11
Mirrors

THE behavior of reflected light within the ray approximation follows from one simple principle – the angle of incidence is equal to the angle of reflection. Everything else we need to know about reflected light just boils down to plane geometry – so far as the physics goes, reflection is from our point of view a solved problem! Nonetheless, we can use the law of reflection along with some carefully applied geometry to derive the behavior of reflected light for a number of important and often-encountered cases.

In this chapter, we will deal with the perfect reflection of light from mirrors. Given an object and a particular sort of mirror, we will learn how to deduce what the nature of the image formed by the mirror will be. If we can first learn how to do this for a single point source of light, we can then build up any more complicated object out many point sources. Our most important example mirrors will be a simple flat mirror, a convex spherical mirror, and a concave spherical mirror. In passing, we will also investigate other technologically important geometries, such as the parabolic reflectors used in satellite dishes.

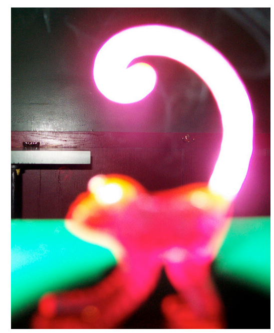


Figure 11.1: *Total internal reflection in the tail of a plastic monkey. Photo by the author.*

More broadly, by treating the problem of reflection in various specific geometries, we will begin to learn about the projection, focusing, and manipulation of light. Combined with what we will learn about refraction in lenses in the next chapter, we will be able to understand in detail a great number of optical instruments, such as microscopes, telescopes, and projectors.

11.1 Flat Mirrors

The most simple reflecting object is just a flat mirror, as shown in Fig. 11.2. What happens if we take a point source of light at position O, a distance p in front of the mirror? A point source of light is just what it sounds like – a single point from which light rays leave radially in straight lines. When the light rays exiting the source (blue) reach the surface of the mirror, we apply the law of reflection to determine where the reflected rays go (orange). Only a few of the rays leaving the source are drawn here.

11.1.1 Image formation

Some rays leaving the point source source are reflected off of the surface of the mirror, and reach an observer. The rays reflected off of the mirror in this case appear to come from a point I behind

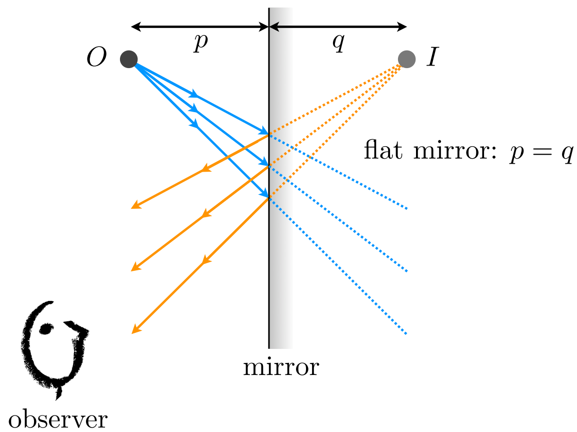


Figure 11.2: *Reflection from a flat mirror. An image is formed by light rays from an object reflecting off of the mirror's surface. The object is located at O, a distance p from the mirror, while the image location I is behind the mirror at a distance q. For a flat mirror, $p = q$. Solid lines indicate actual light rays, dotted lines indicate 'virtual' rays, whose apparent point of convergence determines where the observer sees the image.*

the mirror, if we extrapolate where these diverging rays *appear* to come from (dotted orange lines).[i] Any time we have an intersection of light rays, or a point where light rays appear to originate from, an **image** of the object which was the source of the rays is formed. From the observer's point of view, the rays reflected off of the source object at O appear to come from a point I behind the mirror, so we would say that the view sees an *image* of the object at point I, a distance q behind the mirror.

Remember, for reflection and refraction, we have to be able to run the rays forwards or backwards and get the same result. If we trace the light rays from the object to the observers eyes, this is of course the real path the rays take. Tracing the orange rays backward through the mirror to find their point of convergence tells us *where we would need a second point source to reproduce the image observed.* All real and virtual light rays fall into two categories – ones that converge onto a point (either the image or the object), and ones that diverge.

> **Image formation:**
> Images are formed where light rays converge to a point (intersect), or where they *appear* to originate from.

If the original point source is a distance p from the mirror, straightforward geometry tells us that the image distance q must be the same, $p\!=\!q$. The image observed is exactly as far behind the mirror as the object is in front of it. The image in this case is what is known as a *virtual image* – light doesn't actually pass through the point where the image is created, but only *appears* to come from that point. A *real* image is formed when light actually passes through some point. Real images can be projected onto a screen, for example, since they result from real light sources, while virtual images cannot (hence the term "virtual").

[i] Since this is not a real light ray anyway, we do not worry about refraction in the glass making up the mirror. We further assume the mirrors to be negligibly thin in any case.

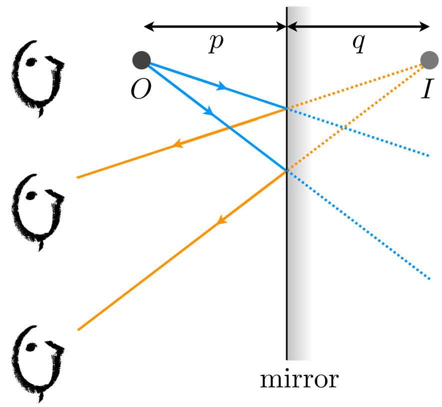


Figure 11.3: *Different observers see the same image from a flat mirror. Even though the three observers are at different positions, geometry tells us that they will all see the image formed at the same location.*

Virtual image: Light rays don't actually pass through an image point, but appear to originate from there.
Real image: Light rays actually pass through a point. Only real images can be projected onto a screen.

