4 Electrical Energy and Capacitance

POTENTIAL energy and the principle of conservation of energy often let us solve difficult problems without dealing with the forces involved directly. More to the point, using an energy-based approach to problem solving let us work with *scalars* instead of *vectors*. This way we get to deal with just plain numbers, which is nice.

In this chapter, we will learn that, as with the gravitational field, the electric field has an associated **potential** and **potential energy**. The electric potential will, in many cases, let us solve problems more easily than with the electric field and, as it turns out, electric potential is what we normally identify with 'voltage' in everyday life.

Figure 4.1: *Michael Faraday (1791 - 1867), an English physicist and chemist who contributed significantly to the field of electromagnetism.*[16]

4.1 Electrical Potential Energy

The work done on an object by a conservative force, such as the electric force, depends only on the initial and final positions of the object, *not on the path taken between initial and final states.* For example, the work done by gravity depends only on the change in height. When a force is conservative, it means that there exists a **potential energy** function, PE, which gives the potential energy of an object subject to this conservative force which depends *only on the object's position. Potential energy* is sometimes called the "energy of configuration" since it only depends on the position of objects in a system. Thus, for the conservative electric force, we can find a change in electrical potential energy just by knowing the starting and final configurations of the system we are studying – nothing in between matters.

As you know, **potential energy is a scalar quantity**, and the *change* in potential energy is equal to the work done by a conservative force.

Potential energy difference, ΔPE

$$\Delta PE = PE_f - PE_i = -W_F \tag{4.1}$$

where the subscripts $f(i)$ refer to the final (initial position), and W_F is the work done by the conservative force $\vec{\mathbf{F}}$.

This is just how you dealt with gravity – moving an object of mass m through a vertical displacement h gives a changes in potential energy $\Delta PE = mgh$. Electrical forces and gravitational forces have a number of useful similarities, as you now know, and the same is true for their respective potential energies.

The Electric Force is Conservative:

1. We can define an *electrical potential*
2. There is potential energy associated with the presence of an electric field
3. Electric potential is *potential energy per unit charge*

Consider a small positive test charge q in a uniform electric field $\vec{\mathbf{E}}$, as shown in Figure 4.2. As the charge moves from point A to point B, covering a displacement $\Delta x = x_f - x_i$, the work done on the charge by the electric field is the component of the force $\vec{\mathbf{F}}_e = q\vec{\mathbf{E}}$ parallel to the displacement Δx:

[i]**Work done moving a charge q in a constant electric field $\vec{\mathbf{E}}$:**

$$\Delta W_{AB} = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = |\vec{\mathbf{F}}|\,|\Delta x| \cos\theta = qE_x\,(x_f - x_i) = qE_x\Delta x \tag{4.2}$$

where q is the charge, E_x is the component of the electric field $\vec{\mathbf{E}}$ along the direction of displacement, and θ is the angle between the force $\vec{\mathbf{F}}$ and the displacement $\Delta\vec{\mathbf{x}}$ (of length Δx).

Note that q, E_x, and Δx can all be either positive or negative. Also recall that E_x is the x-*component* of the electric field $\vec{\mathbf{E}}$, *not* the magnitude! Equation 4.2 is valid for the work done on a charge by *any constant electric field*, no matter what the direction of the field, or sign of the charge. Just remember that the angle between the field and displacement does matter!

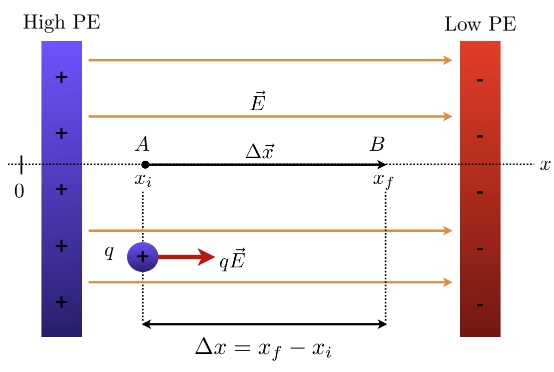


Figure 4.2: *When a charge q moves in a uniform electric field $\vec{\mathbf{E}}$ from point A to point B, covering a distance Δx, the work done on the charge by the electric force is $qE_x\Delta x$.*

Now that we have found the work done by the electric field, the work-energy theorem gives us the potential energy change:

[i]At this point you may want to remind yourself about the scalar or "dot" product, $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |A||B| \cos\theta_{AB}$, where θ_{AB} is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

The change in electric potential energy ΔPE of an object with charge q moving through a displacement Δx in a constant electric field $\vec{\mathbf{E}}$ is:

$$\Delta PE = -W_{AB} = -q|\vec{\mathbf{E}}|\,|\Delta\vec{\mathbf{x}}|\cos\theta = -qE_x\Delta x \tag{4.3}$$

where the quantities are defined as in Eq. 4.2.

Remember, just like any other work, the work done involving the electric force only counts the displacement *parallel to the force.* You can find the component of the field parallel to the full displacement, or find the component of the displacement parallel to the field – it is the same thing. Figure 4.3 compares a charge moving in an electric field to a mass moving in a gravitational field. A positive charge moving in an electric field acts much like a mass moving in a gravitational field: the positive charge at point A falls in the direction of the field, just as the mass does. This lowers its potential energy, and increases its kinetic energy.

Assuming other forces are absent, we can also find the kinetic energy change through conservation of energy. Since both the electrical and gravitational forces are conservative, we can find the changes in kinetic and potential energy in both cases and compare them. In both situations, the change in potential energy must be equal and opposite the change in kinetic energy for energy to be conserved[ii]:

$$\begin{aligned} KE_i + PE_i &= KE_f + PE_f && (4.4)\\ (KE_f - KE_i) &= -(PE_f - PE_i) && (4.5)\\ \Delta KE &= -\Delta PE && (4.6)\\ \Delta KE + \Delta PE &= 0 && (4.7) \end{aligned}$$

For the gravitational case, we have done this a million times for an object of mass m starting at a height d and ending at a height defined as 0:

$$\begin{aligned} \Delta KE + \Delta PE_G &= \Delta KE + (0 - mgd) = 0 && (4.8)\\ &\Longrightarrow \Delta KE = mgd && (4.9) \end{aligned}$$

For the electrical case, it is not much more difficult. We will move a charge q through an electric field E:

[ii] The subscripts i and f refer to initial and final, as usual.

$$\Delta KE + \Delta PE_E = \Delta KE + (0 - qE_d d) = 0 \tag{4.10}$$

$$\implies \Delta KE = qE_d d \tag{4.11}$$

Here d is the distance moved in the electric field $\vec{\mathbf{E}}$, and E_d is the component of the electric field parallel to the direction of motion. For positive charges, electric potential energy works just like gravitational potential energy. Since mass comes only in one flavor, while charge comes in positive and negative varieties, this is not the whole story, however. For a negative charge, we have to substitute $-q$ for q in the equations above - rather than falling in the electric field like the positive charge, the negative charge wants to move *upward*. In other words, the negative charge "falls up" compared to a positive charge.

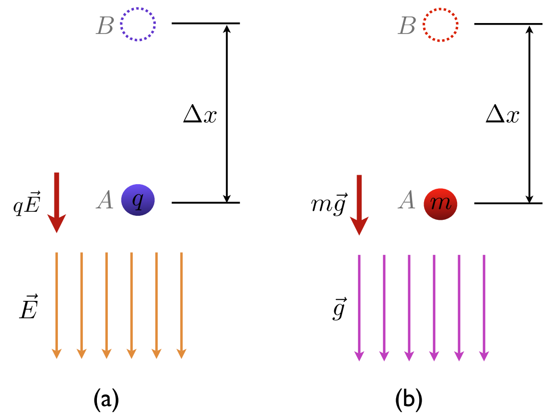


Figure 4.3: (a) *When an electric field $\vec{\mathbf{E}}$ is directed downward, point B has a lower electrical potential energy than point A. As a positive test charge moves from A to B, the electrical potential energy decreases.* **(b)** *An object of mass m moves in the direction of the gravitational field $\vec{\mathbf{g}}$, the gravitational potential energy decreases.*

In order to make a negative charge move downward we would have to do work against the electric field. Remember that positive charges like to follow the direction of the electric field lines, while negative charges like to go against them. For the positive charge in Figure 4.3, we are moving the charge in the direction it wants to go. For a negative charge in the same situation, we are moving the charge *against* the direction it wants to go. The negative charge has a *positive* change in electrical potential energy moving from **point** A **to point** B, meaning kinetic energy has to be lost to make this happen. The positive charge has a *negative* change in potential energy moving from **point** A **to point** B, meaning kinetic energy will be gained by doing this.

4.2 Electric Potential

In Chapter 3, it was convenient to define $\vec{\mathbf{E}}$ related to the electric force, *viz.*, $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$. This let us think about individual charges one at a time, even when our system was a collection of several charges, and discard the idea of "action at a distance." For the same reasons, we would like to define a variation of the electrical potential energy *per unit charge*, so we may think about *how much potential energy would be gained or lost by a single charge present in an electric field.*[iii]

[iii] This is similar to the *chemical potential* in a way, if you are familiar with that.

This quantity is the **electric potential difference** ΔV, and it is related to potential energy by $\Delta PE = q\Delta V$.[iv]

The electric potential difference ΔV between points A and B is the change in electric potential energy between those two points divided by the quantity of charge moving Q:

$$\Delta V = V_B - V_A = \frac{\Delta PE}{q} \qquad \text{or} \qquad q\Delta V = \Delta PE \tag{4.12}$$

where V_B is the potential at point B and V_A is the potential at point A.

Electric potential is measured in Joules per Coulomb, otherwise known as *Volts*. Just like gravitational potential, **electric potential is a scalar quantity.** It is essentially a measure of the change in electric potential energy per unit charge. By definition, it takes 1 J to move 1 C worth of charge between two points with a potential difference of 1 V. If a 1 C charge moves through a potential difference of 1 V, it gains 1 J of potential energy.

Units of V and ΔV: [J/C] (Joules per Coulomb) or **[V]** (Volts)

Consider the special case of a single charge q moving through a region of constant electric field, such as the area between two parallel charged plates (Fig. 3.9). If the displacement of the charge Δx is perfectly parallel to the electric field, we can divide Equation 4.3 by q to find the potential difference ΔV:

Single charge q in a constant electric field $\vec{\mathbf{E}}$

$$\Delta V = \frac{\Delta PE}{q} = -|\vec{\mathbf{E}}|\,|\Delta\vec{\mathbf{x}}|\cos\theta = -E_x\Delta x \tag{4.13}$$

where the quantities are defined as in Eq. 4.2.

This lets us see that potential difference also has units of electric field times distance. This makes sense in a way, since for there to be an electrical potential difference we pretty much have to move through an electric field. Since electric field has the units of newtons per coulomb (N/C), we can make the following observation:

A newton (N) per coulomb (C) equals a volt (V) per meter (m): $1\,\mathbf{N/C} = 1\,\mathbf{V/m}$

If we release a positive charge, it spontaneously accelerates from regions of high potential to low potential - positive charges seek out minima in the electric potential. Conversely, negative charges

[iv] The *gravitational potential* is the potential energy per unit mass, which is just gh for terrestrial cases, or $\frac{-Gm}{r}$ for the more general case. We would say that the *potential energy difference* between two points whose height differs by h is mgh, while the *potential difference* is just gh.

seek out maxima in electric potential. **Work must be done on** positive charges to move them toward higher potential, work must be done on negative charges to move them to regions of lower potential.

4.2.1 Electric Potential and Potential Energy due to Point Charges

As described briefly in Sect. 3.2.1.1, in electric circuits the zero point of electric potential ($V = 0$) is defined by a "ground" wire connecting some point in the circuit to the earth. In a sense, defining a precise point at which $V = 0$ through a ground wire is a bit like choosing an origin in a coördinate system. It can be anywhere you like, but you have to have one! For example, connecting the negative terminal of a 9 V battery to the ground would define the negative terminal as $V = 0$, and the positive terminal would be at +9 V. If, on the other hand, we connected the positive terminal to ground, *it* would have $V = 0$ and the negative terminal would have −9 V. In a way, the potential difference of the battery of 9 V well-defined, but the absolute potentials are *not* until a zero point is chosen.

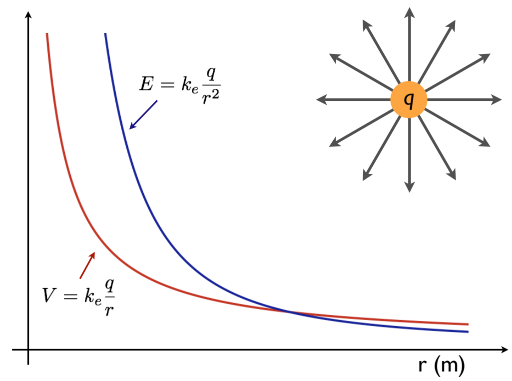


Figure 4.4: *The electric field $\vec{\mathbf{E}}$ and electric potential V versus the distance r from a point charge. Note V is proportional to $1/r$, while E is proportional to $1/r^2$.*

For point charges, the electric field is defined throughout space, except *right at the charge*, and it works the same way for its electric potential. There is no obvious place to call "zero." Further, we cannot connect a tiny ground wire to a single electron! (What could we make the wire out of ...) In the end, we nearly always, we define the potential for a point charge to be zero an infinite distance from the charge itself. This is actually convenient, believe it or not, and it makes clear the fact that the only way to get rid of the potential due to a point charge is to completely banish the charge itself. With this definition and some calculus, the electric potential of a point charge q at a distance r from the charge can be found as:

Electric potential created by a point charge:

$$V = k_e \frac{q}{r} \tag{4.14}$$

where r is the distance from the point charge q, and k_e is Coulomb's constant (Eq. 3.2).

