the region of magnetic field, more of its area is exposed to the field, which increases the total flux through the loop. This increase in magnetic flux will cause an induced potential difference around the loop, via Faraday's law, which will create a current that tries to counteract this change in magnetic flux. Since the flux is increasing, the induced current in the loop will try to act *against* the existing field to reduce the change in flux, which means the current will circulate counterclockwise to create a field out of the page.

Once there is a current flowing in the loop, each current-carrying segment will feel a magnetic force. The left and right segments of the loop will have equal and opposite forces, leading to no net effect, but the current flowing (to the right) in the bottom segment will lead to a force $F_B = BIw$ upward. Again, this is consistent with Faraday's (and Lenz's) law - any magnetic force on the loop must act in such a way to reduce the rate at which the flux changes, which in this case clearly means slowing down the loop. The upward force on the loop will serve to counteract the gravitational force, which is ultimately responsible for the flux change in this case anyway. The faster the loop falls, the larger the upward force it experiences, and at some point the magnetic force will balance the gravitational force perfectly, leading to no net acceleration, and hence constant velocity. This is the "terminal velocity." Of course, once the whole loop is inside the magnetic field, the flux is again constant, and the loop just starts to fall normally again.[i]

Quantitatively, we must first find the induced voltage around the loop, which will give us the current. The current will give us the force, which will finally give us the acceleration. As the loop falls into the magnetic field, at some instant t we will say that a length x of the loop has moved into the field, out of the total length l. At this time, the total flux through the loop is then:

$$\Phi_B = \vec{B} \cdot \vec{A} = BA = Bwx$$

From the flux, we can easily find the induced voltage from Faraday's law.

$$\Delta V = -\frac{\Delta \Phi_B}{\Delta t} = -Bw\frac{\Delta x}{\Delta t} = -Bwv$$

Here we made use of the fact that the rate at which the length of the loop exposed to the magnetic field changes is simply the instantaneous velocity, $\Delta x/\Delta t = v$. Once we have the induced voltage, given the resistance of the loop R, we know the current via Ohm's law:

$$I = \frac{\Delta V}{R} = -\frac{Bwv}{R}$$

From Lenz's law we know the current circulates counterclockwise. In the right-most segment of the loop, the current is flowing up, and the magnetic field into the page. The right-hand rule then dictates that the force on this current-carrying segment must be to the left. The left-most segment of the loop has a force equal in magnitude, since the current I, the length of wire, and the magnetic field are the same, but the force is in the opposite direction. Thus, taken together, the left and right segments of the loop contribute no net force. The bottom segment, however, experiences an upward force, since the current is to the right. For a constant magnetic field and constant current (true at least instantaneously), the force is easily found:

$$F_B = BIw$$

[i] We would still have eddy currents, which would provide some retarding force, but for thin wires eddy current forces are probably going to be negligible. This is basically what we demonstrated with our conducting pendulums swinging through a magnetic field. The pendulums that had only thin segments of conductor (it looked like a fork) experienced very little damping compared to a plain flat plate.

We can substitute our expression for I above:

$$F_B = BIw = -\frac{B^2 w^2 v}{R}$$

At the terminal velocity v_T, this upward force will exactly balance the downward gravitational force:

$$\sum F = mg - \frac{B^2 w^2 v_T}{R} = 0$$
$$\implies \quad v_T = \frac{mgR}{B^2 w^2}$$

2. An ocean current flows at a speed of $1.4\,\mathrm{m/s}$ in a region where the vertical component of the earth's magnetic field is $3.5\times10^{-5}\,\mathrm{T}$. The resistivity of seawater in that region is about $\rho = 0.25\,\Omega\,\mathrm{m}$. On the assumption that there is no other horizontal component of $\vec{E}$ other than the motional term $\vec{v}\times\vec{B}$, find the horizontal current density J in $\mathrm{A/m^2}$? *NB – recall the general version of Ohm's law, viz.* $E = \rho J$. If you carried a bottle of seawater through the earth's field at this speed, would such a current be flowing in it?

Ions in the seawater in motion experience a magnetic force qvB, which will separate positive and negative ions. This results in an electric force qE. In equilibrium, the two will balance, giving $E = vB$. Using Ohm's law, $J = E/\rho = vB/\rho$.

More formally, let the ocean current be along the $\hat{\boldsymbol{y}}$ direction and the magnetic field along the $\hat{\boldsymbol{z}}$ direction. The electric field in the frame of the moving ions is then

$$\vec{E} = \vec{v}\times\vec{B} = vB\,\hat{\boldsymbol{x}} \tag{1}$$

The current density is then given by Ohm's law:

$$\vec{J} = \frac{1}{\rho}\vec{E} = \frac{vB}{\rho}\,\hat{\boldsymbol{x}} \approx 1.96\times10^{-4}\,\mathrm{A/m^2} \tag{2}$$

3. Consider an MRI (magnetic resonance imaging) magnet that produces a magnetic field $B = 1.5\,\mathrm{T}$

at a current of $I = 140\,\mathrm{A}$. Assume the magnet is a solenoid with a radius of $0.30\,\mathrm{m}$ and a length of $2.0\,\mathrm{m}$. **(a)** What is the number of turns of the solenoid? **(b)** What is its inductance? **(c)** How much energy is stored in this magnet? **(d)** If all the energy in part (b) were converted to the kinetic energy of a car ($m = 1000\,\mathrm{kg}$), what would the speed of the car be?