Our flat mirror forms a virtual image, since the image an observer sees is behind the mirror, and does not result from real light rays coming from the point of the image. The virtual image is just where the actual object *appears* to be after the mirror reflects light rays coming from it. Images from flat mirrors are *always* virtual. Can we determine anything else about the image? Is the image of the same size and shape as the object? Can we more rigorously prove our assertion that $p=q$. Sure. How do we deal with more complicated objects, as opposed to simple point sources?

11.1.2 Ray Diagrams

If we know how to handle single light rays and point sources, we can handle any more complicated object by *building it out of point sources.* We can consider any object to be made up of a series of points (or pixels, if you like), and trace the light rays from each point on the object. Usually it is not necessary to trace rays from *every* point on the object, it is enough to trace rays from a few crucial points and fill in the blanks by symmetry and common sense. As an example, consider the upright blue arrow in front of a flat mirror in Fig. 11.4. Our usual example object will be an arrow, since it is a simple shape that lets us easily determine whether images formed are inverted or magnified. As we shall see, another advantage is that all we need to do is trace out the rays from the very tip of the arrow, and the rest fills in naturally.

We place the arrow of height h at point P, a distance p from the mirror. Simple geometric techniques will let us figure out exactly what the image is like. First, we trace a ray outward from the tip of the arrow which intersects the mirror at a perfect $90°$, intersecting the mirror at point Q. This ray will just be reflected right back – if the angle of incidence is $90°$, then so must be the angle of reflection. For an observer sitting directly behind the object, this ray would *appear* to

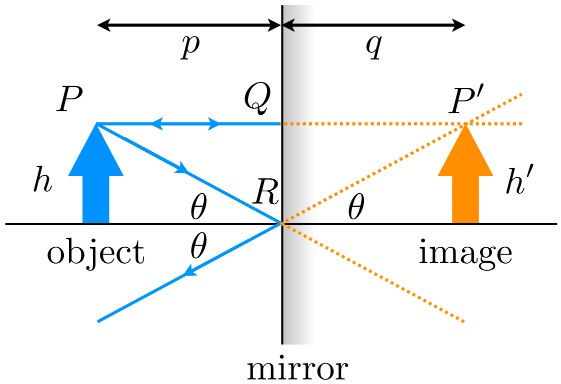


Figure 11.4: *The location and size of a reflected image from a flat mirror can be found with a simple geometric construction. Trace one ray from the object perpendicular to the mirror's surface and one ray from the object through the origin. Real light rays reflect off of the mirror, virtual light rays continue on through the mirror. The convergence of virtual rays behind the mirror gives the image location. Since triangles PQR and $P'QR$ are identical, the image and object heights are equal, $h = h'$, as are the image and object distances, $p = |q|$.*

come from behind the mirror, so we continue tracing a virtual ray (dotted orange line) behind the mirror.

Now, we need to trace at least one more ray to uniquely determine what the image looks like. We need to find an intersection of real or virtual rays in order to have an image, so we have to have at least two, and in general three is safer. For the second ray, we will trace a line from the tip of the arrow to a point on the mirror at the same vertical position as the bottom of the arrow. The use of two extremal rays gives us more confidence in the position of the resulting image – if two such extreme rays find an intersecting point, we are fairly sure we have found the image location. If we chose two rays at similar angles, small inaccuracies in our drawing become more important, and we have a harder time discerning the image position and size with any accuracy. Try tracing some ray diagrams for yourself, you will quickly find this to be true.

This second ray is reflected downward from point R on the mirror at the same angle θ at which it impinges on the mirror. Extrapolating the reflected ray back through the mirror as a virtual ray (dotted orange line), we see that it converges with the first virtual ray at point P'. This point of convergence, then, must be the location of the image. Furthermore, since we are tracing out rays from the *tip* of the arrow, this must be the tip of the *image's* arrow. Symmetry alone tells us that the image arrow must be upright, like the real one. If you are not convinced, trace out the same two types of rays from the *bottom* of the arrow, and you will see!

We have established, then, that the image is virtual, and upright (not inverted). What about its size? The virtual ray from R to P', $\overrightarrow{RP'}$ clearly must make an angle θ with the horizontal axis, since it is just a continuation of the reflected ray at point R. The lines $\overline{PQ}$ and $\overline{QP'}$ are horizontal, so the angles $\angle RPQ$ and $\angle RP'Q$ must also be θ, since they are alternate interior angles to the θ drawn in the figure. The triangles $\triangle RPQ$ and $\triangle RP'Q$ must therefore be equivalent, since they share $\overline{RQ}$ as a side. If these two triangles are equivalent, it clear that $h = h'$, and $p = q$. Now we have proved our assertion that **the image formed by an object placed in front of a flat mirror is as far behind the mirror as the object is in front of it**. We have further proved that **the image is the same size as the object.** The images formed by flat mirrors faithfully reproduce objects.

Flat Mirrors:

1. The image is as far behind the mirror as the object is in front of it.
2. The image is the same size as the object.
3. The image is upright and virtual.