This gives us the electric potential – work per unit charge – required to move the charge q from an infinite distance away to a point r. Figure 4.4 plots for comparison the electric field and electric potential for a point charge as a function of the distance from the charge.

Keep in mind: you can only measure *differences* in electric potential. Some reference point must always be defined as $V = 0$. For a point charge, this is $r = \infty$, for a circuit it is a specific point in the circuit.

One quick point, to clear up any later confusion: when dealing with point charges like electrons in electric fields, or atoms in a crystal (*e.g.,* in nuclear or atomic physics, and sometimes inorganic chemistry), we often use a more convenient unit of energy, the *electron volt*.

An **Electron Volt [eV]** is the kinetic energy an electron gains when accelerated through a potential difference of 1 V.

$$1\,\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{C{\cdot}V} = 1.60 \times 10^{-19}\,\mathrm{J}$$

We will encounter the electron volt more and more as time goes on, it turns out to be quite convenient when worrying about small numbers of charges.

4.2.2 Energy of a System of Charges

Electric potential also obeys the superposition principle, just like the electric force. **The total electric potential at some point due to several point charges is just the sum of the electric potentials due to the individual point charges.** Since electric potential is a scalar, we do not need to worry about components, electric potentials are just numbers.

Figure 4.5 shows a "3-d" plot of the electric potential of an electric dipole (one positive charge and one negative charge close together, as in Fig. 3.6), where the color height scale represents the magnitude of the electric potential. As expected from the superposition principle, the potential is zero right between the two charges, and becomes very large near each charge, as does the electric field (Fig. 3.6).

From Eq. 4.12, we can see that it is easy to convert between electric potential and electric potential energy. What about the potential energy of two charges? If V_1 is the potential due to a charge q_1 at a point P, the work required to bring a charge q_2 from infinity to the point P is q_2V_1, as shown in Fig. 4.6. That is, q_2V_1 is the energy it took to configure our system with charge q_2 at point P, and how much energy would be gained or lost by completely removing q_2. Similarly, if q_2 is fixed in place, it takes q_1V_2 to bring q_1 in from an infinite distance to its final position.

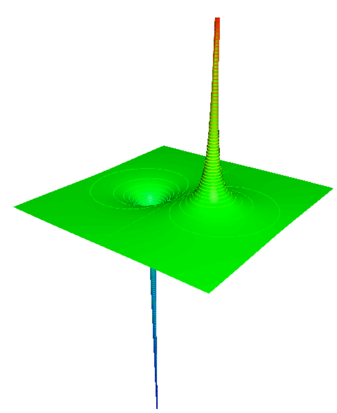


Figure 4.5: *The electric potential in a plane containing an electric dipole. The height (color) scale gives the electric potential. The lines represent equipotential contours.*

This means that configuring two charges close to one another entails a gain or loss of energy – each charges feels the potential from the other. Bringing charges close together means energy is gained or lost to make that happen, and that energy is the potential energy of the pair of charges – how much energy is tied up in keeping those two charges where they are. For example, if two positive charges are to be kept close together against their natural repulsion, energy should be supplied to keep them together. If a positive and negative charge are to be kept together, energy should be supplied to keep them *apart.*

Now we see that potential energy really is the energy it takes to configure the system under study. Figure 4.6 also illustrates the difference between the *potential of a the separate point charges*, and the *potential energy of the pair of point charges.* If q_1 is already fixed its position, but q_2 is at infinity, the work that must be done to bring q_2 from infinity to its position near q_1 is $PE = q_2V_1 = k_eq_1q_2/r_{12}$. That is what the potential energy is, the energy of this configuration of charges *relative to just having q_1 all by itself.* If q_2 is fixed, it also takes $PE = k_eq_1q_2/r_{12}$ to bring in q_1. Thus, it takes $PE = k_eq_1q_2/r_{12}$ to build our system of two charges, no matter how we do it:

$$PE_{\text{two charges}} = PE_{(1\text{ due to }2)} = PE_{(2\text{ due to }1)} = q_2V_1 = q_1V_2 = \frac{k_eq_2q_1}{r_{12}} \tag{4.15}$$

As mentioned above, if the charges are of the same sign, PE is positive, and work must be done by an external force to bring the charges together. If they are of opposite charges, PE is negative, and negative work must be done to keep the charges from accelerating toward each other as they are brought together. In other words, work must be done to *keep* the charges apart. Another way to view the potential energy of the pair of charges is to think about how much kinetic energy would be gained if we let one of them loose again. If we have a pair of charges with an electrical potential energy of, say, 1 J with both charges fixed, the charges can gain between them 1 J of kinetic energy after being let loose. If one stays fixed, the other gets a full 1 J. If both charges are identical and both move, they each get 0.5 J.

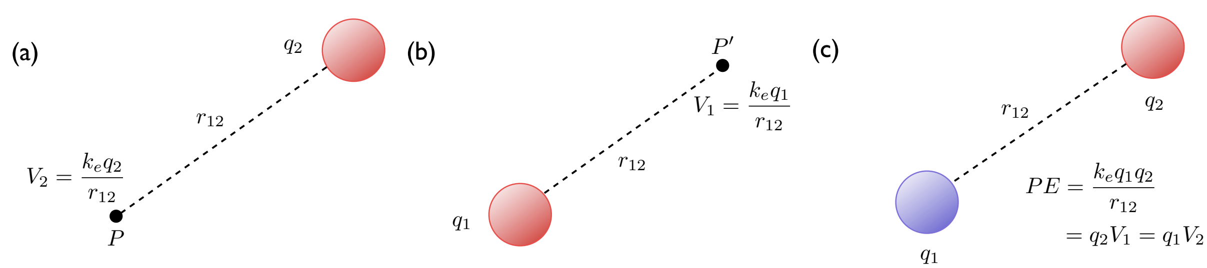


Figure 4.6: **(a)** *If the charge q_1 is removed, a potential k_eq_2/r_{12} exists at point P due to* **charge** *q_2* **(b)** *Similarly, the charge q_1 gives a potential k_eq_1/r_{12} at point P'.* **(c)** *Either way we build our system of charges, the potential energy of the system of two charges is just $q_2V_1 = q_1V_2$, or $k_eq_1q_2/r_{12}$.*

What if we have several charges? Just to be concrete, take the system of three point charges in Figure 4.7. We can obtain the total potential energy of this system by calculating the PE for every pair combination of charges and adding the results together. Since potential and potential energy are scalars, we don't need to worry about components – this is just an algebraic sum:

$$PE = PE_{1\&2} + PE_{2\&3} + PE_{1\&3} = PE_{2\&1} + PE_{3\&2} + PE_{3\&1} = k_e\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) \tag{4.16}$$

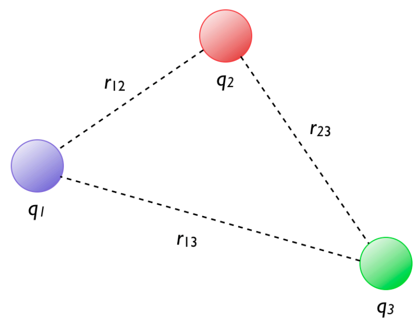


Figure 4.7: *A system of three point charges. Finding the total potential energy is just a matter of adding up the potential of pair combinations of charges.*

Note that it doesn't matter what the order we sum them in, or if we transpose the labels – $PE_{1\&2}$ is the same thing as $PE_{2\&1}$, and r_{13} is the same as r_{31}, just like the example with two charges above.[v]

What does this really mean, physically? It is the same whether we have two charges or three or a million. What we are really summing up is the energy required to *build* this particular configuration of charges. Imagine that q_1 is fixed at the position shown in Figure 4.7, but that q_2 and q_3 are at infinity. The work that must be done to bring q_2 from infinity to its position near q_1 is $PE_{1\&2} = k_eq_1q_2/r_{12}$, which is the first term in Equation 4.16. The last two terms represent the work required to bring q_3 from infinity to its position near q_1 and q_2, which involves the interaction with q_1 (the second term in Equation 4.16) and the interaction with q_2 (the third term in Equation 4.16). Compare this with Equation 4.15. Again, the result is independent of the order in which the charges are moved in from infinity.

We can write this more succinctly as a sum over all the charges:

$$PE = \frac{1}{2}\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3}\frac{k_eq_iq_j}{r_ij} \tag{4.17}$$

$$= \frac{1}{2}\left(\frac{k_eq_2q_1}{r_{21}} + \frac{k_eq_3q_1}{r_{31}} + \frac{k_eq_1q_2}{r_{12}} + \frac{k_eq_3q_2}{r_{32}} + \frac{k_eq_1q_3}{r_{13}} + \frac{k_eq_2q_3}{r_{23}}\right) \tag{4.18}$$

$$= k_e\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) \tag{4.19}$$

Here we color-coded the like terms for clarity. Basically, first we pick some charge j, and sum over all its pairings with the other charges i, making sure not to pair the charge with itself. Here we

[v] If you are into the math, that means we sum over all possible *combinations*, ${}^nC_k$, not *permutations*, ${}^nP_k$, so *we do not count any pair more than once.*

have the factor $\frac{1}{2}$ because the sum as written would count every pair of charges *twice* – since the pair 1&3 is the same as the pair 3&1. Think about that for a second, and reassure yourself that the factor $\frac{1}{2}$ is necessary. (If you are not familiar with summations, don't worry. We will only ever deal with a few charges at once.) For any arbitrary number of charges N, we can just change the limits on the sum:

$$PE_{\text{total}} = \frac{1}{2}\sum_{j}^{N}\sum_{i\neq j}^{N}\frac{k_e q_i q_j}{r_i j} \tag{4.20}$$

The double-sum notation above means "take the charge $j=1$, and sum over all the other charges $i=2,3,4,\ldots N$, then take the charge $j=2$, and sum over the other charges $i=1,3,4\ldots$N, and so on, until $j=N$." Again, this counts every pair *twice*, hence the factor $\frac{1}{2}$.

4.2.2.1 Electrical Energy in a Crystal Lattice

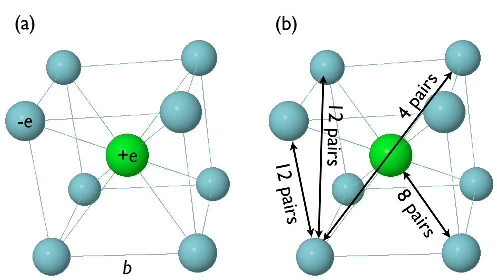


Figure 4.8: **(a)** *A crystal consisting of a cube of $-e$ negative charges, with a single $+e$ charge at the center of the cube. The potential energy of the arrangement of nine charges is a sum over potential energy of all pairs.* **(b)** *There are four types of pairs involved in the sum.*[17]

What good is being able to find the energy of a large number of charges? Well, for one, this is one way to compute the stability of various crystal lattices. As an example, let us calculate the potential energy of eight negative charges on the corners of a cube of side b, with a single positive charge in the center. We will say each negative charge has $-e$, while the single positive charge is $+e$, Fig. 4.8 We can readily sum over all the possible pair interactions in the crystal, after a bit of geometry to figure out the distances between pairs.

For this crystal, we have 12 pairs of negative charges that are just one edge of the cube apart, twelve pairings between negative charges sideways across the cube faces, eight pairings between the negative corner charges and the central positive charge, and four corner to corner pairings of negative charges. This is illustrated in Fig. 4.8. Standard geometry tells us that the distance between edge charges is just b, the distance from corner to center is $\frac{\sqrt{3}}{2}b$, the corner-corner distance across a cube face is $\sqrt{2}b$, and finally the distance between opposite corner charges is $\sqrt{3}b$. The sum over all pairs is then:

$$PE_{\text{crystal}} = 8\times\left[\frac{k_e(-e\cdot e)}{(\sqrt{3}/2)b}\right] + 12\times\left[\frac{k_e e^2}{b}\right] + 12\times\left[\frac{k_e e^2}{\sqrt{2}\,b}\right] + 4\times\left[\frac{k_e e^2}{\sqrt{3}\,b}\right] \approx \frac{13.55 k_e e^2}{b} \tag{4.21}$$

Figure 4.8 shows where each term in the sum comes from. Though this seems a bit complicated, think about how hard it would be to compute the *forces* for every pair of charges and find the resultant vector force! We would have to do that for every stage of construction of the crystal, a tedious task at best. The relatively simple potential energy calculation above is a powerful way to address the amount of energy tied up in maintaining a particular charge distribution.

In this case, note that the total energy of this crystal lattice is *positive*, representing the fact that work had to be done on the crystal to assemble it in the first place. Left to its own devices, the charges in the crystal would want to disassemble. If we did let these charges move apart again, they would recover the potential energy as kinetic energy and speed away. This makes sense – it is silly to expect that real crystals are made of mostly negative charges, when we know that they are neutral overall. In reality, crystals are made of an equal number of positive and negative charges, which in many cases leads to a *negative* potential energy, indicating that the charges actually lower their energy by assembling into a crystal, and therefore favor doing so.

It is also curious that the potential energy sum for our cubic crystal ends up being a constant factor (about 13.55 times) what it would be for just a single pair of point charges separated by a distance b. In general, this is true for nearly any crystal lattice we can construct – the energy will always be some multiple of what for a single pair of charges. The multiple itself – in this case 13.55 – divided by the total number of charges is known as *Madelung's constant*, and every sort of crystal lattice has its own particular Madelung constant. The Madelung constant only depends on the geometric arrangement of the constituent ions in the crystal structure. Basically, the Madelung constant is something you look up in a table that takes care of all the nasty summing for you – someone has already done it! In general we can the potential energy of a crystal like this:

$$PE_{\text{crystal}} = \frac{1}{2} MN \frac{k_e z^2 e^2}{r} \tag{4.22}$$

here M is the Madelung constant, N is the number of charges we are considering, z is the charge of the ions in the lattice (± 1 in this case), and r is their separation. By inspection, you can see that for our cubic crystal, $13.55 = \frac{1}{2}MN$. Since there are $N = 9$ charges in our example, our Madelung constant is $2(13.55)/10 = 2.71$.

