Lenses

12

LENSES

12.1 Spherical refracting surfaces

In order to start discussing lenses *quantitatively*, it is useful to consider a simple spherical surface, as shown in Fig. 12.1. Our 'lens' is a semi-infinte rod with one spherical surface, made of a material of refractive index n_2 greater than the surrounding material ($n_1 < n_2$). Qualitatively, we know what will happen based on the law of refraction. Rays emanating from a distant object placed at O will impinge on the spherical surface, bend toward the principle axis (toward the surface normal), and converge at a point inside the rod, forming a real image. But where?

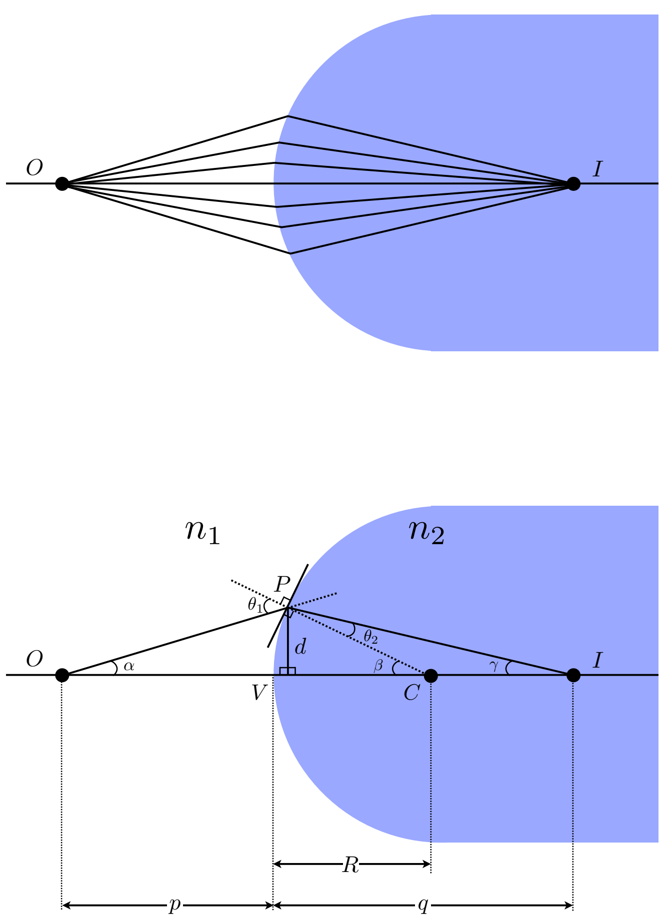

Figure 12.1: *A spherical refracting surface.* **upper:** *Rays incident from a distant object O are refracted toward the principle axis, and focused at a point I.* **lower:** *Construction for determining the relative image and object distances in terms of the radius of curvature and refractive indices.*

With a bit of geometry, we can figure out exactly where the image must form, given the object distance and the radius of the spherical surface. Referring to the second portion of Fig. 12.1, let the object (O) and image (I) distances be p and q, respectively, measured from the intersection of the principle axis with the spherical surface (V). The center of the sphere of radius R making up the surface is at C. Trivially, a ray drawn from O through the principle axis pass through V, C, and I.

Now, draw a ray leaving the object and intersecting the surface at point P, $\overrightarrow{OP}$. At the point P, we draw surface normal and tangent lines to define the angle of incidence θ_1 and the angle of refraction θ_2. The refracted ray will be bent toward the principle axis, intersecting it at point I. This ray $\overrightarrow{PI}$ makes an angle α with the principle axis. Recall that any line perpendicular to the surface of a circle must pass through the center of the circle. Thus, if we extend the normal drawn at point P, it must intersect point C, forming ray $\overrightarrow{PC}$, which makes an angle β with the principle axis. Now we have everything labeled that we need, "all" that is left is to find a relationship between p, q, and R.