We know the field B, current I, and length l. For a solenoid as such with N turns of wire, we must have:

$$B = \mu_o \left(\frac{N}{l}\right) I \tag{3}$$

$$N = \frac{Bl}{\mu_o I} \approx 17,000 \tag{4}$$

The inductance for a solenoid of cross sectional area $A = \pi r^2$ (given the radius r) can be found in your textbook:

$$L = \frac{\mu_o N^2 A}{l} = \frac{\mu_o \pi r^2 N^2}{l} \approx 51\,\mathrm{H} \tag{5}$$

The stored energy is $U = \frac{1}{2}LI^2 \approx 5 \times 10^5\,\mathrm{J}$. If this energy were converted to kinetic energy $\frac{1}{2}mv^2$ of a car with mass 1000 kg, we have $U = \frac{1}{2}mv^2 = \frac{1}{2}LI^2$, and the speed would be $v = \sqrt{2U/m} = I\sqrt{L/m} \approx 32\,\mathrm{m/s}$.

4. Very large magnetic fields can be produced using a procedure called *flux compression.* A metallic cylindrical tube of radius R is placed coaxially in a long solenoid of somewhat larger radius. The space between the tube and the solenoid is filled with a highly explosive material. When the explosive is set off, it collapses the tube to a cylinder of radius $r < R$. If the collapse happens very rapidly, induced current in the tube maintains the magnetic flux nearly constant inside the tube, even though the area shrinks. If the initial magnetic field in the solenoid is 2.50 T, and $R/r = 12.0$, what is the maximum field that can be reached?

The basic idea here is that the *flux* through the tube is the same before and after the explosion. Since after the explosion the cross-sectional area is severely reduced, the field must be much larger in order to make the flux the same. Here's a bit from the Wikipedia about flux compression, explaining things in more detail:

> Magneto-explosive generators use a technique called "magnetic flux compression", which will be described in detail later. The technique is made possible when the time scales over which the device operates are sufficiently brief that resistive current loss is negligible, and the magnetic flux on any surface surrounded by a conductor (copper wire, for example) remains constant, even though the size and shape of the surface may change.
>
> This flux conservation can be demonstrated from Maxwell's equations. The most intuitive explanation of this conservation of enclosed flux follows from the principle that any change in an electromagnetic system provokes an effect in order to oppose the change. For this reason, reducing the area of the surface enclosed by a conductor, which would reduce the magnetic flux, results in the induction of current in the electrical conductor, which tends to return the enclosed flux to its original value. In magneto-explosive generators, this phenomenon is obtained by various techniques which depend on powerful explosives. The compression process allows the chemical energy of the explosives to be (partially) transformed into the energy of an intense

magnetic field surrounded by a correspondingly large electric current.

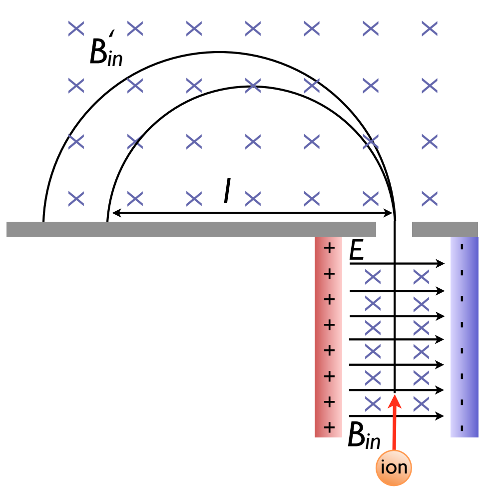
– Wikipedia, "Flux Compression"

So: all we need to do is calculate the flux before and after the explosion, set them equal to each other, and solve for the field after the explosion. Quantities with 'i' subscripts refer to before the explosion, those with 'f' after the explosion, and all symbols have their usual meanings.

$$\begin{aligned} \Phi_{B,i} &= B_i A_i = B_i \cdot \pi R^2 \\ \Phi_{B,f} &= B_f A_f = B_f \cdot \pi r^2 \\ \Phi_{B,i} &= \Phi_{B,f} \\ \Longrightarrow \quad B_i \pi R^2 &= B_f \pi r^2 \\ B_f &= \left(\frac{R}{r}\right)^2 B_i = (12.0)^2 \cdot 2.50\,\mathrm{T} = 360\,\mathrm{T} \end{aligned}$$

5. In a mass spectrometer, a beam of ions is first made to pass through a *velocity selector* with perpendicular $\vec{E}$ and $\vec{B}$ fields. Here, the electric field $\vec{E}$ is to the right, between parallel charged plates, and the magnetic field $\vec{B}$ in the same region is into the page. The selected ions are then made to enter a region of different magnetic field $\vec{B}'$, where they move in arcs of circles. The radii of these circles depend on the masses of the ions. Assume that each ion has a single charge e. Show that in terms of the given field values and the impact distance l the mass of the ion is

$$m = \frac{eBB'l}{2E}$$



Standard mass spectrometer problem from the text/notes/lecture, just be sure to use the new field B' in the region with no electric field, and note that $l = 2r$.