If we take the structure of NaCl (common salt or rocksalt), the so-called face-centered cubic structure shown in Fig. 4.9a, the Madelung number ends up being about -1.75 if you carefully take the limit of the sum for very large N. The rocksalt structure has alternating positive Na^+ and Cl^- ions, arranged in a face-centered cubic structure. Overall, it is electrically neutral, and the negative potential energy reflects the stability of the structure. **The negative sign shows that** *work would have to be done to take the NaCl crystal apart – it is intrinsically stable.* This is in contrast to our ficticious body-centered cubic case above. Since our cubic crystal is mostly made of negative charges, it is not stable, and work has to be done to assemble it. The NaCl structure, however, has an equal number of positive and negative charges, and the negative potential energy sum explains the cohesion of the crystal and the fact that NaCl spontaneously assembles when Na and Cl are mixed. The Na and Cl constituents can lower their overall energy by assembling into a crystal, and that is what they do when given half a chance. The more negative the Madelung constant, the more stable the crystal is, if everything else is the same.

As another example, consider the Rutile (TiO_2) structure in Fig. 4.9b. In this case, the Madelung number is -4.82, suggesting that rutile structure materials should be quite stable, and they generally are. There is one problem with all of this, however. Based on the analysis above, shrinking the distance b between charges in the crystal should make the potential energy even *more* negative. In other words, the smaller the spacing, the more stable the crystal would be. If that were true, why would the crystal not just keep shrinking shrinking until it collapsed? In fact, it can be shown that no system of stationary charges can be in a stable equilibrium according to classical physics. We need quantum physics to explain why, *e.g.,* salt crystals do not spontaneously shrink, and how crystals are stable in the first place.

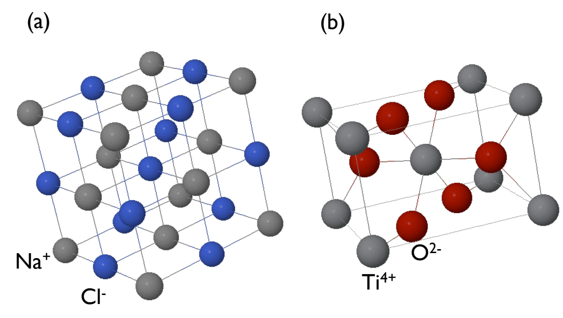


Figure 4.9: (a) *The NaCl or rocksalt structure. There are an equal number of Na^+ and Cl^- ions, the crystal is neutral overall.* **(b)** *The Rutile (TiO_2) structure. There are twice as many O^{2-} ions as Ti^{4+} to maintain neutrality.*[17]

4.3 Potentials and charged conductors

So the work done on a charge by an electric force is related to the change in electric potential energy of the charge. We also know that the change in electric potential energy between points A and B must be related to the potential difference between those two points. Putting these two facts together, we can easily relate work and potential difference:

Work and electrical potential for a charge moving from point A to B:

$$-W = \Delta PE = q\left(V_B - V_A\right) \tag{4.23}$$

where V_B is the electrical potential B, and V_A is the electrical potential A.

In Chapter 3, we said that for a conductor in electrostatic equilibrium, net charge resides only on the conductor's surface. Moreover, we said that the electric field just outside the surface of the conductor is perpendicular to the surface, and that the field inside the conductor is zero. This also means that **all points on the surface of a charged conductor in electrostatic equilibrium are at the same potential.**

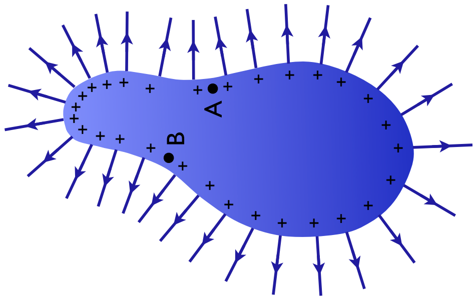


Figure 4.10: *An arbitrarily shaped conductor carrying a positive charge. When the conductor is in electrostatic equilibrium, all of the charge resides at the surface, $\vec{\mathbf{E}} = 0$ inside the conductor, and the direction of $\vec{\mathbf{E}}$ just outside the conductor is perpendicular to the surface. The electric potential is constant inside the conductor and is equal to the potential at the surface. Note from the spacing of the positive signs that the surface charge density is nonuniform.*

Equation 4.23 gives us a very general result: **no net work is required to move a charge between two points which are at the same electric potential.** Mathematically, $W = 0$ whenever $V_B = V_A$.

Consider the path connecting points A and B along the surface of the conductor in Figure 4.10. If we move only along the conductor's surface, the electric field $\vec{\mathbf{E}}$ is always perpendicular to our path. Since the electric field and displacement are always perpendicular, no work is done when moving along the surface of a conductor. Equation 4.23 then tells us that if the work is zero, points A and B must be at the same potential, $V_B - V_A = 0$. Since the path we have chosen is completely arbitrary, this means it is true for any two points on the surface.

Potentials and charged conductors

1. electric potential is a constant on the surface
2. electric potential is constant inside, and has the same as the value at the surface
3. no work is required to move a charge from the interior to the surface, or between two points on the surface

Of course, this only holds for perfect conductors. If other dissipative (or non-conservative) forces are present, this is not true, and work *is* required to move the charge in the presence of a dissipative force. The electrical analog of friction or viscosity is *resistance*, which will be treated in the next chapter.

4.4 Equipotential Surfaces

A surface on which all points are at the same electric potential is called an *equipotential surface*. The potential difference between any two points on the surface is zero, hence, **no work is required to move a charge at constant speed on an equipotential surface**. The surface of a conductor is therefore an equipotential surface. Equipotential surfaces have a simple relationship to the $\vec{\mathbf{E}}$ field: the field is perpendicular to the equipotential surface at every point. Figure 4.11 shows equipotential surfaces and electric field lines for a single point charge, a dipole, and two like charges. Notice that once you have drawn electric field lines, drawing equipotential surfaces is trivial, and *vice versa*.

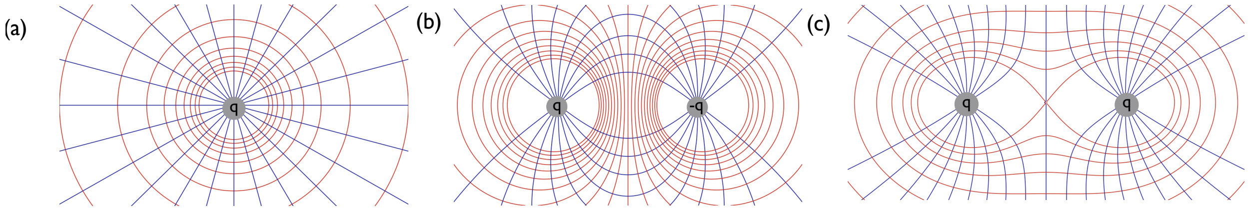


Figure 4.11: *The blue lines electric field lines, and the red lines are equipotential surfaces for* **(a)** *a single point charge,* **(b)** *an electric dipole, and* **(c)** *two like charges. In each case, the equipotential surfaces are perpendicular to the electric field lines at every point. (Again, arrows are left off of the field lines for simplicity. Equipotential lines do not need arrows, since potential is a scalar.)*

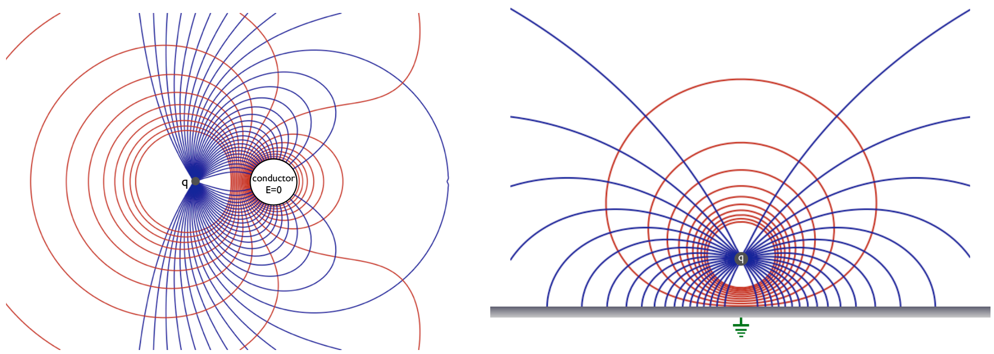


Figure 4.12: *The blue lines electric field lines, and the red lines are equipotential surfaces for* **left** *a conducting sphere near a point charge q, and* **right** *a point charge suspended above a long grounded conducting plate.*

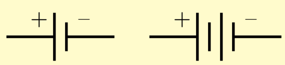
More examples are given in Fig. 4.12, which include conductors. For a conductor, we know the electric field inside is zero, and the electric potential is constant. Add to this the fact that electric field lines and equipotential lines are *always* perpendicular where they meet, and you should be able to explain all of the examples shown here. This why in the right-hand example, a single charge above a ground plane, the electric field lines all intersect the ground plane at perfect right angles, and in the left-hand example, there are no lines inside the conducting sphere. Compare these figures with Fig. 3.9 – the relationship between electric field lines and equipotential lines should be clear. Appendix B might give you a bit more insight as to why the electric field lines and equipotential lines behave the way they do. Recall from Sect. 3.5 that *a conductors are mirrors for electric field lines, the same is true for the equipotential lines.*

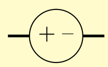
4.5 Voltage Sources as Circuit Elements

How do we actually change the potential or voltage of one object relative to another? Charging by induction or conduction are two ways, but somewhat cumbersome. A device known as a *voltage source* is a circuit element with two terminals, where a constant voltage difference is supplied between these two terminals. Whatever you connect to the "negative" terminal of the voltage source will have a voltage ΔV lower than the "positive" terminal. Using a "ground" point (recall Sect. 3.2.1.1), one can also experimentally define one of the terminals as $V = 0$. If we "ground" the negative terminal, then the negative terminal is $V_{\text{neg}} = 0$, and the positive terminal has $V_{\text{pos}} = \Delta V$. We will see much more of this in the coming chapters, and it will begin to make more sense!

Batteries are one example of a constant voltage source, which we will cover in more detail in Chapter 6, and the wall outlets in your house are another example of a voltage source (though this voltage is not strictly constant, see Chapter 9). Ideal textbook voltage sources *always* supply a constant potential difference, ΔV. Real voltage sources always have restrictions, a primary one being the amount of power that can be sourced. Below are circuit diagram symbols for constant voltage sources: the first two represents batteries, the last is a generic symbol for any more complicated sort of voltage source:

Circuit diagram symbol for voltage sources:

Batteries: 

General constant voltage source: 

Now that we know a bit about voltage and conductors, we are moving closer to being able to describe simple electric circuits. Presently, we will introduce our first real circuit element, the capacitor.

4.6 Capacitance

A **capacitor** is an electronic component used to store electric charge, it is used in essentially any electric circuit you can name. Capacitors are at the heart of both Random Access Memory (RAM) and flash memory, besides being crucial for nearly any sort of power supply. It is one of the fundamental building blocks for electronics, and the first we will meet. Figure 4.13 shows a typical design for a capacitor – two metal plates with some special stuff in between. It is hard to believe complicated devices like computers rely on such a simple construction, but it is true!

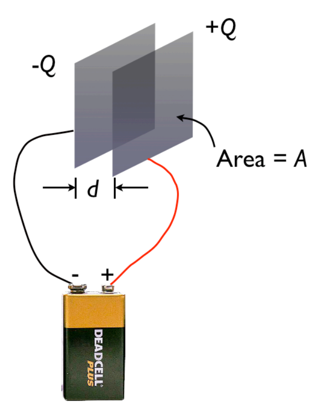


Figure 4.13: *A parallel-plate capacitor consists of two conducting plates of area A, separated by a distance d. The capacitance of this structure is $C = \epsilon_0 A/d$.*

A typical capacitor consist of two parallel metal plates, separated by a distance d. When used in a circuit, the plates are connected to the positive and negative terminals of a voltage source such as a battery. An ideal voltage source insists that the two plates have a voltage difference of ΔV, and this has the effect of pulling electrons off of one plate, leaving it with a net positive charge $+Q$, and transferring these electrons to the second plate, leaving it with a net negative charge $-Q$. The charge on both plates is equal, but opposite in sign. Essentially, putting the two plates at different potentials means electrons want to migrate to the plate with higher potential, and leave the plate with lower potential deficient.

The transfer of charge between the plates stops when the potential difference across the plates is the same as the potential difference of the voltage source. *The capacitor stores this potential difference, and hence stores electrical energy, until some later time when it can be reclaimed for a specific application.* You can think of this as energy storage from one point of view, or a time-delayed response from another.

Keep in mind (again): you can only *measure* differences in electric potential. Some reference point must always be defined as $V = 0$. In the case of the capacitor connected only to a battery (without any ground points), the potential is zero half way between the two plates.[vi]

Definition of Capacitance:
The capacitance C is the ratio of the charge stored on one conductor (or the other) to the potential difference between the conductors:

$$C \equiv \left|\frac{Q}{\Delta V}\right| \tag{4.24}$$

C is always positive, and has units of farads **[F]**, or coulombs per volt **[C/V]**.