First, we can use right triangle $\triangle OPC$. The angles $\angle OPC$, α, and β making up this triangle must add up to 180°. We also know that that the angles θ_1 and $\angle OPC$ by themselves define a straight line, and must therefore add up to 180° as well. Thus:

$$\alpha + \beta + \angle OPC = 180^\circ \tag{12.1}$$

$$\theta_1 + \angle OPC = 180^\circ \tag{12.2}$$

$$\implies \quad \theta_1 = \alpha + \beta \tag{12.3}$$

Slowly, we are reducing the number of unknown quantities. Now examine the triangle $\triangle PCI$. We know that the angles θ_2, γ, and $\angle PCI$ must add up to 180°. Further, we know that β and $\angle PCI$ must together make 180°, since they define the line $\overline{OI}$. Putting these facts together:

$$\theta_2 + \gamma + \angle PCI = 180 \tag{12.4}$$

$$\beta + \angle PCI = 170 \tag{12.5}$$

$$\implies \quad \theta_2 = \beta - \gamma \tag{12.6}$$

Equations 12.3 and 12.6 give us the angles of incidence (θ_1) and refraction (θ_2) in terms of the interior angles α, β, and γ which can be more easily related to the distances of interest, *viz.*, p, q, and R. Before we can do that, we have one trick up our sleeve: we haven't yet used Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{12.7}$$

If we substitute equations 12.3 and 12.6 into this expression, we have:

$$n_1 \sin(\alpha + \beta) = n_2 \sin(\beta - \gamma) \tag{12.8}$$

We can apply the sum and difference identities for $\sin(a \pm b)$ to this, which yields the following:

$$\begin{aligned}
n_1 \sin\alpha\cos\beta + n_1\cos\alpha\sin\beta &= n_2 \sin\beta\cos\gamma - n_2\cos\beta\sin\gamma && (12.9)\\
n_1\cos\alpha\left(\tan\alpha\cos\beta + \sin\beta\right) &= n_2\cos\gamma\left(\sin\beta - \cos\beta\tan\gamma\right)\\
n_1\cos\alpha\sin\beta\left(\frac{\tan\alpha}{\tan\beta}+1\right) &= n_2\cos\gamma\sin\beta\left(1-\frac{\tan\gamma}{\tan\beta}\right)\\
n_1\cos\alpha\cancel{\sin\beta}\left(\frac{\tan\alpha}{\tan\beta}+1\right) &= n_2\cos\gamma\cancel{\sin\beta}\left(1-\frac{\tan\gamma}{\tan\beta}\right) \qquad (\beta\neq 0)\\
n_1\cos\alpha\left(\frac{\tan\alpha}{\tan\beta}+1\right) &= n_2\cos\gamma\left(1-\frac{\tan\gamma}{\tan\beta}\right) \qquad (\beta\neq 0) && (12.10)
\end{aligned}$$

For the last line, we must take care that $\beta \neq 0$, otherwise canceling the $\sin\beta$ terms would be division by zero - strictly not allowed. This is not a problem - β is only zero for the trivial case of the ray traveling on the principle axis, which we already know how to deal with. In order to proceed further, we need to make a crucial approximation. Namely, we assume that the object is very distant relative to the radius of the spherical surface, $p \gg R$, and we only consider rays incident near the principle axis, $d \ll R$. If this is true, then the tangents of α, β, and γ can be nicely approximated:

$$\begin{aligned}
\tan\alpha &\approx \frac{d}{\overline{OV}} = \frac{d}{p}\\
\tan\beta &\approx \frac{d}{\overline{VC}} = \frac{d}{R}\\
\tan\gamma &\approx \frac{d}{q}
\end{aligned}$$

Basically, we have just decided to ignore the tiny distance between point V and the intersection of $\overline{PV}$ with the principle axis. Qualitatively, these approximations seem reasonable. It would be equivalent to say that we only consider large p and small α - the same approximations result – if α is small, so too are β and γ. Using these approximations, Eq. 12.10 reduces to:

$$n_1 \cos\alpha \left(1 + \frac{d/p}{d/R}\right) = n_2 \cos\gamma \left(1 - \frac{d/q}{d/R}\right)$$
$$n_1 \cos\alpha \left(1 + \frac{R}{p}\right) = n_2 \cos\gamma \left(1 - \frac{R}{q}\right) \tag{12.11}$$

Now, given that the angles α and β are supposed to be tiny and the object distance large, we know that $p \gg d$ and $q \gg d$. Thus, the ratios d/p and d/q will be very small compared to 1. We can use this fact to simplify things even further. Using the same logic behind the tangent approximations, we find $\cos\alpha \approx 1$, and $\cos\gamma \approx 1$

$$\cos\alpha \approx \frac{p}{\sqrt{d^2+p^2}} = \frac{p}{p\sqrt{1+d^2/p^2}} = \frac{1}{\sqrt{1+d^2/p^2}} \approx 1$$
$$\cos\gamma \approx \frac{q}{\sqrt{d^2+q^2}} = \frac{q}{q\sqrt{1+d^2/q^2}} = \frac{1}{\sqrt{1+d^2/q^2}} \approx 1$$