11.1.3 Conventions for Ray Diagrams

For flat mirrors, we now know almost everything we need to. Other types of mirrors will not always give images that are the same size as the object, however, and will not always be the same distance away. If the image is not the same size as the object, we say that it is *magnified*. Magnified can mean either larger *or smaller*. The degree of magnification is nothing more than the ratio of the image height to the object heigh – how much larger or smaller is the image compared to the object?

Lateral Magnification of a Mirror:

$$M \equiv \frac{\text{image height}}{\text{object height}} \equiv \frac{h'}{h} \tag{11.1}$$

where h is the object height and h' the image height. For a flat mirror, $M = 1$.

For future convenience, we should also lay down some conventions for our ray diagrams. First, we will always treat the mirror as the 'zero' for our horizontal axis. Distance is positive in front of the mirror, and negative behind it. Real images are formed in front of the mirror, while virtual images are formed behind the mirror (since no light goes through the mirror). The distance from the mirror to the object will always be p, the distance to the image always q. The height of the image will be h, the height of the object h'.

Conventions for Mirror Ray Diagrams:

1. The distance between the object and the mirror is p.
2. The distance between the image and the mirror is q.
3. The object's height is h, the image's height is h'.
4. In front of the mirror, p and q are positive.
5. The front of the mirror is where real rays propagate, the back is where virtual rays are formed.
6. Behind the mirror, p and q are negative.
7. Real light rays are solid lines, virtual rays are dotted.

11.1.4 Handedness

Before we move on to different mirror geometries, one last word about mirrors and handedness. You may remember that we discussed the difference between left- and right-handed coordinate systems in Sect. 7.1.4. You already know of course that when you look in a mirror your sense of left and right are reversed. If you wave your right hand in the mirror, the image seems to wave its left. Similarly, a mirror reflection is what relates left-handed and right-handed coordinate systems, or right-handed and left-handed corkscrews. Examine Fig. 11.5, and convince yourself once again that there is an intrinsic *handedness* or *chirality* to certain things. Only a mirror reflection can change a left-handed to a right-handed coordinate system, no number of simple rotations will do it.

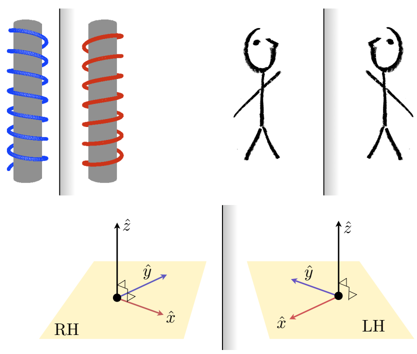


Figure 11.5: *Reflected images have reversed handedness. Clockwise, from upper left: a right-handed corkscrew becomes a left-handed one in reflection, a left hand becomes a right, and more generally a right-handed coördinate system transforms to a left-handed one.*

11.2 Spherical Mirrors

Spherical mirrors are just what they sound like: the reflective surface has the shape of an arc of a circle. Spherical mirrors can be uniquely described by the radius of the circle R making up the arc, and whether they are *concave* or *convex*. **Concave** mirrors are made by putting a reflective coating on the *inside* surface of the circle, while **convex** mirrors are made by putting a reflective coating on the *outside* surface of the circle.

11.2.1 Concave Mirrors

An example of a concave mirror is shown in Fig. 11.6a. The point C is the center of curvature of the mirror (the center of the circular arc), and is a distance R from any point on the mirror's surface. The line drawn through the center of curvature C and a point V at the center of the arc defines the **principle axis** of the mirror. How do we figure out what images look like using such a mirror? Just like before, we trace light rays and apply the law of reflection and geometry.

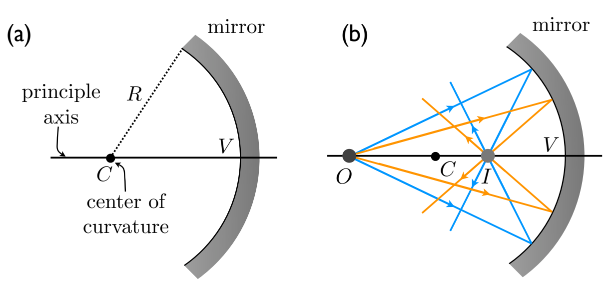


Figure 11.6: **(a)** *Reflection from a concave spherical mirror. The center of curvature C is the center of the spherical arc of radius R making up the mirror. The principle axis passes through the center of curvature as well as the middle of the mirror, V.* **(b)** *If we place an object O anywhere on the principle axis farther away from the mirror than C, a real image is formed at I. If the distance from O to the mirror is relatively large compared to R (such that the rays come off of the principle axis at small angles), all rays reflect through the same point.*

Figure 11.6b shows a point source O placed relatively far from a spherical mirror, outside the center of curvature. Rays leaving point O with a sufficiently small angle intersect the mirror, and are all reflected back through a common convergence point I. The point I is the *image point*, and the convergence of rays indicates that an image will form there, as though there were a copy of the source at that point. Since real light rays are passing through the point I, the image formed is *real.*

For spherical mirrors in particular, we will usually assume that the light rays from the source make a small angle with the principle axis. When this condition is met, all incident rays will reflect back through the image point. On the other hand, when some rays reaching the mirror make a relatively large angle with the principle axis – when the object is relatively close to the spherical mirror – this is no longer true, as shown in Fig. 11.7. When the object is too close to the mirror, some of the rays making a large angle with the principle axis no longer reflect back through the image point, and no single point of convergence exists. This means that the image formed is not clearly focused on one point, but spread out – the image is blurry. This phenomena is known as *spherical aberration.* It is quite important for, *e.g.,* telescopes and cameras – since spherical shapes the easiest to produce, most lenses have spherical shapes and will suffer from this phenomena, as we will see in more detail in the following chapter.