4.6.1 Parallel-Plate Capacitors

The capacitance of a particular arrangement of two conductors depends on their geometry and relative arrangement. One common (and simple) structure is the **parallel plate** capacitor, as shown in Figure 4.13. In Chapter 3, we stated without proof (but not without good reason) that the electric field between two parallel plates is constant. But what is the field in between the plates?

First, we assume that the two plates are identical, such that they have the same charge on them – one has $+Q$ and one has $-Q$. Second, we assume the plates area A is large compared to their spacing d, such that we can ignore the edge regions where the field "fringes" (see Fig. 3.9 and 4.14). Finally, we will connected the plates to a battery with total voltage V.

In Sect. 3.8.4, we found that the electric field above a flat conducting plate is given by $E = \sigma_E/\epsilon_0$, where σ_E is the charge per unit area on the plate. Since the total charge on each plate is just Q, the charge per unit area is $\sigma_E = Q/A$, and $Q = \sigma_E A$. This leads us to a more useful expression for the field: $E = \frac{Q}{A\epsilon_0}$. Again, this is not valid near the edges of the plates where the field is not really constant.

[vi] The potential is also zero infinitely far away of course, but this is hardly useful or reassuring when wiring a circuit.

Now where the field *is* constant, we know that the potential difference between the two plates is $\Delta V = Ed$, where d is the distance between the two plates. Combining this with the facts above, we can find the capacitance of the parallel plate capacitor from Equation 4.24:

$$C = \frac{Q}{\Delta V} = \frac{\sigma_E A}{Ed} = \frac{\sigma_E A}{(\sigma_E/\epsilon_0)\, d} = \frac{\cancel{\sigma_E} A}{(\cancel{\sigma_E}/\epsilon_0)\, d} = \epsilon_0 \frac{A}{d} \tag{4.25}$$

Capacitance of a parallel plate capacitor:

$$C = \epsilon_0 \frac{A}{d} \tag{4.26}$$

where d is the spacing between the plates, and A is the area of the plates.

We can see from Equation 4.26 that capacitors can store more charge when the plates become larger. The same is true when the plates get close together. When the plates are closer together, the opposing charges exert a stronger force on each other, allowing more charge to be stored on the plates. From Equation 4.24, **a capacitor of value C at a potential difference of ΔV stores a charge $Q = C\Delta V$.**

Figure 4.14 shows more realistic field lines for a parallel plate capacitor. In between the two plates, the field is very nearly constant, but much less so near the edges of the plates. So long as the plates are relatively large compared to their separation, we can for practical purposes ignore this complication, and our capacitance calculated from Eq. 4.26 will be very accurate.

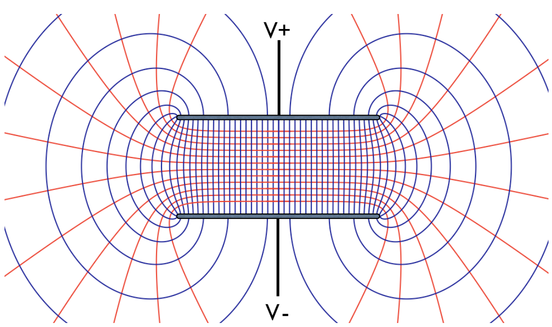


Figure 4.14: (a) *The electric field (blue) and equipotential (red) lines near and between the plates of a parallel-plate capacitor. The potential and field are both uniform near the center, but nonuniform near the edges.*

Capacitors form the basis for several types of Random Access Memory (RAM) in modern computers. Dynamic random access memory (DRAM) is one type of random access memory that stores each bit of data in a separate capacitor. One capacitor in a DRAM structure holds one *bit* of information (a "1" or a "0"). When the capacitor has charge stored in it, the bit is a "1," and when there is no charge stored the bit is a "0." Flash memory works in a roughly similar manner.

4.6.2 Energy stored in capacitors

Capacitors store electrical energy. Anyone who has worked with electronic equipment long enough has verified this one painful way or another.[vii] If the plates of a charged capacitor are connected to a conducting object, the capacitor will transfer charge from one plate to another until it is discharged. This is often seen as a "spark" if the capacitor was charged to a high enough voltage. Given that humans are reasonably good conductors at high voltages, this can be a problem.

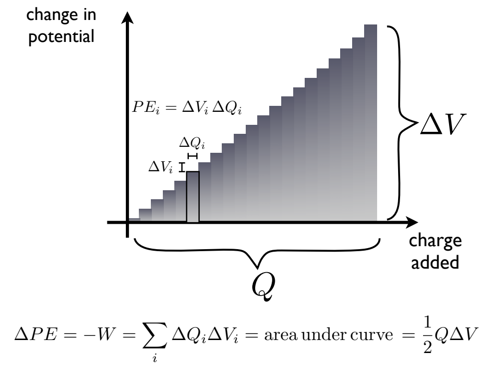


Figure 4.15: *Each bit of charge ΔQ_i transferred through a voltage ΔV_i contributes a bit of potential energy $PE_i = \Delta V_i \Delta Q_i$. Summing all those contributions to get the total energy stored is the same as finding the total area of the shaded region. If we make ΔV_i and ΔQ_i tiny enough, the area is basically a triangle, and in total $PE = \frac{1}{2}Q\Delta V$.*

Charged capacitors store energy, and that energy is the work required to move the charge onto the plates. If a capacitor is initially uncharged (both plates neutral), very little work is required to move a charge ΔQ from one plate to another across the separation d. As soon as this charge is moved, however, a potential difference $\Delta V = \Delta Q/C$ appears between the plates. This potential difference means that work must be done to move additional charges onto the plates. Combining what we know so far, and assuming a constant electric field between the plates, the work that needs to be done to move the first bit of charge ΔQ has to be:

$$\begin{aligned}
\Delta PE &= -\Delta W && (4.27)\\
&= \Delta Q \cdot E\Delta x && (4.28)\\
&= \Delta Q \cdot E\ d && (4.29)\\
&= \frac{1}{\epsilon_0}\Delta Q \sigma_E d && (4.30)
\end{aligned}$$

But we know that $\sigma_E = \frac{\Delta Q}{A}$, and thus $\Delta Q = \sigma_E A$, which simplifies things:

$$\Delta PE = \Delta Q \Delta Q \frac{d}{A\epsilon_0} \tag{4.31}$$

Since $C = \frac{\epsilon_0 A}{d}$ for our parallel plate capacitor,

$$\Delta PE = \frac{(\Delta Q)(\Delta Q)}{C} \tag{4.32}$$

[vii] I once burned a small hole in my thumb by accidentally discharging a high-voltage capacitor across it while repairing a TV, for example. Capacitors can store dangerous amounts of energy if released at the wrong time!

If we keep doing this with more and more ΔQs, until we build up the total charge Q, we can find the total work. As illustrated in Fig. 4.15, each little bit of charge ΔQ_i adds a bit of potential energy $\Delta V_i \Delta Q_i$. If we sum up all those contributions, we are really just finding the shaded area of the triangle on the graph. The area of a triangle is just $\frac{1}{2}$(base)(height), so the total change in potential energy is just:[viii]

$$|W| = |\Delta PE| = \frac{1}{2} Q \Delta V \tag{4.33}$$

Remember that $Q = C\Delta V$ must still be true, so we can write the energy stored in the capacitor in three different ways, as shown below (noting that energy stored = work done). For example, you can verify that a $5\,\mu\text{F}$ capacitor charged with a $120\,\text{V}$ source stores $3.6\,\text{mJ}$ ($3.6 \times 10^{-3}\,\text{J}$).

Energy stored in a capacitor:

$$\text{Energy stored} = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2 = \frac{Q^2}{2C} \tag{4.34}$$

Remember that the units of energy are **Joules**.

Is there an analogy for electrical energy storage? One way to store *gravitational energy* is simply to pump a large mass m of water up to a height Δy, see Figure 4.16. Releasing the water at a later time releases the stored potential energy $mg\Delta y$, which could be used to, *e.g.*, rotate a turbine. In fact, this is one way to store excess energy generated at off-peak times in power plants for later reclamation.

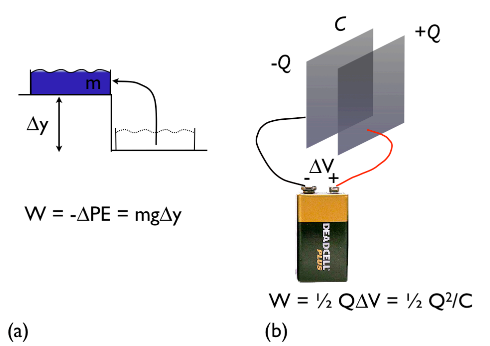


Figure 4.16: **(a)** *Raising a mass m of water to a height Δy above the ground stores an energy $mg\Delta y$.* **(b)** *Charging a capacitor C with a potential difference ΔV stores an energy $\frac{1}{2} Q\Delta V = \frac{Q^2}{2C}$.*

4.6.3 Capacitors as Circuit Elements

Now that we know about a second circuit element, we can begin to make some simple circuits. As you might have gathered above, capacitors are often used in electrical circuits as energy-storage

[viii] This is a bit of a hand-waving derivation, but it doesn't require any calculus like the more rigorous version does.

devices. As we will find out later, they can also used to filter out high- and low-frequency signal selectively. The circuit diagram symbol for a capacitor is a reminder of the parallel plate geometry:

Circuit diagram symbol for a capacitor: —| |—

What can we do only knowing about two circuit components, capacitors and batteries? Well, we can hook up a capacitor to a battery, as shown in Fig. 4.17

What does this circuit do? The moment we connect the battery to the capacitor, charges will start to flow from one plate to another for time, until both plates are fully charged. Fully charged means that the potential difference between the two plates is the same as that at the battery terminals, ΔV. After that ... nothing. The capacitor will just happily store these charges. If the capacitor is disconnected from the battery, the charges will remain on the two plates since they have no path to escape. The capacitor stays charged, thereby storing energy, so long as it is truly isolated. If one of the plates had a path to ground, for instance, the charges would leak away *via* this ground connection, and the energy would dissipate. In a rough sense, FLASH memory works by storing charges on very tiny, isolated conducting plates.

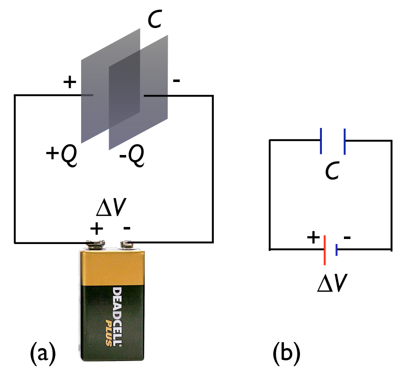


Figure 4.17: **(a)** *A parallel plate capacitor of value C connected to a battery supplying a voltage difference ΔV* **(b)** *Circuit diagram for this configuration.*

We cannot do very much with only capacitors and batteries, but we will remedy this in subsequent chapters. For now, there are a few more things we can figure out about capacitors.

4.6.4 Combinations of Capacitors

Two or more capacitors can be combined in circuits in many possible ways, but most reduce to two simple configurations: **parallel** and **series**. Two capacitors in series or in parallel can be reduced to a single equivalent capacitance, and more complicated arrangements can be viewed as combinations of series and parallel capacitors.

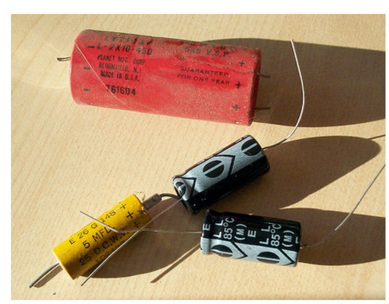


Figure 4.18: *A picture of several common types of capacitors.*[18]

4.6.4.1 Parallel Capacitors

Capacitors are manufactured with standard values, and by combining them in different ways, any non-standard value of capacitance can be realized. Figure 4.19 shows a parallel arrangement of capacitors. The left plate of each capacitor is connected by a wire (black lines) to the positive terminal of a battery, while the right plate of each capacitor is connected to the negative terminal of the battery.[ix] This means that **the capacitors in parallel both have the same potential difference ΔV across them,** the voltage supplied by the battery.

When the capacitors are first connected, electrons leave the positive plates and go to the negative plates until equilibrium is reached - when the voltage on the capacitors is equal to the voltage of the battery. The internal (chemical) energy of the battery is the source of energy for this transfer. In this configuration, both capacitors charge independently, and the total charge stored is the sum of the charge stored in C_1 and the charge stored in C_2. We can write the charge on the capacitors using Equation 4.24:

$$\begin{aligned} Q_1 &= C_1 \Delta V \\ Q_2 &= C_2 \Delta V \\ Q_{\text{total}} &= Q_1 + Q_2 = C_1 \Delta V + C_2 \Delta V = (C_1 + C_2)\,\Delta V \end{aligned}$$

What this equation shows is that **two capacitors in parallel behave as** *one single capacitor* with a value of C_1+C_2. In other words, "capacitors add to each other in parallel." We call C_1+C_2 the *"equivalent capacitance"*, $C_{\text{eq}}=C_1+C_2$

Two Capacitors in Parallel:

$$C_{\text{eq}} = C_1 + C_2 \tag{4.35}$$

Three or More Capacitors in Parallel:

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \ldots \tag{4.36}$$

The key point for capacitors in parallel is that **the *voltage* on each capacitor is the same**. One way to see this is that they are both connected to the battery by the same perfect wires, so they pretty much *have* to have the same voltage. This is true in general, as we will find out, so long as we have perfect textbook wires. It follows readily that **the equivalent capacitance of a parallel combination is always more than either of the individual capacitors**.