Thus, so long as d/p and d/q are very small (and their squares are even smaller), we can simply ignore the cosine terms, which leaves us:

$$n_1 \left(1 + \frac{R}{p}\right) = n_2 \left(1 - \frac{R}{q}\right) \tag{12.12}$$
$$n_1 \frac{R}{p} + n_2 \frac{R}{q} = n_2 - n_1 \tag{12.13}$$
$$\implies \quad \frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \tag{12.14}$$

This is the result we desire: the image and object distances are simply related by the radius of curvature of the spherical surface, and the indices of refraction of the lens material and its surrounding.

Spherical refracting surfaces:

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \tag{12.15}$$

Here q is the image distance inside the dense material n_2, and p is the object distance in the less dense material n_1 ($n_1 < n_2$). The results holds for rays not far from the principle axis.

12.1.1 Flat Refracting Surfaces

If we let R tend toward infinity, $R \to \infty$, our spherical surface becomes a flat one.[i] If R tends toward infinity, then $1/R$ tends toward zero, and our spherical lens equation reduces to:

Flat refracting surfaces:

$$q = -\frac{n_2}{n_1} p \tag{12.16}$$

Here q is the image distance inside the dense material n_2, and p is the object distance in the less dense material n_1 ($n_1 < n_2$).

This derives a result with important everyday consequences: since $n_2 \neq n_1$, then $p \neq q$. This is why, when looking into a pool of water, objects are actually much farther below the surface than we think they are.

12.2 Spherical Lenses

Armed with a knowledge of spherical refracting surfaces, we can move on to *spherical lenses*. All of the lenses we will consider can be defined only by the surfaces of spheres, hence the name. Figure 12.2 shows how one can construct either biconvex (upper) or biconcave (lower) spherical lenses, defined by the intersection and region between two spheres, respectively.

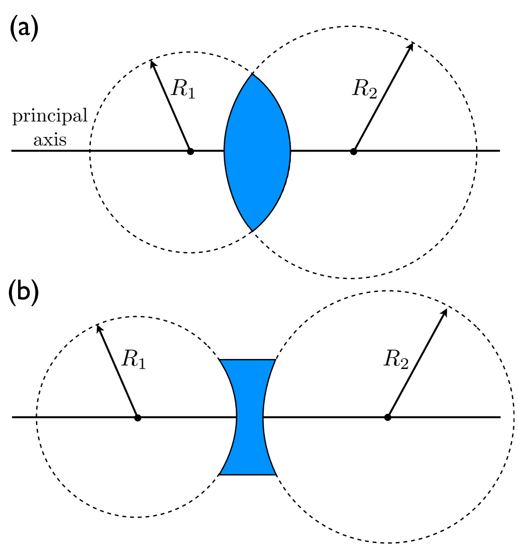

Figure 12.2: *Spherical lenses can also be either concave or convex, and their surfaces are defined by the surfaces of two spheres.* **(a)** *Biconvex lenses are formed by the intersection of two spheres, and* **(b)** *biconcave lenses are formed by the region between two spheres. When $R_1 = R_2$, the lens is spherically symmetric.*

How can we analyze a lens like this? A lens can be considered the combination of two spherical interfaces, so all we need to do is use our solution to the case of the spherical refracting surface and

[i] One can say that the radius of curvature of a flat plane is infinite, or equivalently, that a plane is just the surface of a sphere with infinite radius.

apply it twice. First, we find the image due to (for instance) the left-hand spherical surface, and the image formed by that surface serves as the object for the right-hand spherical surface. This is shown in Fig. 12.3, where we consider a lens of thickness d formed by overlapping spheres of radii R_1 and R_2, both of which are made of a material of refractive index n_2. Surrounding the model lens is a material of refractive index n_1.