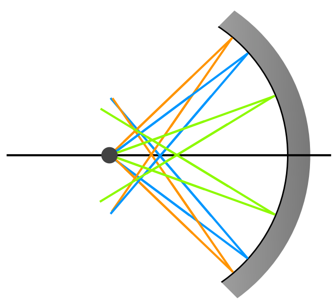


Figure 11.7: *Rays at large angles from the principle axis do not all reflect back to intersect the principle axis at the same point. As a result, when objects are too close to a spherical mirror, the image formed is "fuzzy" since the convergence of rays is now spread out. This effect is known as spherical aberration.*

If we ensure that the object is sufficiently far from the mirror to avoid spherical aberration, what will the image look like? Just like with flat mirrors, we will trace the rays coming from the tip of an arrow placed in front of the mirror, as shown in Fig. 11.8. Again the arrow of height h is placed a distance p from the mirror, at point O. The center of curvature for the mirror is C, and the center of the mirror is at V.

First, we trace a ray from the tip of the arrow through the center of curvature at C. Since the mirror is the arc of a circle, any line passing through the center of curvature must be normal to the surface of the arc – that is, it must intersect the surface of the arc at a 90° angle. Therefore, the ray drawn through the center of curvature reflects back along the same path. We will call the angle this ray makes with the principle axis α.

Next, we draw a second ray from the tip of the arrow through the center of the mirror at V. This ray makes an angle θ with the principle axis, and will reflect off the mirror at V with the same angle. This ray intersects the first at the point I, and defines the tip of the image arrow. Since the intersection point lies below the principle axis, *the image is inverted.* Further, we can already see

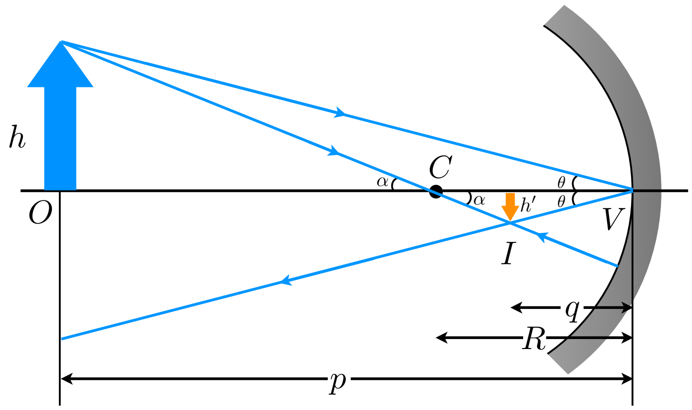


Figure 11.8: *The image formed by a spherical concave mirror for objects placed outside of the center of curvature C. The image is real, magnified, and inverted.*

that it is not the same size as the original arrow, so the image is also *magnified.* Finally, it is real light rays that are intersecting in front of the mirror, so the image formed is real.

Concave spherical mirrors:
Images are *real*, *inverted*, and *magnified*.

Still, it would be nice to know *exactly* how big the image is, and where it is. This much we can figure out with a bit of geometry. First, we can use the two θ angles and relate the object height h and the image height h'. From the triangle formed by the object arrow and the uppermost ray:

$$\tan\theta = \frac{h}{p} \tag{11.2}$$

Similarly, from the triangle formed by the reflection of that ray and the image arrow:

$$\tan\theta = \frac{-h'}{q} \tag{11.3}$$

Note that since the image arrow points downward below the principle axis, the height of the image is negative. Some simple algebra yields the magnification of the mirror:

$$\tan\theta = \frac{h}{p} = \frac{h'}{q} \tag{11.4}$$

$$\implies M = \frac{h'}{h} = -\frac{q}{p} \tag{11.5}$$

Magnification for a concave spherical mirror:

$$M = \frac{h'}{h} = -\frac{q}{p} \tag{11.6}$$

Here h is the height of the object, h' is the height of the image, p is the object distance, q is the image distance. Negative M means the image is *inverted*.

Assuming we know h and p to begin with, we still need one more equation in order to uniquely determine h' and q, the height and position of the image. For that, we can use the α angles. From the triangle defined by the left-most α and the object,

$$\tan\alpha = \frac{h}{p-R} \tag{11.7}$$

Using the triangle defined by the right-most α and the image,

$$\tan\alpha = -\frac{h'}{R-q} \tag{11.8}$$

We can now use the above equations for $\tan\alpha$ along with Eq. 11.6 to find another useful equation relating p and q alone:

$$\begin{aligned}
\tan\alpha = \frac{h}{p-R} &= -\frac{h'}{R-q} \\
\frac{h'}{h} &= -\frac{R-q}{p-R} = -\frac{q}{p} \qquad \text{(using Eq. 11.6)} \\
p(R-q) &= q(p-R) \\
pR - pq &= qp - qR \\
pR + qR &= 2qp \\
R(p+q) &= 2qp \\
\frac{R}{2} &= \frac{qp}{p+q} = \frac{1}{\frac{1}{q}+\frac{1}{p}} \\
\frac{2}{R} &= \frac{1}{p} + \frac{1}{q}
\end{aligned}$$

This last expression is known as the *mirror equation*, relates the image and object distances to the physical radius of curvature of the mirror alone. As we shall find out shortly, this equation is far more general than our simple derivation of it would imply. Coupled with the expression for magnification, we can now deduce the behavior of any object with any concave spherical mirror . . . so long as the object isn't too close to the mirror.