4.6.4.2 Series Capacitors

Figure 4.20a shows the second simple combination, two capacitors connected in *series.* **For series capacitors, the magnitude of charge is be the same on all plates.** Consider the left-most plate of C_1 and right-most plate of C_2 in Figure 4.20. Since they are connected directly to the battery, they must have the same magnitude of charge, $+Q$ and $-Q$ respectively.

Since the middle two plates (the right plate of C_1 and the left plate of C_2) are not connected to the battery at all, *together they must have no net charge.* On the other hand, the left and right

[ix] In circuit diagrams like these, the wires are assumed to be perfect.

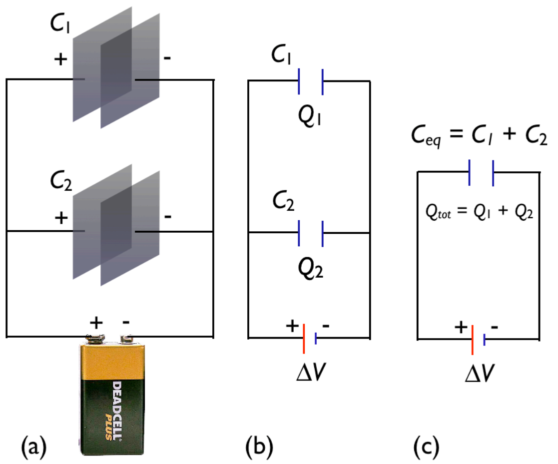


Figure 4.19: **(a)** *A parallel connection of two capacitors to a battery* **(b)** *The circuit diagram for the parallel combination.* **(c)** *The potential differences across the capacitors are the same, and the equivalent capacitance is* $C_{\rm eq} = C_1 + C_2$

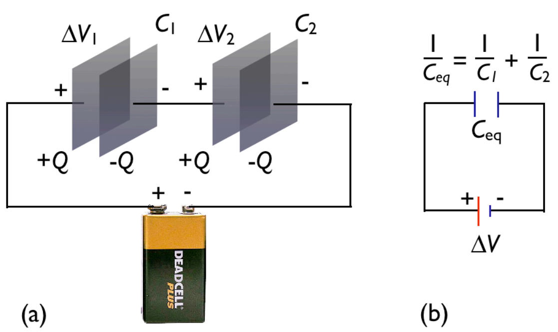


Figure 4.20: **(a)** *A series combination of two capacitors. The charges on the capacitors are the same.* **(b)** *Circuit diagram corresponding to (a). The equivalent capacitance be calculated from the reciprocal relationship,* $\frac{1}{C_{\rm eq}} = \frac{1}{C_1} + \frac{1}{C_2}$

plates of the *same capacitor* have to have the same magnitude of charge, so this means *all plates have a charge of either* $+Q$ *or* $-Q$ *stored on them.* **All of the right plates have charge** $-Q$**, and all the left plates have a charge** $+Q$

Can we reduce this series combination to a single equivalent capacitor, like we did for the parallel case? Sure, with a little math. A single capacitor equivalent to the series capacitors, Figure 4.20b, must have a charge of $+Q$ on its right plate, and $-Q$ on its left plate, so the total charge stored is still $\pm Q$ on each plate.. Further, it must have a potential difference equal to that of the battery, ΔV. Using Equation 4.24:

$$\Delta V = \frac{Q}{C_{\rm eq}} \tag{4.37}$$

We can also apply Equation 4.24 to each of the individual capacitors:

$$\Delta V_1 = \frac{Q}{C_1} \qquad\qquad \Delta V_2 = \frac{Q}{C_2} \tag{4.38}$$

Conservation of energy requires that all of the potential difference of the battery ΔV **be**

"used up" somewhere. Since our wires are assumed to be perfect, the only place the potential can go is onto the capacitors. Therefore, for the series case the voltage on C_1 and C_2 must together total that of the battery:

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{4.39}$$

This, combined with Equations 4.37 and 4.38, gives us:

$$\Delta V = \frac{Q}{C_{\text{eq}}} = \frac{Q}{C_1} + \frac{Q}{C_2} \tag{4.40}$$

Canceling the Q's, we can come up with the equivalent capacitance for series capacitors:

Two Capacitors in Series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{4.41}$$

Three or More Capacitors in Series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \dots \tag{4.42}$$

It follows that **the equivalent capacitance of a series combination is always less than either of the individual capacitors**. The key point for capacitors in series is that **the *charge* on each capacitor is the same, and the same as the charge on the equivalent capacitor.**

What to do for more complex combinations of capacitors?

1. **Combine** capacitors that are in parallel or series in to single equivalent capacitors, using (4.35) and (4.41).
2. **Parallel** capacitors all have the same potential difference ΔV across them.
3. **Series** capacitors all have the same charge Q, which is the same as the charge on their equivalent capacitor.
4. **Redraw** the circuit after every combination.
5. **Repeat** the first two steps until there is only equivalent one capacitor left.
6. **Find the charge** on this equivalent capacitor using (4.24).
7. **Reverse** your steps one by one to find the charge and voltage drop on each equivalent capacitor along the way, until you recreate the original diagram.

4.6.4.3 Example of a complex capacitor combination

The easiest way to see how one can use the rules for series and parallel capacitors to reduce *any* complex combination of capacitors to a single equivalent capacitor is by example. For example, consider the combination of capacitors in Figure 4.21 below.

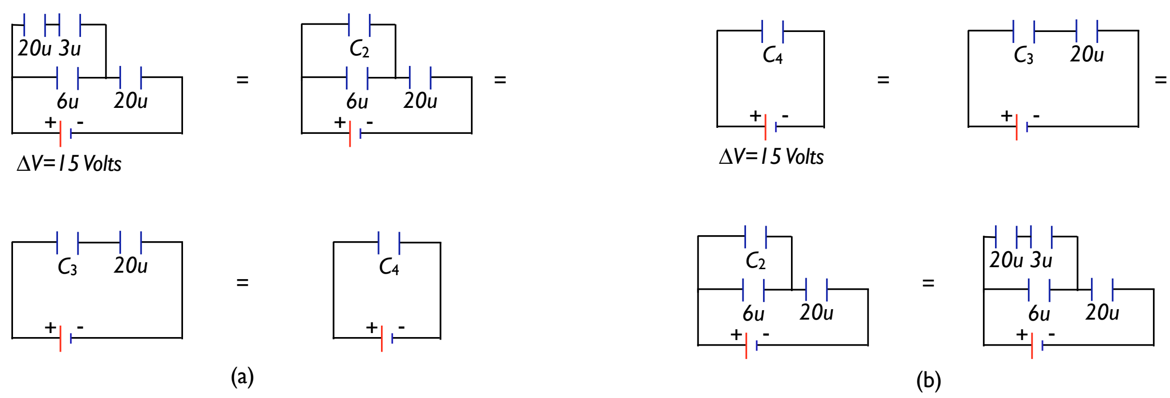


Figure 4.21: (a) *Reducing the complex combination to a single equivalent capacitor.* **(b)** *Working backwards to find the charge on each capacitor.*

Finding the equivalent capacitor First, we notice from Figure 4.21a that the only purely series or parallel combination to start with is the $20\,\mu\text{F}$ and $3\,\mu\text{F}$ capacitors in series. We can combine those into an equivalent capacitance, C_2, using Equation 4.41:

$$\frac{1}{C_2} = \frac{1}{20\,\mu\text{F}} + \frac{1}{3\,\mu\text{F}} \tag{4.43}$$

$$C_2 = \frac{1}{\frac{1}{20\,\mu\text{F}} + \frac{1}{3\,\mu\text{F}}} = \frac{3 \cdot 20}{3 + 20} \tag{4.44}$$

$$C_2 = 2.6\,\mu\text{F} \tag{4.45}$$

Redraw the circuit to reflect this change, and we arrive at the second diagram in Figure 4.21a. Now we have the equivalent capacitor C_2 purely in parallel with the $6\,\mu\text{F}$ capacitor. Using Equation 4.35, we can combine those two into another equivalent capacitance C_3:

$$C_3 = C_2 + 6\,\mu\text{F} = 8.6\,\mu\,\text{F} \tag{4.46}$$

Redraw the circuit, and we arrive at the third diagram in Figure 4.21a. Now we only have C_3 in parallel with $20\,\mu\text{F}$ left, which we can now combine into a final overall equivalent capacitance C_4.

Again using Equation 4.41, we have

$$\frac{1}{C_4} = \frac{1}{C_3} + \frac{1}{20\,\mu\text{F}} \quad \text{or} \quad C_4 = 6.02\,\mu\text{F} \tag{4.47}$$

So the equivalent capacitance of the four capacitors we started with is about $6\,\mu$F.

Finding the charge on each capacitor Now we have to work *backwards* from our single equivalent capacitor and deduce the charge and voltage on each individual capacitor, following Figure 4.21b. First, we know the charge on C_4, the equivalent capacitor, once we know the value of C_4 (above) and ΔV(given):

$$Q_4 = C_4\Delta V = (6.02\mu\,\text{F})(15\,\text{V}) = 90.3\,\mu\text{C}$$

Now C_3 and the $20\,\mu$F are in series. Two series capacitors must both have the *same charge but different voltages*. Further, the charge on series capacitors is *the same as the charge on the equivalent capacitor*. Therefore, both the $20\,\mu$F and C_3 have to have the same charge that C_4 has. So

$$Q_3 = Q_{20\,\mu} = Q_4 = 90.3\,\mu\text{C}$$

Now we get to the third diagram. We know that the $6\,\mu$F and C_2 *together* have Q_4 worth of charge. *Parallel capacitors both have the same voltage, but different charges.* If we call the voltage on these two capacitors V, the charge on the $6\,\mu$F is $6\,\mu\text{F}{\cdot}V$, and the charge on C_2 is $C_2 \cdot V$, which gives us Q_4:

$$Q_4 = 90.3\,\mu\text{C} = (C_2)V + (6\,\mu\text{F})V$$

Since $C_2 = 2.6\,\mu$F, this gives $V = 10.47$ Volts, so

$$Q_{6\,\mu} = (6\,\mu\text{F})V = 62.9\,\mu\text{C} \quad \text{and} \quad Q_2 = (C_2)V = 27.4\,\mu\text{C}$$

Note that the voltage V and the voltage on the lower $20\,\mu$F capacitor must together equal the battery voltage, so the voltage on the lower $20\,\mu$F capacitor must be $15.00 - 10.47 = 4.53$ V. Now for the last step. You now know the charge on C_2, which is the same as the total charge on the $20\,\mu$F and $3\,\mu$F capacitors. Since they are in series, they both have the same charge, and the both have to have Q_2. Thus $Q_{3\,\mu} = Q_{20\,\mu} = 27.4\,\mu$C. We can find the voltage on each by noting that

$$V_{3\,\mu} = \frac{Q_{3\,\mu}}{C_{3\,\mu}} = 9.13\,\mathrm{V} \quad \text{and} \quad V_{20\,\mu} = \frac{Q_{20\,\mu}}{C_{20\,\mu}} = 1.37\,\mathrm{V}$$

Further, we know that that $V_{3\,\mu} + V_{20\,\mu}$ has to equal the voltage on the equivalent capacitor C_2, *viz.* 10.47 V. So, in the end, the charge on the 20 µF is the same as that on the effective capacitance, the charge on 20 µF and the 3 µF are the same, and the charge on the 6 µF is about halfway in between either of those. The charge, capacitance, and voltages are summarized in Table 4.1.

Table 4.1: *Equivalent capacitances, charges, and voltages for Figure 4.21.*

Capacitor [µF]	Charge [µC]	Voltage [V]
top 20µ	27.4	1.37
$C_2 = 2.6$	27.4	10.47
$C_3 = 8.6$	90.3	10.47
$C_4 = 6.02$	90.3	15
6µ	63	10.47
3µ	27.4	9.13
lower 20	90	4.53

4.6.5 Capacitors with (non-conducting) stuff inside

What if we separate the plates of our parallel plate capacitor with something other than air? As you might expect, this changes the capacitance. A **dielectric** is another name for an insulating material (like rubber, or most ceramics and plastics). When we put a dielectric between the plates of our capacitor, the capacitance increases. If the dielectric totally fills the region between the plates, the increase is proportional to a constant κ, the dielectric constant. We note that sometimes you will see the dielectric constant is written as ϵ_r rather than κ, but it is the same thing.

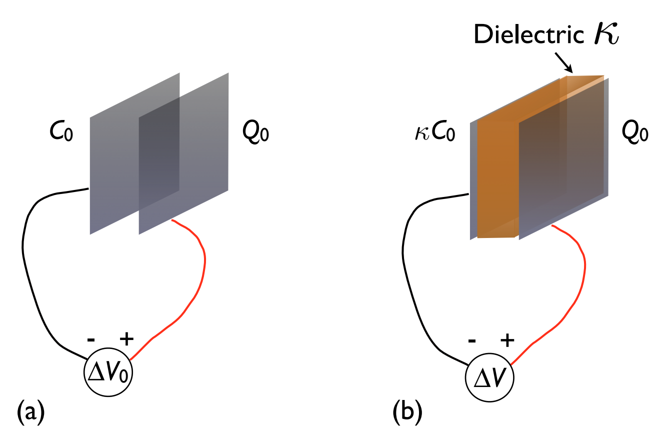


Figure 4.22: **(a)** *With air between the plates, the voltage across the capacitor is ΔV_0, the capacitance is C_0, and the charge is Q_0.* **(b)** *With a dielectric inside, the charge is still Q_0, but the voltage and capacitance change.*

Figure 4.22 shows the effect of a dielectric inserted in a parallel plate capacitor. Without the dielectric, we know that

$\Delta V_0 = Q_0/C_0$. If we now insert the dielectric, the voltage is *reduced* to:

$$\Delta V = \frac{\Delta V_0}{\kappa} = \frac{\Delta V_0}{\epsilon_r} \tag{4.48}$$

What happens is that part of the potential difference originally across the plates of the capacitor is now spent on the dielectric itself. Being an insulator, the dielectric can support regions of charge, unlike a conductor. When it is inserted into the capacitor, the part of the dielectric near the $+Q_0$ plate builds up a partial *negative* charge in response, and the part near the $-Q_0$ plate builds up a partial *positive* charge. This has the effect of "canceling" part of the $+Q$ and $-Q$ charges on the plates, so the battery supplies more charges to compensate! This goes on until an equilibrium is reached, and the dielectric can steal no more charge.