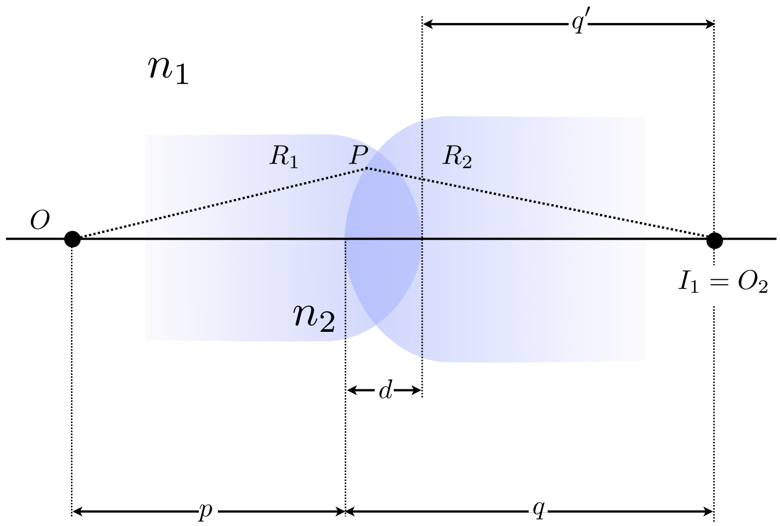

Figure 12.3: *Our model spherical lens is built out of two separate spherical refracting surfaces.*

First, consider only the object on the right-hand side by itself. Light from point O a distance p from the spherical surface, reaches the spherical interface at point P. Since we are only worrying in the end about the region where the two spherical surfaces overlap, we presume that the light is not refracted on the way from O to P. After refraction, the ray is refracted toward point I_1 on the principle axis. Since this is just refraction from a spherical surface as we solved above, we know

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R_1} \tag{12.17}$$

This forms an image at point I_1. This image now serves as an *object* for the second spherical surface - $I_1 = O_2$. Now ignore the right-hand side and consider only the left-hand side. Light from the image formed at O_2 will be incident on the spherical surface defined by R_2 in this case. Now, since point O_2 is on the *right* side of the lens, the object distance is *negative*, $p' < 0$. This distance is related to the object distance of the first lens, q, by the thickness of the lens:

$$p' = d - q \tag{12.18}$$

where we made sure to carefully follow our sign convention. Refraction from the spherical surface R_2 can be calculated in the same way:

$$\frac{n_2}{p'} + \frac{n_1}{q} = \frac{n_1 - n_2}{R_2} \tag{12.19}$$

$$\frac{n_2}{d-q} + \frac{n_1}{q} = \frac{n_1 - n_2}{R_2} \tag{12.20}$$

Now, add Eqns. 12.17 and 12.20:

$$\frac{n_1}{p} + \frac{n_2}{q} + \frac{n_2}{d-q} + \frac{n_1}{q} = \frac{n_1 - n_2}{R_2} + \frac{n_2 - n_1}{R_1} \tag{12.21}$$

$$\frac{n_1}{p} + \frac{n_2 + n_1}{q} + \frac{n_2}{d-q} = (n_2 - n_1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{12.22}$$

This is the general equation for a spherical lens.

General equation for a spherical lens:

$$\frac{n_1}{p} + \frac{n_2 + n_1}{q} + \frac{n_2}{d-q} = (n_2 - n_1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{12.23}$$

Here R_1 and R_2 are the radii of the spherical sections making up the lens, d is the thickness of the lens, n_2 the refractive index of the lens material, and n_1 of the surrounding material. The result holds for rays not far from the principle axis.

Most of the time, we are interested in the so-called *thin lens approximation*, in which we neglect the thickness of the lens. That is, we presume that the image and object distances are so large compared to the thickness of the lens, $p, q \gg d$, that we can safely neglect d. If we let $d \to 0$, we have what is known as the *lensmaker's formula*:

$$\frac{n_1}{p} + \frac{n_1}{q} = (n_2 - n_1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{12.24}$$

We can find the focal length of the lens by considering the case of an extremely distant object, where we let p tend toward infinity. In that case, parallel rays will be converged on to a single focal point, just as with a spherical mirror, which we define to be the focal length f. Thus, we let p tend toward infinity (which makes $1/p$ tend toward zero), and find the corresponding value of $q = f$. This yields the more common form of the lensmaker's equation:

Lensmaker's equation:

$$\frac{1}{f} = \left(\frac{n_2 - n_1}{n_1}\right)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{12.25}$$

here n_1 is the index of refraction of the surrounding material, n_2 of the lens. The lens is defined by the surfaces of spheres of radius R_1 and R_2.