Mirror equation:

$$\frac{2}{R} = \frac{1}{p} + \frac{1}{q} \tag{11.9}$$

where p is the object distance, q is the image distance, and R is the radius of curvature of the mirror.

11.2.1.1 Concave spherical mirrors and distant objects

We have already seen that forming sharp images from a concave spherical mirror requires the object to be relatively far from the mirror (at least outside the radius of curvature). What happens if the object is *really, really* far away? Say, far enough compared to R that p is essentially infinite? When the object is very, very far away, the incident rays are all very nearly parallel to the principle axis. For very distant sources, any small angle away from the principle axis will result in the rays diverging too far to hit the mirror, only those rays at tiny angles relative to the principle axis will hit the mirror. For all intents and purposes, we can assume all rays from a very distant object impinge on the mirror parallel to the principle axis, as shown in Fig. 11.9.

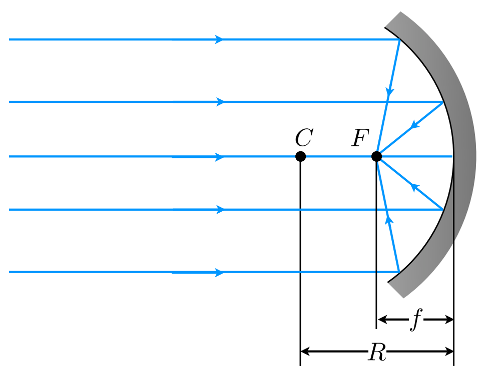


Figure 11.9: *For very distant objects ($p \leftarrow \infty$), incident light rays are essentially parallel, and all reflect through the focal point of the mirror F. For very distant objects, the image distance is $q \approx f \approx R/2$, where f is the focal length of the mirror (the position of F).*

The mirror equation gives us yet more insight. If we let p tend toward infinity, then $1/p$ tends toward zero. In this case, $q \approx R/2$ – the image is formed exactly half way between the center of curvature and the mirror when the object is very far away compared to R. In this special case of a distant object, all the incident rays converge at the same point F (Fig. 11.9), which we call the *focal point* of the mirror. The **focal length** f of a mirror is just the distance between the mirror and the focal point on the principle axis where light from a distant object would converge. Put another way, it is the image distance q when we allow p to tend toward infinity. Thus, for our concave spherical mirror, $f = \frac{R}{2}$

Though the focal length and radius of curvature are simply related, it is the former that you will hear more often in optics. The focal length of a mirror is where light would focus if we had a point source infinitely far away, and is one way of comparing the properties of different mirrors (or lenses, as we shall see). Even though we can't actually realize this situation, we can get far enough away from a mirror to approximate it, and in fact, this is the regime in which we try to operate most optical instruments. If you have any experience with photography, you are no doubt already familiar with focal lengths. In any case: the focal length is a characteristic of a spherical mirror, just half its radius of curvature, and it allows us to re-write the mirror equation in an ostensibly more useful way:

Mirror equation in terms of the focal length:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \tag{11.10}$$

where p is the image distance, q is the object distance, and f is the focal length. For a concave spherical mirror, the focal length is half the radius of curvature, $2f = R$.

The fact that spherical mirrors focus all distant light onto a single point makes them potentially useful for, *e.g.*, solar heating or focusing antennas. As we shall see in subsequent sections, however, there is a still more clever geometry which is much better for light harvesting applications.

11.2.2 Convex Spherical Mirrors

A convex spherical mirror is shown in Fig. 11.10, in which the *outer* surface of the spherical arc has a reflective coating. While a concave spherical mirror tends to focus distant light on to a single point, a convex spherical mirror tends to *diverge* incident rays. Nearly all incident rays on the surface of the convex spherical mirror diverge after reflection, as if they are coming from *behind* the mirror itself. Analyzing the image formed by this type of mirror is not much more difficult than the other cases we have dealt with, we just have to construct a ray diagram.

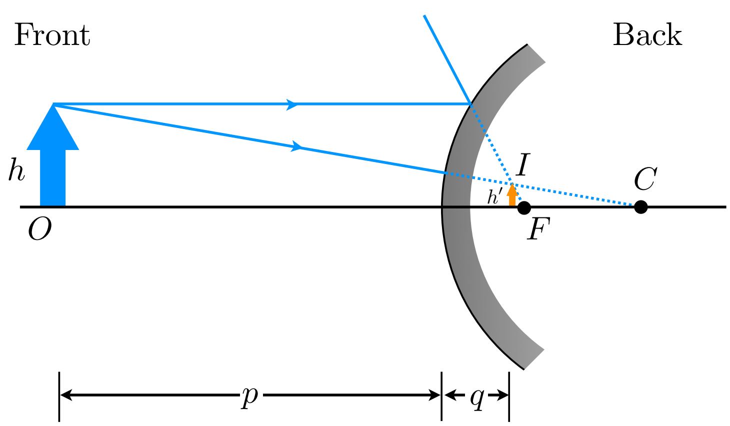


Figure 11.10: *The image formed by a spherical convex mirror is virtual, magnified, and upright.*

For the moment, two rays are enough to grasp the nature of image formation for a convex mirror. First, we draw a ray horizontally from the tip of our object arrow in Fig. 11.10. This ray is reflected upward away from the object and mirror. If we trace the reflected ray backward through the mirror, it intersects the principle axis exactly at the focal point of the mirror. Next, we draw a ray from the tip of the arrow through the center of curvature of the mirror. In front of the mirror, it is a real ray, while in back of the mirror it is a virtual ray. The intersection of our two virtual rays behind the mirror gives the image location.