In the end, since the dielectric "steals" a bit of extra charge, the capacitor with a dielectric inside stores more charge than the capacitor without the dielectric. The total amount of charge present, including the "extra" bit "stolen" by the dielectric, is proportional to κ, so the capacitance of the new structure is *increased* by a factor of κ:

$$C = \frac{Q_0}{\Delta V} = \frac{\kappa Q_0}{\Delta V_0} = \frac{\epsilon_r Q_0}{\Delta V_0} \tag{4.49}$$

For a parallel plate capacitor, this means:

Parallel plate capacitor with a dielectric between the plates:

$$C = \kappa \epsilon_0 \frac{A}{d} = \epsilon_r \epsilon_0 \frac{A}{d} \tag{4.50}$$

the dielectric *increases* the capacitance by a factor κ, the dielectric constant. The dielectric constant is also sometimes called ϵ_r.

This is not an insignificant effect - the value of κ can range from ~ 1 for air to a few thousands – adding a good dielectric layer can increase the amount of charge stored by hundreds or thousands! For vacuum, the value is exactly 1, so Equation 4.50 just reduces to Equation 4.26. The value of κ is **always** greater than 1 ($\kappa > 1$), so the capacitance always increases when a dielectric is included. Why this is true microscopically is treated in the next section. Table 4.2 lists the dielectric constants for a few common materials.

This trick for making larger capacitors does not work indefinitely. Every dielectric has a "dielectric strength," the maximum tolerated value of the electric field inside that particular material. If the electric field inside the dielectric exceeds this value, the dielectric breaks down, which usually means a spark jumps across (or through) it. Exceeding the dielectric strength is a catastrophic failure, and usually results in "magic smoke" being released from the device in question.

Table 4.2: *Dielectric constants of materials at $T_0 = 20° C$*[19]

Material	κ	Material	κ
Vacuum	1		
Air	1.00054	Teflon®	2.1
Polyethylene	2.25	Paper	3.5
Silicon dioxide	3.7	Pyrex	4.7
Rubber	7	Methanol	30
Silicon	11.68	Water (distilled)	80.1
$SrTiO_3$	310	$BaTiO_3$	~ 1000

4.7 Dielectrics in Electric Fields

Somehow or another, dielectrics inside a capacitor are able to dramatically increase the amount of charge that can be stored and *decrease* the voltage across the capacitor. Our explanation so far is that the dielectric itself partly charges, which both increases the amount of charge stored and decreases the net voltage. How does this work? In order to understand what is really going on, we have to think a bit about the microscopic nature of the dielectric.

The dielectric itself contains a large number of atomic nuclei and electrons, but overall there are equal numbers of positive nuclei and electrons to make the dielectric overall neutral. We have said that charges in insulators are not mobile, so electrons and nuclei remain bound. What, then, are the induced charges in the dielectric? Despite being bound, both electrons and nuclei in a dielectric can move very slightly without breaking their bonds. Electrons will attempt to move in the direction opposite the electric field between the plates, and nuclei will attempt to move in the opposite direction. As a result, tiny dipoles are formed inside the dielectric, which will be aligned along the direction of the electric field (see Figure 4.23). Random thermal motion of the atoms or molecules will limit the degree of alignment to an extent. In most materials the degree of alignment and the induced dipole strength are directly proportional to the external electric field. Essentially, an electric field induces a charge separation within the atom or molecule.

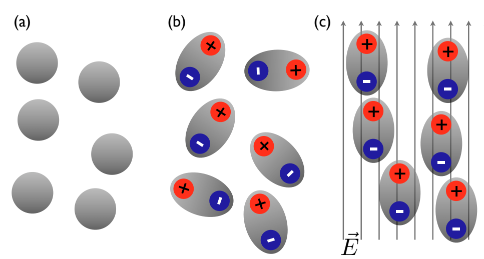


Figure 4.23: **(a)** *Atoms and many nuclei have no net charge separation without an electric field present.* **(b)** *Some "polar" molecules have a permanent electric dipole moment. Usually, these moments are oriented randomly from molecule to molecule, and the net moment is zero.* **(c)** *In an electric field, non-polar molecules can have an induced dipole moment, due to electrons and nuclei wanting to move in opposite directions in response to the field. Permanent dipoles remain bound, but can move or rotate slightly to align with the electric field. Either way, an overall dipole moment results.*

Some molecules have a natural charge separation or dipole moment already built in, so-called *polar* molecules such as water or CO_2. In these kinds of dielectrics, the built-in dipole moments are usually randomly aligned, and cancel each other out overall. An electric field exerts a torque on the dipoles, which tries to orients them along the electric field. Once again, random thermal motion works against this alignment, but the overall effect of the electric field is a net alignment, the degree of which is proportional to the applied electric field. Thus, in both polar and non-polar dielectrics, there is a net orientation of dipoles when an electric field is applied. The net dipole strength is far stronger in polar materials, and in the rest of the discussion below we will assume that our dielectric is made of polar molecules.

Now, what happens when we place our dielectric between two conducting plates? With no voltage applied between the plates, there is no electric field, and the tiny dipoles are randomly oriented, Fig. 4.24b. Once a voltage is applied to the plates, a constant electric is created between them, which serves to align the dipoles, Fig. 4.24c. The net alignment of dipoles within a dielectric leads to the surfaces of the dielectric being slightly charged, Fig. 4.24. Within the bulk of the dielectric, dipoles will be aligned head-to-tail, and their electric fields will mostly cancel (Fig. 4.24a). At the surfaces of the dielectric, however, there will be an excess of positive charge on one side, and an excess of negative charge on the other. In this situation, the dielectric is said to be *polarized*. The dielectric is still electrically neutral on the whole, an equal number of positive and negative charges still exist, they have only separated due to the applied electric field.

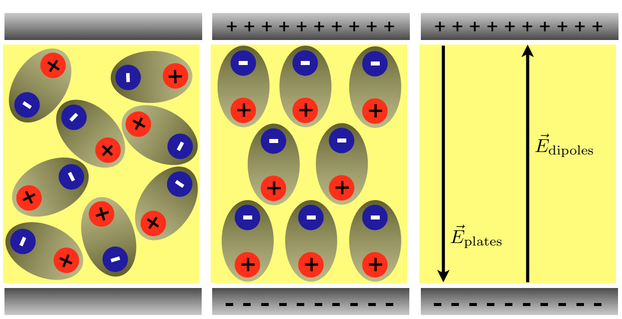


Figure 4.24: **(a)** *When no voltage is applied between the plates, the polar molecules align randomly, and there is no net dipole moment.* **(b)** *A voltage applied across the plates creates an electric field, which aligns the molecules.* **(c)** *The electric field from the voltage applied across the plates is partially canceled by the field due to the aligned dipoles.*

These surface charges from the aligned dipoles look just like sheets of charge, in fact. This is the origin of our earlier statement that the dielectric picks up an induced charge on its surface – the part of the dielectric near the positive plate *does* build up a partial *negative* charge, and the part near the negative plate *does* build up a partial *positive* charge. What we missed in our initial analysis was the fact that in reality we are *aligning charges throughout the dielectric, even though only the surfaces have a net charge.* Not only are we storing energy in the surface charges, we are also storing energy by creating the aligned configuration of the dipoles! It took energy to orient them, so keeping them aligned is in a sense storing energy for later release. In a sense, we actually store energy in the whole volume of the dielectric, not just at the surfaces.

The electric field due to these effective sheets of charge is *opposite* that of the applied electric field, and thus the total electric field – the sum of the applied and induced field – is smaller than if there were no dielectric. Thus, the dielectric reduces both the applied voltage and the electric field. The electric field due to the oriented dipoles inside the dielectric is usually proportional to the *total* electric field they experience:

$$E_{\text{dipoles}} = \chi_E E_{\text{total}} \tag{4.51}$$

where the constant of proportionality χ_E is called the *electric susceptibility.* It represents the relative strength of the dipoles within the material, or more accurately, how easily a material polarizes in response to an electric field. The total electric field the dipoles experience is not just the field due to voltage applied across the plates, but must also include *the field of all the other dipoles* as well:

$$E_{\text{total}} = E_{\text{plates}} - E_{\text{dipoles}} \tag{4.52}$$

$$E_{\text{total}} = E_{\text{plates}} - \chi_E E_{\text{total}} \tag{4.53}$$

$$(1+\chi_E)E_{\text{total}} = E_{\text{plates}} \tag{4.54}$$

$$\Longrightarrow E_{\text{total}} = \frac{1}{1+\chi_E} E_{\text{plates}} \tag{4.55}$$

Thus, the field inside the plates is *reduced* by a factor $\frac{1}{1+\chi_E}$ by the presence of the dielectric (χ_E is always positive). We already know that for a parallel plate capacitor, $\Delta V = Ed$, where d is the spacing between the plates, so we can also readily find the effect of the dielectric on the voltage between the plates:

$$\Delta V_{\text{total}} = \frac{1}{1+\chi_E} E_{\text{plates}} d = \frac{1}{1+\chi_E} \Delta V_0 = \frac{\Delta V_0}{\kappa} \tag{4.56}$$

here we again use ΔV_0 for the voltage on the plates without the dielectric. This result agrees precisely with Eq. 4.48, if we make the substitution $\kappa = 1+\chi_E$, as we have in the last term in the equation above. We can go further and calculate the capacitance, just as we did for Eq. 4.50:

$$C = (1+\chi_E)\epsilon_0 \frac{A}{d} = \kappa\epsilon_0 \frac{A}{d} = \kappa C_0 \tag{4.57}$$

where C_0 is the capacitance without the dielectric present. Thus, our "dielectric constant" is simply related to the dielectric susceptibility, the ability of the dielectric to polarize in response to an electric field. This makes sense in a way – the more easily polarized the dielectric, the more easily it affects the capacitance. Also, since $\kappa = 1$ for vacuum, $\chi_E = 0$, which also makes sense as the vacuum is not polarizable (so far as we know). The result we obtain using this more sophisticated model is exactly the same as earlier, but now we have a plausible *microscopic* origin for the effect of dielectrics in capacitors, and we know *why* the electric field and voltage are reduced, and the capacitance increased.

4.8 Quick Questions

1. Capacitors connected in parallel *must always* have the same:

◯ Charge
◯ Potential difference
◯ Energy stored
◯ None of the above

2. An ideal parallel plate capacitor is completely charged up, and then disconnected from a battery. The plates are then pulled a small difference apart. What happens to the capacitance, C, and charge stored, Q, respectively?

◯ decreases; increases
◯ increases; decreases
◯ decreases; stays the same
◯ stays the same; decreases

3. An isolated conductor has a surface electric potential of 10 Volts. An electron on the surface is moved by 0.1 m. How much work must be done to move the charge? (e is the electron charge.)

◯ $1e$ Joules
◯ $0.1e$ Joules
◯ $10e$ Joules
◯ 0

4. An electron initially at rest is accelerated through a potential difference of 1 V, and gains kinetic energy KE_e. A proton, also initially at rest, is accelerated through a potential difference of -1 V, and gains kinetic energy KE_p. Which of the following must be true?

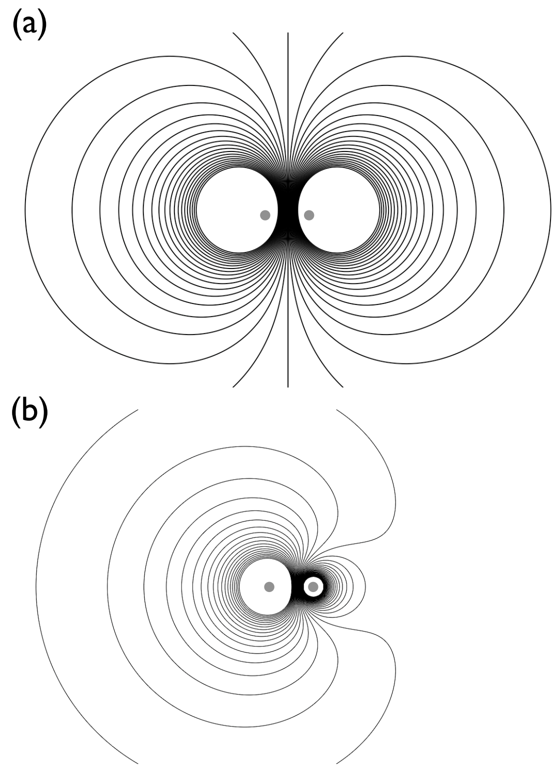
◯ $KE_e < KE_p$
◯ $KE_e = KE_p$
◯ $KE_e > KE_p$
◯ not enough information

5. A parallel plate capacitor is shrunk by a factor of two in every dimension – the separation between the plates, as well as the plates' length and width are all two times smaller. If the original capacitance is C_0, what is the capacitance after all dimensions are shrunk?

◯ $2C_0$
◯ $\frac{1}{2}C_0$
◯ $4C_0$
◯ $\frac{1}{4}C_0$

6. The figure at right shows the **equipotential** lines for two different configurations of two charges (the charges are the solid grey circles). Which of the following is true?