Comparing this to the preceding equation, we can also immediately relate the focal length to the image and object distance, which yields the 'lens equation':

Lens equation:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q} \tag{12.26}$$

Surprise, surprise, the mirror equation is the same as the lens equation! A convex lens like the one we just considered will have a positive focal length f. Even though we derived these lens equations for the case of a convex lens, they are valid for thin concave lenses as well, so long as they are spherical. We will consider some other types of lenses shortly, but we have one bit of pressing business: we still don't know the magnification factor of the lens!

In order to determine the image magnification, it is easier at this point to construct a ray diagram, just as we did with mirrors. The rules are only slightly different:

How to construct ray diagrams:

Ray 1 is drawn parallel to the principle axis, and refracts through one focal point.

Ray 2 is drawn through the (other) focal point, and refracts parallel to the axis.

Ray 3 is drawn through the center of the lens, and continues in a straight line.

Figure **??** shows a ray diagram for a simple convex lens. Using the geometry of this figure, we can readily figure out the magnification factor, and verify our lens equation above to boot.

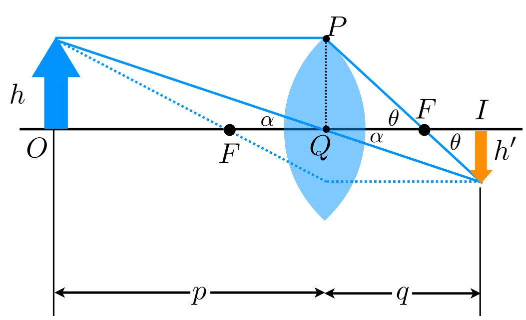

Figure 12.4: *Image construction with a biconcave lens.*

Consider the triangle formed by points O, Q, and the tip of the object arrow. The tangent of the angle α is the object height over the object distance:

$$\tan\alpha = \frac{h}{p} \tag{12.27}$$

The triangle formed by points I, Q, and the tip of the *image* arrow give us another expression for $\tan\alpha$:

$$\tan\alpha = \frac{-h'}{q} \tag{12.28}$$

Comparing these two expressions, and using the definition of the magnification factor, we have our answer:

Magnification for a spherical lens:

$$M \equiv \frac{h'}{h} = -\frac{q}{p} = \frac{f-q}{f} = \frac{f}{f-p} \tag{12.29}$$

The last two forms are derived below. They follow by using the lens equation (**??**) in the first relationship.

Once again, the lens and mirror equations are the same - same spherical geometry, same equations. This formula is also much more general than our derivation suggests - it is valid for any spherical lens, not just the symmetric concave one we considered here.

We can also verify the lens equation by using the geometry of the uppermost ray. The triangle $\triangle PQF$ gives us another relationship, noting that the distance from the center of the lens (Q) to the focal point (F) is by definition the focal length ($\overline{QF}=f$) and $\overline{PQ}=h$:

$$\tan\theta = \frac{\overline{PQ}}{f} = \frac{h}{f} \tag{12.30}$$

The triangle defined by F, I, and the tip of the object arrow gives us one more equation:

$$\tan\theta = \frac{-h'}{q-f} \tag{12.31}$$

Comparing the last two equations, we have

$$\frac{h}{f} = \frac{-h'}{q-f} \tag{12.32}$$

$$\implies \quad \frac{h'}{h} = -\frac{q-f}{f} \equiv M \tag{12.33}$$

Now we have two different expressions for M, which we can combine:

$$M = -\frac{q}{p} = -\frac{q-f}{f} \tag{12.34}$$
$$\frac{q}{p} = \frac{q}{f} - 1 \tag{12.35}$$
$$\frac{q}{p} + 1 = \frac{q}{f} \tag{12.36}$$
$$\frac{q}{p} + \frac{q}{q} = \frac{q}{f} \tag{12.37}$$
$$\Longrightarrow \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{f} \tag{12.38}$$

A result that should be reassuring: we have now independently derived the lens equation. We can derive a third relationship between the magnification and focal length using the lens equation and our result above:

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} \tag{12.39}$$
$$\Longrightarrow \quad q = \frac{fp}{p-f} \tag{12.40}$$
$$M = \frac{f-q}{f} \tag{12.41}$$
$$= \frac{f - \frac{fp}{p-f}}{f} \tag{12.42}$$
$$= \frac{fp - f^2 - fp}{f\left(p-f\right)} \tag{12.43}$$
$$= \frac{-f^2}{f\left(p-f\right)} \tag{12.44}$$
$$= \frac{f}{f-p} \tag{12.45}$$

This gives us three different relationships for the magnification factor, each one involving only two of the three quantities f, p, and q.