Table 11.1: *Sign Conventions for Mirrors*

Quantity	Symbol	Front	Back	Upright	Inverted
Object location	p	$+$	$-$		
Image location	q	$+$	$-$		
Focal length	f	$+$	$-$		
Object height	h'			$+$	$-$
Image height	h'			$+$	$-$
Magnification	M			$+$	$-$

In this case, we can see that the image is *upright*, *virtual*, and *magnified*. What is the actual image position and magnification factor? As it turns out, if we work through the geometry, **the same mirror equation is valid for convex spherical mirrors, if we keep in mind that p and q are *negative* when we are behind the mirror.** In this particular case for convex spherical mirrors, h and h' are positive, p is positive, and q is negative. Table 11.1 is a reminder of the sign conventions we use for mirrors. Parenthetically, we note that the mirror equation also works for flat mirrors! The radius of curvature of a flat plane is infinite, and applying this to Eq. 11.9 readily gives $p\!=\!q$.

11.3 Ray Diagrams for Mirrors

So far, we have constructed *ad hoc* ray diagrams for the different mirrors under consideration. The ray diagrams are nothing more than graphical constructions to give us an overall impression of the image formed. We tried to choose rays that gave extremal cases, in the hopes that this would give a more accurate image. In fact, we can come up with a set of general rules for constructing a ray diagram for any simple mirror, so long as we know the object location and the mirror's center of curvature. In the end, we need only three rays. So far we have used only two, and that has worked fine. In some sense the third ray is a 'sanity check.' With only two rays, it is almost certain that we will have an intersection *somewhere*, even if make some small mistakes in our ray tracing. The odds of a third ray spuriously intersecting the other two at the same point is *tiny*, so if all three rays intersect at the same point, we can be sure that our diagram is reasonably correct.

How to construct ray diagrams:
Ray 1 is drawn parallel to the principle axis, and reflects back through the focal point.
Ray 2 is drawn through the focal point, and reflects back parallel to the principle axis.
Ray 3 is drawn through the center of curvature, and reflects back on itself.

In using these rules and analyzing different situations for spherical mirrors, we can make the some generalizations to serve as rules-of-thumb:

Images from Spherical Mirrors:

1. Concave Mirrors (Fig. 11.11):
   (a) $p > R$: object *outside* center of curvature,
   gives a real, inverted, and reduced image
   (b) $R > p > f$: object *outside* focal point and *inside* center of curvature,
   gives a real, inverted, enlarged image
   (c) $p < f$: object inside focal length
   gives virtual, upright image
2. Convex Mirrors:
   (a) image is always virtual and upright

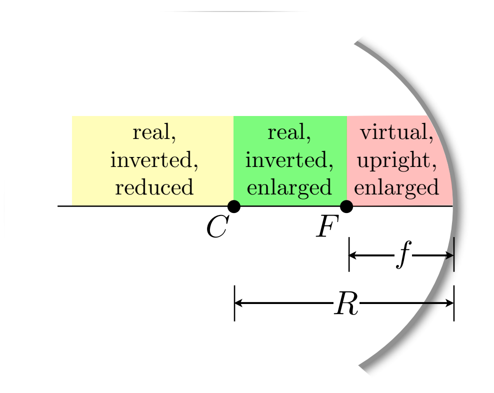


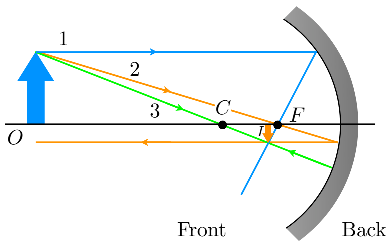
Figure 11.11: *The type of image formed by a spherical mirror depends on the location of the object relative to the center of curvature and the focus of the mirror. For objects outside the center of curvature, the image is real, inverted, and reduced. For objects between the center of curvature and focus, the image is real, inverted, and enlarged. For objects inside the focus, the images are virtual, upright, and enlarged.*

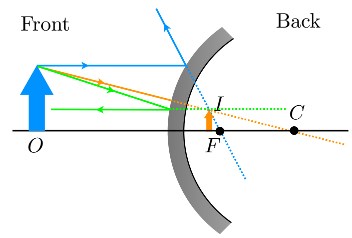
Figure 11.12 shows these three rules applied to concave and convex spherical mirrors. The first rule just follows from our discussion of discussion of very distant rays incident on a spherical mirror – the *definition* of the focal point is the point at which rays parallel to the principle axis reflect through (virtual rays in the case of convex mirrors). The second rule follows in the same way. The third rule is essentially the definition of the radius of curvature – any line passing through the radius of curvature is incident normal on the surface of the mirror, and must reflect back on itself.

11.4 Parabolic Mirrors

Circular mirrors are just fine, but isn't there something more efficient? Is there a shape of mirror we could make such that *all* distant rays are focused onto a single point, not just those close to the central axis? Indeed, there is just such a curve, and you are already familiar with it: the parabola. In fact, the parabola is unique in this regard. It is the only curve such that all incident parallel rays will be reflected and focused on to a single point, the *focus* of the parabola.