◯ The charges in (a) are of the same sign and magnitude, the charges in (b) are of the same sign and different magnitude.
◯ The charges in (a) are of opposite sign and of the same magnitude, the charges in (b) are of the opposite sign and different magnitude.
◯ The charges in (a) are of the same sign and magnitude, the charges in (b) are of the opposite sign and the same magnitude.
◯ The charges in (a) are of the opposite sign and different magnitude, the charges in (b) are of the same sign and different magnitude.



7. A capacitor with air between its plates is charged to 120 V and then disconnected from the battery. When a piece of glass is placed between the plates, the voltage across the capacitor drops to 30 V. What is the dielectric constant of the glass? (Assume the glass completely fills the space between the plates.)

◯ 4
◯ 2
◯ 1/4
◯ 1/2

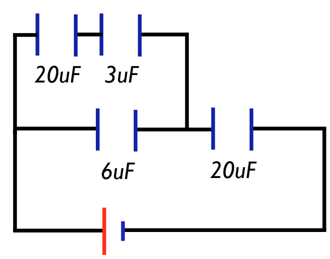
4.9 Problems

1. Electrons in a TV tube are accelerated from rest through a potential difference of 2.00×10^4 V from an electrode towards the screen 25.0 cm away. What is the magnitude of the electric field, if it assumed to be constant over the whole distance? You may assume that the electron moves parallel to the electric field at all times.

2. A proton moves 1.5 cm parallel to a uniform electric field of $E = 240$ N/C. How much work is done by the field on the proton?

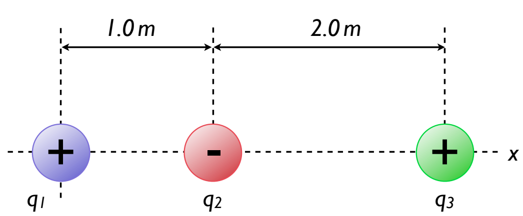
3. It takes 3×10^6 J of energy to fully recharge a 9 V battery. How many electrons must be moved across the 9 V potential difference to fully recharge the battery?

4. What is the effective capacitance of the four capacitors shown at right?



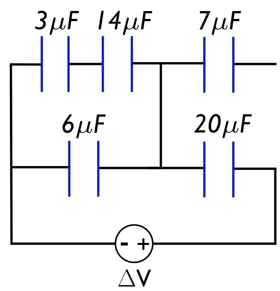
5. Calculate the speed of a proton that is accelerated from rest through a potential difference of 104 V.

6. A proton at rest is accelerated parallel to a uniform electric field of magnitude 8.36 V/m over a distance of 1.10 m. If the electric force is the only one acting on the proton, what is its velocity in km/s after it has been accelerated over 1.10 m?

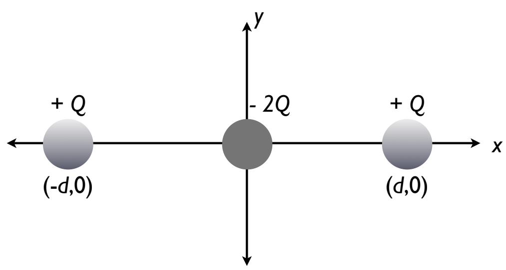


7. Three charges are positioned along the x axis, as shown at left. All three charges have the same *magnitude* of charge, $|q_1| = |q_2| = |q_3| = 10^{-9}$ C (note that q_2 is negative though). What is the total **potential energy** of this system of charges? We define potential energy zero to be all charges infinitely far apart.

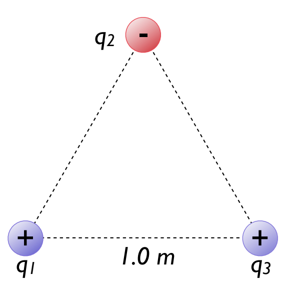
8. Two identical point charges $+q$ are located on the y axis at $y = +a$ and $y = -a$. What is the electric potential for an arbitrary point (x,y)?



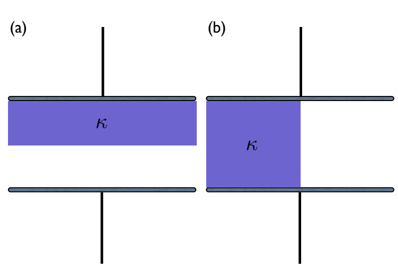
9. What is the equivalent capacitance for the five capacitors at left (approximately)?



10. The charge distribution shown is referred to as a linear quadrupole. What is the electric potential at a point on the y axis?



◯ **11.** Three charges are arranged in an equilateral triangle, as shown at left. All three charges have the same *magnitude* of charge, $|q_1| = |q_2| = |q_3| = 10^{-9}\,\mathrm{C}$ (note that q_2 is negative though). What is the total **potential energy** of this system of charges? Take the zero of potential energy to be when all charges are infinitely far apart.



◯ **12.** A parallel plate capacitor has a capacitance C when there is vacuum between the plates. The gap between the plates is half filled with a dielectric with dielectric constant κ in two different ways, as shown below. Calculate the effective capacitance, in terms of C and κ, for both situations. *Hint: try breaking each situation up into two equivalent capacitors.*

4.10 Solutions to Quick Questions

1. Potential difference.

2. Decreases; stays the same. The capacitance of a parallel plate capacitor is $C = \frac{\epsilon_0 A}{d}$. If we pull the plates apart and increase the spacing d, the capacitance decreases. Nothing happens to the charges already on the plates if the capacitor is disconnected, though – they have no where to go!

3. 0. The charge is moved along the surface of the conductor, which is always at the same electric potential. Since the charge has moved through no net potential difference, no work has been done.

4. $KE_e = KE_p$**.** All of the potential energy gained by the proton and electron has to be converted into kinetic energy, and both particles lose the same potential energy by moving through the potential difference. Both particles have equal but opposite charges and move through equal and opposite potential differences – since the negatively charged electron moves through a positive potential difference, and the positively charged proton moves through a negative potential difference, the net loss of potential energy $q\Delta V$ is the same. Therefore, the amount of kinetic energy gained by each particle is the same. Since both particles started at rest, their resulting kinetic energies have to be the same. The *velocity* of the electron will be much greater, however, owing to its smaller mass – recall that kinetic energy is $\frac{1}{2}mv^2$.

5. $\frac{1}{2}C_0$. The capacitance of a parallel plate capacitor whose plates have an area A and a separation d is $C = \frac{\epsilon_0 A}{d}$. If we imagine the plates to be rectangular of length l and width w, the area A is $A = lw$. Let the capacitance of the capacitor be $C_0 = \frac{\epsilon_0 lw}{d}$ *before* dimensions are shrunk. Once we reduce the length, width, and separation by two times, we have:

$$C = \frac{\epsilon_0 \left(\frac{1}{2}l\right)\left(\frac{1}{2}w\right)}{\left(\frac{1}{2}d\right)} = \frac{\epsilon_0 \frac{1}{2} lw}{d} = \frac{1}{2}C_0$$

It is easy to prove that if we chose, *e.g.,* circular plates, the answer would be the same – for any reasonable shape, the area goes down as the *square* of the dimensional decrease, while the separation just goes down as the factor itself.

6. This is probably another question most easily answered by elimination. In (a), the charges are clearly of the same magnitude, since the graph is perfectly symmetric, while in (b) the charges must be of different magnitude to explain the asymmetric graph. Therefore, the third answer cannot be correct.

In (a), the potential is constant along a vertical line separating the two charges (since there is a perfectly vertical line running halfway between the charges). This would only be true if they are of *opposite* signs. If the charges were of the same sign, there would be equipotential lines running horizontally from charge to charge. Similarly, the charges must also be of opposite sign in (b). This also rules out the first answer.

Based on similarity of (a) and (b), it must be that if (a) has charges of opposite magnitude, then so does (b). This also means that the fourth answer is out, which leaves only the second answer as a possibility. If you are still not clear on why the correct answer must be the second one, you may want to look carefully at the examples of equipotential lines in different situations presented in this chapter.

7. 4. Without the piece of glass, our capacitor has a value we'll call C. The charge stored on the capacitor is $Q = CV = 120C$ when the initial voltage is $V_{\text{initial}} = 120\,\text{V}$. The piece of glass acts as a dielectric, which increases the capacitance to κC (κ is always greater than 1).

Since the battery was disconnected, after inserting the piece of glass the total amount of charge Q stays the same - there is no source for additional charge to enter the capacitor. Now, however, the voltage V_{final} is less and the capacitance is more. We can set the initial amount of charge before inserting the glass equal to the final charge after inserting the glass, and solve for κ:

$$\begin{aligned} Q &= CV_{\text{initial}} = \kappa CV_{\text{final}} \\ &\text{or} \quad \cancel{C}V_{\text{initial}} = \kappa \cancel{C}V_{\text{final}} \\ \Longrightarrow V_{\text{initial}} &= \kappa V_{\text{final}} \\ \kappa &= \frac{V_{\text{initial}}}{V_{\text{final}}} = \frac{120}{30} = 4 \end{aligned}$$

4.11 Solutions to Problems

1. 8.00×10^4 **V/m.** In a constant electric field, the electric field, potential difference and displacement are related by:

$$\Delta V = -|\vec{\mathbf{E}}||\Delta\vec{\mathbf{x}}|\cos\theta \tag{4.58}$$

Since the displacement and electric field are parallel everywhere, $\theta = 0$, and we have just $\Delta V = E\Delta x$. We have a potential difference $\Delta V = 2 \times 10^4$ V developed over a displacement of $\Delta x = 25$ cm (0.25 m). Plugging in the numbers:

$$\Delta V = -E\Delta x \tag{4.59}$$

$$2 \times 10^4\,\text{V} = -E\,(0.25\,\text{m}) \tag{4.60}$$

$$\Longrightarrow E = -\frac{2 \times 10^4\,\text{V}}{0.25\,\text{m}} = -8.00 \times 10^4\,\text{V/m} \tag{4.61}$$

Since we want only the magnitude of the electric field, it is sufficient to write 8.00×10^4 V/m.

2. 5.8×10^{-19} **J** The work done in moving a single charge through a constant electric field is given by:

$$W = qE_x\Delta x \tag{4.62}$$

where E_x is the component of the electric field parallel to the displacement. In this case, the displacement is always parallel to the electric field, so E_x is just the total field and Δx

the displacement. Now we just plug in the numbers, remembering to put the displacement in meters:

$$\begin{aligned}
W &= qE\Delta x && (4.63)\\
&= \left(1.6\times10^{-19}\,\mathrm{C}\right)(240\,\mathrm{N/C})\,(0.015\,\mathrm{m}) && (4.64)\\
&\approx 5.8\times10^{-19}\,\mathrm{N\cdot m} = 5.8\times10^{-19}\,\mathrm{J} && (4.65)
\end{aligned}$$

In the last line we used the fact that one Joule is defined to be one Newton times one meter.

3. 2×10^{24} electrons. The energy required to charge the battery is just the amount that the potential energy of all the charges changes by. Each electron is moved through 9 V, which means each electron changes its potential energy by $-e\cdot 9\,\mathrm{V}$, where e is the charge on one electron. The total potential energy is the potential energy per electron times the number of electrons, n. Basically, this is conservation of energy: the total energy into the battery has to equal the amount of energy to move one electron across 9 V times the number of electrons.

$$\begin{aligned}
\Delta E_{in} + \Delta PE &= 0\\
3.6\times10^{6}\,\mathrm{J} + n(-e\cdot 9\,\mathrm{V}) &= 0\\
ne\cdot 9\,\mathrm{V} &= 3.6\times10^{6}\,\mathrm{J}\\
n &= \frac{3.6\times10^{6}\,\mathrm{J}}{e\cdot 9\,\mathrm{V}}\\
&= \frac{3.6\times10^{6}\,\mathrm{J}}{(1.6\times10^{-19}\,\mathrm{C})\,(9\,\mathrm{V})}\\
&= \frac{3.6\times10^{6}}{(1.6\times10^{-19})\,(9)}\\
&\approx 2\times10^{24}
\end{aligned}$$

Here we make use of the fact that Coulombs times Volts is Joules. As usual, if you just use proper SI units throughout, the units will work out on their own.

4. $6.02\,\mu$**F.** See page 129, this is the same capacitor layout!

5. 1.41×10^{5} **m/s.** When the proton is accelerated through a potential difference ΔV, it loses a potential energy of $e\Delta V$, which is converted into kinetic energy. We only need to apply conservation of energy, noting that the proton started at rest, and choosing our zero of potential energy such that the final potential energy is zero:

$$
\begin{aligned}
E_{\text{initial}} &= E_{\text{final}} \\
KE_{\text{initial}} + PE_{\text{initial}} &= KE_{\text{final}} + PE_{\text{final}} \\
0 + q\Delta V &= \frac{1}{2} m_p v_f^2 + 0 \\
\implies v_f^2 &= \frac{2q\Delta V}{m_p} \\
v_f &= \sqrt{\frac{2q\Delta V}{m_p}} = \sqrt{\frac{2 \cdot 1.6 \times 10^{-19}\,\text{C} \cdot 104\,\text{V}}{1.67 \times 10^{-27}\,\text{kg}}} \\
&\approx 1.41 \times 10^{-5}\ [\text{C} \cdot \text{V/kg}]^{\frac{1}{2}} \\
&= 1.41 \times 10^{-5}\ [\text{J/kg}]^{\frac{1}{2}} = 1.41 \times 10^{-5}\ \left[\text{kg} \cdot \text{m}^2/\text{s}^2 \cdot \text{kg}\right]^{\frac{1}{2}} \\
&= 1.41 \times 10^{-5}\,\text{m/s}
\end{aligned}
$$

The units are a bit tricky here, but remember that if you keep everything in proper SI units from the start, they will always work out ok. Remember from the definition of electrical potential that one Volt is equal to one Joule per Coulomb, $1\,\text{V} = 1\,\text{J/C}$ – it then follows that $1\,\text{C} \cdot \text{V} = 1\,\text{J}$.