We now have all the mathematical and geometric ammunition we need for spherical lenses of any kind. Though we derived our results for the special case of convex lenses, they are more generally valid (it would take much more mathematics and geometry to demonstrate this, however), and hold for any spherical lenses we wish to consider. What we need to do next is figure out how different sorts of spherical lenses behave and what sorts of images the form on a qualitative level.

12.3 Types of spherical lenses

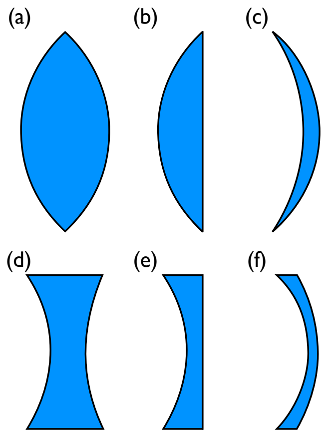

Figure 12.5: *There are a variety of common lens shapes, all essentially based on the intersection of two spheres or the space between two spheres.* **(a)** *Double convex,* **(b)** *plano-convex,* **(c)** *convex meniscus,* **(d)** *double concave,* **(e)** *plano-concave,* **(f)** *and concave meniscus lenses.*

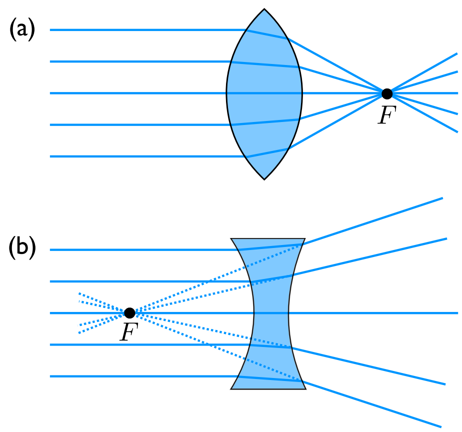

Figure 12.6: **(a)** *A biconvex lens converges distant light rays and focuses them onto a point – hence the name 'focusing lens.'* **(a)** *A biconcave lens causes distant light rays to diverge. They appear to diverge outward from a focal point on the incident side of the lens.*

12.4 Quick Questions

1. An object is placed to the left of a converging lens. Which of the following statements are true and which are false?

1. The image is always to the right of the lens
2. The image can be upright or inverted
3. The image is always smaller or the same size as the object

□ 1 and 2 are true, 3 is true
□ 2 and 3 are false, 1 is true
□ 1 and 3 are false, 2 is true
□ 2 and 3 are true, 1 is false

12.5 Problems

1. A contact lens is made of a plastic with an index of refraction of 1.60. The lens has an inner radius of curvature of 1.99 cm and an inner radius of curvature of 2.56 cm. What is the focal length of the contact lens?

12.6 Solutions to Quick Questions

1. 1 and 3 are false, 2 is true.

12.7 Solutions to Problems

1. 14.9 cm. Qualitatively, we know that the lens must form a real image in order for contact lenses to function properly. Therefore, we know that in the end the lens must have a positive focal length. Further, we know the order of magnitude of the lens, based on the size of an average human head: it must be centimeters, clearly not meters, kilometers, or micrometers! In order to attack the problem quantitatively, we need the lensmaker's equation.

$$\frac{1}{f} = (n_2 - n_1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \tag{12.46}$$

Since the outer surface of the lens is exposed to plain air, we may assume $n_1 = 1.00$ there. Since we just want the refractive index of the contact lens itself, not the contact lens in combination with the eye, we will assume the other surface is exposed to air as well. Given the refractive index of the lens material $n_2 = 1.60$ and the two radii, we need only solve for f. Since we know the answer has to be positive, we know that $R_2 = 2.56$ cm and $R_1 = 1.99$ cm, not the other way around:

$$\frac{1}{f} = (1.60 - 1.00)\left[\frac{1}{1.99} - \frac{1}{2.56}\right] \tag{12.47}$$

$$\frac{1}{f} = (0.60)\,[0.112]\ \text{cm}^{-1} \tag{12.48}$$

$$\frac{1}{f} = 0.0617 \tag{12.49}$$

$$\implies \quad f = 14.9\,\text{cm} \tag{12.50}$$