This is illustrated in Fig. 11.13. If a series of parallel rays is incident downard on the parabola, they will all converge at the focus F. Equivalently, since we can always run our ray diagrams





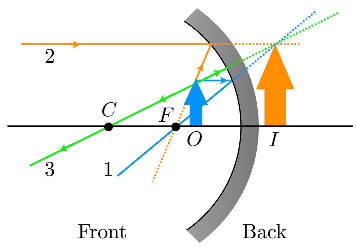


Figure 11.12: *Ray diagrams for spherical mirrors.* **top left:** *An object outside the center of curvature for a concave spherical mirror. The image formed is real, inverted, and smaller.* **bottom left:** *An object inside the center of curvature of a concave spherical mirror gives a virtual image which is upright and larger than the object.* **above:** *A convex spherical mirror always gives an image which is upright, virtual, and smaller than the object.*

'forward' or 'backward,' a point source of light placed at F will produce a parallel beam of light. Incidentally, this works in three dimensions too. A circular paraboloid, made by rotating a parabola about its axis, is the only 3D surface for which all rays parallel to a given ray pass through the same point after reflection by the surface. What good is this property? Well, this is how modern car headlights use a single bulb to produce a beam of light, and it is how satellite antennas ('dishes') manage to focus an extremely tiny amount of radiation into a usable signal. Make the parabola as large as possible, collecting radiation from as large an area as possible, and it all gets focused to a single point, enormously amplifying the intensity. The same principle is used for radio astronomy and solar ovens.

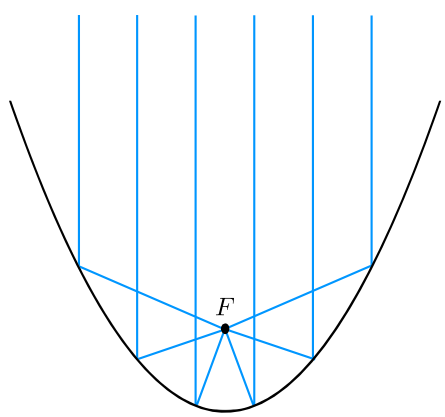


Figure 11.13: *Focusing of light by a parabolic mirror. A distant light source providing incident rays which are parallel will be reflected by the parabola and focused onto a single point F. Conversely, a point source located at the focus F will produce a beam of parallel rays. Parabolic mirrors offer some advantages over spherical mirrors for focusing – the parallel rays can come it at an angle to the parabola and still be focused, and spherical aberrations can be significantly reduced.*

How does this work? Geometrically, a parabola is a conic section defined as the locus of points equidistant from a single point (the focus) and a straight line (the directrix). This is shown in Fig. 11.14. Without loss of generality, we will take the parabola centered on the origin of an $x-y$ coordinate system. Let the focus F be at the point $(0, f)$, and the directrix be the line $y=-f$.

This is still perfectly general - an arbitrary point and line, since we can make f whatever we want. Our parabola is 'between' the focus and directrix.

Construct a line connecting F with an arbitrary point $P(x_0, y_0)$ on the parabola, and a vertical line intersecting the directrix at point $D(x_0, -f)$. A parabola is, as stated above, geometrically defined as the locus of all points for which $\overline{FP} = \overline{PD}$. If we didn't already know that, could we figure out what curve satisfies this relationship? We can, simply calculate the lengths $\overline{FP}$ and $\overline{PD}$ with the distance formula:

$$\begin{aligned}
\overline{FP} &= \overline{PD} \\
\sqrt{(x_0 - 0)^2 + (y_0 - f)^2} &= \sqrt{(x_0 - x_0)^2 + (y_0 + f)^2} \\
x_0^2 + y_0^2 - 2fy_0 + f^2 &= y_0^2 + 2fy_0 + f^2 \\
x_0^2 &= 4fy_0 \\
y_0 &= \frac{1}{4f}x_0^2
\end{aligned}$$

Lo and behold, the curve is a parabola. One can easily repeat this calculation for a parabola centered on an arbitrary point, the same conclusion holds: a parabola is the only curve for which all points are equidistant from a single line and a single point. For a parabola centered on (x_0, y_0) symmetric about the y axis (*i.e.*, pointing upward or downward), one finds $(y - y_0) = \frac{1}{4f}(x - x_0)^2$.

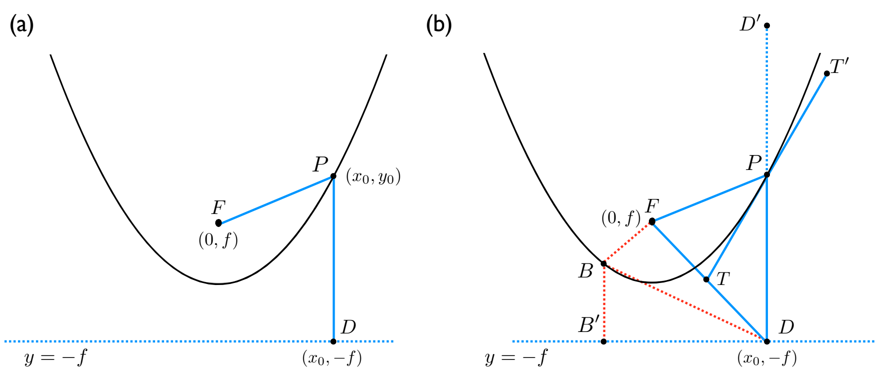


Figure 11.14: *left: Construction of a parabola. A parabola is the locus of points equidistant from the focus $F(0, f)$ and the directrix line $y = -f$. right: Any ray directed along the parabola's axis of symmetry is reflected and passes through the focus.*