6. 42.0 km/s. Of course, 42 is the answer to life, the universe, and everything.[x]

Anyway. The proton starts from rest, and hence has no kinetic energy. It is accelerated by an electric field, and thus gains kinetic energy. The kinetic energy gained must come from the electric field. A charge q moving parallel to a constant electric field E over a distance Δx changes its potential energy by:

$$\Delta PE = qE\Delta x$$

The charge on a proton is just $+e$, and E and Δx are given. The change in kinetic energy is just the final kinetic energy of the proton, since it started from rest. The gain in kinetic energy must equal the change in potential energy:

$$
\begin{aligned}
\Delta PE = PE_{\text{initial}} - PE_{\text{final}} &= -\Delta KE = -\left(KE_{\text{initial}} - KE_{\text{final}}\right) \\
eE\Delta x - 0 &= -\left(0 - \frac{1}{2} m_p v_{\text{final}}^2\right) \\
eE\Delta x &= \frac{1}{2} m_p v_{\text{final}}^2 \\
\implies v_{\text{final}}^2 &= \frac{2eE\Delta x}{m_p} \\
v_{\text{final}} &= \sqrt{\frac{2eE\Delta x}{m_p}}
\end{aligned}
$$

Plugging in what we are given ...

[x]From *Hitchhiker's Guide to the Galaxy ... there are often nerd jokes on physics exams.*

$$\begin{aligned}
v_{\text{final}} &= \sqrt{\frac{2\,(1.6\times10^{-19}\,\text{C})\,(8.36\,\text{V/m})\,(1.10\,\text{m})}{1.67\times10^{-27}\,\text{kg}}}\\
&\approx 42000\,\sqrt{\text{C}\cdot\text{V/kg}}\\
&= 42000\,\sqrt{\text{J/kg}}\\
&= 42000\,\sqrt{\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2\cdot\text{kg}}}\\
&= 42\,\text{km/s}
\end{aligned}$$

Making absolutely sure that the units work out, one should note that Coulombs times Volts is Joules, or kg$\cdot$m^2/s^2. If you always use proper SI units, it will work out though, and you won't have to remember lots of unit conversions.

7. The potential energy of a system of charges can be found by superposition, by adding together the potential energy of all *unique* pairs of charges. In this case, we have three distinct pairs of charges – (1,2), (1,3), and (2,3). The potential energy of the pair (1,2) is the electric potential that charge 2 feels due to charge 1, times charge 2:

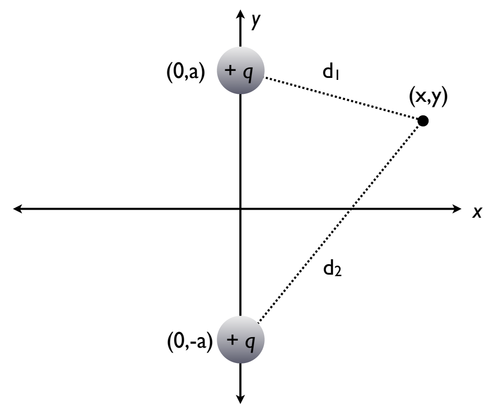
$$PE_{(1,2)} = k_e q_2 \frac{q_1}{r_{12}^2} = k_e \frac{q_1 q_2}{r_{12}^2}$$

Here r_{12} is the separation between charges 1 and 2, or just 1.0 m in this case. We do the same for the other two pairs of charges, and add all three energies together (being very careful with signs):

$$\begin{aligned}
PE_{\text{total}} &= PE_{(1,2)} + PE_{(1,3)} + PE_{(2,3)}\\
&= k_e \frac{q_1 q_2}{r_{12}} + k_e \frac{q_1 q_3}{r_{13}} + k_e \frac{q_2 q_3}{r_{23}}\\
&= k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)\\
&= \left(9\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{\left(-10^{-9}\,\text{C}\right)\left(10^{-9}\,\text{C}\right)}{1\,\text{m}} + \frac{\left(10^{-9}\,\text{C}\right)\left(10^{-9}\,\text{C}\right)}{3\,\text{m}} + \frac{\left(-10^{-9}\,\text{C}\right)\left(10^{-9}\,\text{C}\right)}{2\,\text{m}}\right]\\
&= \left(9\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left(\frac{10^{-18}\,\text{C}^2}{\text{m}}\right)\left[-1+\frac{1}{3}-\frac{1}{2}\right]\\
&= \left(9\times10^{-9}\,\text{N}\cdot\text{m}\right)\left[\frac{-7}{6}\right]\\
&\approx -1.1\times10^{-8}\,\text{J}
\end{aligned}$$

Here we used the fact that a 1 J$\equiv$1 N $\cdot$ m.

8. $\frac{k_e q}{\sqrt{x^2+(a-y)^2}} + \frac{k_e q}{\sqrt{x^2+(a+y)^2}}$. For this one, it is perhaps easier to draw ourselves a picture:
We will label the upper charge 1, and the lower charge 2. The principle of superposition tells us that we only need to find the potential at point (x, y) due to each separately, and then add the results together. First, we focus on charge 1, located at $(0, a)$. First, we need the distance



d_1 from charge 1 to the point (x, y). The horizontal distance is just x, and the vertical distance has to be $a-y$. Therefore,

$$d_1 = \sqrt{x^2 + (a-y)^2} \tag{4.66}$$

The potential due the first charge, which we'll call V_1 is then found from Eq. 4.14:

$$V_1 = \frac{k_e q}{d_1} = \frac{k_e q}{\sqrt{x^2 + (a-y)^2}} \tag{4.67}$$

The potential due to the second charge at $(0, -a)$ is found in an identical manner, only noting that the vertical distance is now $a+y$:

$$d_2 = \sqrt{x^2 + (a+y)^2} \tag{4.68}$$

$$V_2 = \frac{k_e q}{d_2} = \frac{k_e q}{\sqrt{x^2 + (a+y)^2}} \tag{4.69}$$

Finally, since potential is a scalar quantity (it has only magnitude, not direction), the superposition principle tells us that the total electric potential at point (x, y) is just the sum of the individual potentials due to charges 1 and 2:

$$V_{\text{tot}} = V_1 + V_2 = \frac{k_e q}{\sqrt{x^2 + (a-y)^2}} + \frac{k_e q}{\sqrt{x^2 + (a+y)^2}} \tag{4.70}$$

Without resorting to approximations, there isn't really a much more aesthetically pleasing form for this one.

9. First of all, we should notice that the $7\,\mu$F capacitor has nothing connected to its right wire, so it can't possibly be doing anything in this circuit. We can safely ignore it. Next, the $3\,\mu$F and $14\,\mu$F capacitors are simply in series, so we can readily find their equivalent capacitor:

$$C_{\text{eff},3\&14} = \frac{(3\,\mu\text{F})(14\,\mu\text{F})}{(3\,\mu\text{F}) + (14\,\mu\text{F})} \approx (2.65\,\mu\text{F})$$

This $2.65\,\mu$F effective capacitor is purely in parallel with the $6\,\mu$F capacitor. We can therefore just add the two capacitances together and come up with an equivalent capacitance for the 3, 14, and $6\,\mu$F capacitors:

$$C_{\text{eff},3,14,\&6} = C_{\text{eff},3\&14} + 6\,\mu\text{F} = 8.65\,\mu\text{F}$$

Finally, *that* equivalent capacitance is just in series with the $20\,\mu$F capacitor, so the overall equivalent capacitance is readily found:

$$C_{\text{eff, total}} = \frac{C_{\text{eff},3,14,\&6} 20\,\mu\text{F}}{C_{\text{eff},3,14,\&6} + 20\,\mu\text{F}} \approx 6\,\mu\text{F}$$

10. Once again, we can simply use the principle of superposition. The total electric potential at any point is just the sum of the electric potentials due to each point charge. We'll label the charges 1-3 from left to right, and calculate the potential due to each first.

If we take an arbitrary point on the y axis $(0, y)$, what is the distance to charge 1? The vertical distance will always be just y, and the horizontal distance is just d. Therefore, the distance d_1 to the first charge is:

$$d_1 = \sqrt{d^2 + y^2} \tag{4.71}$$

The electric potential V_1 due to charge 1, $+Q$, is then found from Eq. 4.14:

$$V_1 = \frac{k_e Q}{d_1} = \frac{k_e Q}{\sqrt{d^2 + y^2}} \tag{4.72}$$

The distance to charge 2 is simply y, since it is also located on the y axis. The electric potential V_2 due to charge 2 is then:

$$V_2 = \frac{-2k_e Q}{y} \tag{4.73}$$

Finally, the distance to charge 3 is just the same as the distance to charge 1. Since both charges also have the same magnitude, $V_1 = V_3$. The total potential at a point $(0, y)$ is then just the sum of the potentials from all three individual charges:

$$
\begin{aligned}
V_{\text{tot}} &= V_1 + V_2 + V_3 && (4.74)\\
&= \frac{k_e Q}{d_1} = \frac{k_e Q}{\sqrt{d^2+y^2}} + \frac{-2k_e Q}{y} + \frac{k_e Q}{d_1} = \frac{k_e Q}{\sqrt{d^2+y^2}} && (4.75)\\
&= \frac{2k_e Q}{\sqrt{d^2+y^2}} + \frac{-2k_e Q}{y} && (4.76)\\
&= 2k_e Q\left[\frac{1}{\sqrt{d^2+y^2}} - \frac{1}{y}\right] && (4.77)
\end{aligned}
$$

11. $-9\,\mu$**J.** The potential energy of a system of charges can be found by calculating the potential energy for every unique pair of charges and adding the results together. In this case, we have three unique pairings: charges 1 and 2, charges 2 and 3, and charges 1 and 3:

$$
\begin{aligned}
PE &= PE_{1\&2} + PE_{2\&3} + PE_{1\&3} && (4.78)\\
&= k_e\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) && (4.79)
\end{aligned}
$$

Here r_{12} is the distance between charge 1 and 2, and so on. Since we have an equilateral triangle, all distances are 1 m. Since all charges are equal in magnitude, we can simplify this quite a bit once we plug in what we know - we just need to keep track of the signs of the charges:

$$
\begin{aligned}
PE_{\text{total}} &= k_e\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)\\
&= \left(9\times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\left[\frac{(10^{-9}\,\mathrm{C})\,(-10^{-9}\,\mathrm{C})}{1\,\mathrm{m}} + \frac{(10^{-9}\,\mathrm{C})\,(10^{-9}\,\mathrm{C})}{1\,\mathrm{m}} + \frac{(-10^{-9}\,\mathrm{C})\,(10^{-9}\,\mathrm{C})}{1\,\mathrm{m}}\right]\\
&= \left(9\times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\left(\frac{10^{-18}\,\mathrm{C^2}}{\mathrm{m}}\right)[-1+1-1]\\
&= \left(9\times 10^{-9}\,\mathrm{N\cdot m}\right)[-1]\\
&\approx -9\times 10^{-9}\,\mathrm{J}
\end{aligned}
$$

Again, we used the conversion $1\,\mathrm{J} \equiv 1\,\mathrm{N\cdot m}$.

12. (a) Dielectric parallel to the plates: $C_{\text{eff}} = \frac{2K}{1+K}C$.

It is easiest to think of this as two capacitors in series, both with half the plate spacing - one filled with dielectric, one with nothing. First, without *any* dielectric, we will say that the original capacitor has plate spacing d and plate area A. The capacitance is then:

$$C_0 = \frac{\epsilon_0 A}{d} \tag{4.80}$$

The upper half capacitor with dielectric then has a capacitance:

$$C_d = \frac{K\epsilon_0 A}{d/2} = \frac{2K\epsilon_0 A}{d} = 2KC_0 \tag{4.81}$$

The half capacitor without then has

$$C_{\text{none}} = \frac{\epsilon_0 A}{d/2} = \frac{2\epsilon_0 A}{d} = 2C_0 \tag{4.82}$$

Now we just add the two like capacitors in series:

$$\begin{aligned}
\frac{1}{C_{\text{eff}}} &= \frac{1}{2KC_0} + \frac{1}{2C0} && (4.83)\\
C_{\text{eff}} &= \frac{4KC_0^2}{2KC_0 + 2C_0} && (4.84)\\
&= \frac{2K}{1+K} C_0 && (4.85)
\end{aligned}$$

(b) Dielectric "perpendicular" to the plates: $C_{\text{eff}} = \frac{K+1}{2} C$.

In this case, we think of the half-filled capacitor as two capacitors in *parallel*, one filled with dielectric, one with nothing. Now each half capacitor has half the plate *area*, but the same spacing. The upper half capacitor with dielectric then has a capacitance:

$$C_d = \frac{K\epsilon_0 \frac{1}{2} A}{d} = \frac{K\epsilon_0 A}{2d} = \frac{1}{2} K C_0 \tag{4.86}$$

The half capacitor without then has

$$C_{\text{none}} = \frac{\epsilon_0 \frac{1}{2} A}{d} = \frac{\epsilon_0 A}{2d} = \frac{1}{2} C_0 \tag{4.87}$$

Now we just add our parallel capacitors:

$$\begin{aligned}
C_{\text{eff}} &= \frac{1}{2} K C_0 + \frac{1}{2} C_0 && (4.88)\\
&= \frac{1}{2}\left(K+1\right) C_0 && (4.89)\\
&= \frac{K+1}{2} C_0 && (4.90)
\end{aligned}$$