So what? Now we can sketch a proof of the unique focal property of the parabola as well, using the second portion of Fig. 11.14. If we can prove that a tangent line to the parabola at point P will make equal angles with $\overline{PF}$ and $\overline{PD}$, this is enough to prove the focal property. First, we must figure out how to construct a tangent to the parabola at any point.[ii]

[ii]Many of you probably realize how much easier this task would be with a bit of calculus - in fact, it is a trivial problem if we use calculus. The geometric problem is not trivial, but worth working through if for no other reason to emphasize the fact that parabolas are simple *geometric* constructions, not just abstract quadratic equations. In our studies of optics, good geometrical insight will serve you well.

By definition, triangle $\triangle FPD$ is isosceles - for a parabola, $\overline{PF}$ and $\overline{PD}$ are equal. Let point T be the midpoint of the line connecting F and D, $\overline{FD}$. Now the triangles $\triangle FPT$ and $\triangle TPD$ have two equal sides, since $\overline{FP} = \overline{PD}$ and by construction $\overline{FT} = \overline{TD}$. The perpendicular bisector $\overline{FD}$ divides the $x - y$ pane into two sections: all points which are nearer to F than to D, and all points that are nearer to D than to F. Except for point P, every point on the parabola itself lies closer to F than to D by virtue of being above the line $\overline{PT}$.

Let B be any other point on the parabola, and B' the point nearest to it lying on the directrix. The line segment $\overline{BB}'$ is the shortest possible segment connecting the point B on the parabola to the directrix. The segment $\overline{BB}'$ must be vertical and perpendicular to the directrix for this to be true. By construction, then, $\overline{BB}' = \overline{FB} < \overline{BD}$ - a vertical line segment from B to the directrix must be the same length as the line segment from B to F. Since $\overline{BB}'$ is the shortest distance from B to the directrix, it must be shorter than $\overline{BD}$. If this is true, then $\overline{PT}$ can not pass through B, or it would be closer to the directrix than the focus, a contradiction. Thus P is the only point of intersection of the line $\overline{PT}$ and the parabola. Thus, $\overline{PT}$ must be tangent to the parabola at point P.

Whew! Now, if $\overline{PT}$ is tangent to the parabola at P, the angles $\angle FPT$ and $\angle TPD$ must be equal. Further, $\angle TPD$ is equal to angle $\angle D'PT'$. If we imagine $\overline{D'P}$ to be a light ray incident on a parabolic surface reflected toward F, this establishes that the incident and reflected angles are equal. Since the point P was completely arbitrary, this means that *any* incident vertical ray must be reflected through the focus F, and that any light originating at F will be reflected as a vertical ray.

Other conic sections have reflective properties similar to the parabola. For instance, if a light source is placed at one focus of an ellipse, the rays will converge onto the other focus after being reflected. Any wave, including sound waves, may be substituted for light. A nice trick is to make an elliptically-shaped room, known as a 'whispering gallery.' If a sound is created at one focus - even a very quiet one - it will be heard clearly at the second focus. It is a dramatic demonstration. You can stand at one focus and whisper so quietly someone standing next to you cannot hear, and yet be clearly heard at the other focus. Some famous examples of rooms like this are listed in the Wikipedia: `http://en.wikipedia.org/wiki/Whispering_gallery`.

11.5 Quick Questions

1. A concave makeup mirror has a focal length of 15 cm. If an object is placed 25 cm in front of the mirror, determine the signs of the focal length, object distance, and image distance.

- □ $+, -, +$
- □ $+, -, -$
- □ $+, +, -$
- □ $+, +, +$

2. An inverted image of an object is viewed on a screen from the side facing a converging lens. An opaque card is then introduced covering *only the upper half* of the lens. What happens to the image on the screen?

- □ Half the image would disappear.
- □ Half the image would disappear and be dimmer.
- □ The entire image would appear and remain unchanged.
- □ The entire image would appear, but would be dimmer.

3. A concave makeup mirror is designed so that a person 26 cm in front of it sees an upright image magnified by a factor of two. What is the radius of curvature of the mirror?

- □ 1.04 m
- □ 3.78 m
- □ 0.52 m
- □ 2.08 m

11.6 Problems

1. While looking at her image in a cosmetic mirror, Dina notes that her face is highly magnified when she is close to the mirror, but as she backs away from the mirror, her image first becomes blurry, then disappears when she is about 38.0 cm from the mirror, and then inverts when she is beyond 38.0 cm.

(a) What type of mirror does Dina have?
(b) What is the focal length of the mirror?
(c) What is the radius of curvature of the mirror?

11.7 Solutions to Quick Questions

1. +, +, +.

2. The entire image would appear, but would be dimmer.

3. 1.04 **m.**

11.8 Solutions to Problems

1. The fact that the image chagnes from upright to inverted immediately tells us that Dina has a **concave** spherical mirror. A convex mirror always gives an upright image, as does a flat mirror. The point at which the image (briefly) disappears and inverts is the focal length, so $f = 38\,\text{cm}$. For spherical mirrors, we know that $f = 2R$, the radius of curvature is just twice as big: $R = 76\,\text{cm}$.

Figure 11.11 may jog your memory a bit. Right at the focal point, when the image goes from upright and enlarged to inverted and enlarged, the image disappears.