3

Electric Forces and Fields

ELECTRICITY has become ubiquitous in modern life, so much that we rarely think about life without it. Though ancient Greeks first began experimenting with electricity around 700 B.C., it was not until the 18TH and 19TH centuries that we began to clearly understand electricity and how to harness it.

In this chapter, we will discuss electric charges and the electric force, quantified through Coulomb's law, and introduce the electric field associated with charges. With these concepts, we will be able to explain many of the myriad electrostatic phenomena around us.

Figure 3.1: *Charles Augustin de Coulomb (1736 - 1806), a French physicist. He discovered an inverse relationship on the force between charges and the square of their separation, later named after him.*[9]

3.1 Properties of Electric Charges

Probably you have noticed that after running a plastic comb through your hair, the comb will attract bits of paper. Often this attraction is strong enough to suspend the paper from the comb, completely counteracting the force of gravity. Another simple experiment is to rub an inflated balloon against your shirt or hair, with the result that the balloon will then stick to the wall or ceiling.

Both of these situations arise because the materials involved have become **electrically charged**. The same things happens when you get "shocked" after dragging your feet on the carpet – you have built up electric charge on your body. An object that is **electrically charged** has built up an imbalance of electric charge. What is electric charge though? Experiments have demonstrated a few basic facts about electric charges:

Some basic properties of charges:

1. There are two types of electric charge, **positive** and **negative**.
2. Like charges repel one another, unlike charges attract one another.
3. Charge comes in discrete units.
4. *Protons* are the positive charges, *electrons* are the negative charges.
5. Electrically neutral objects have an equal number of positive and negative charges.
6. Electrically neutral objects do not experience an electric force in the presence of electric charges.

Normal objects usually contain equal amounts of positive and negative charges –

they are electrically neutral. Electric forces arise only when there is an imbalance in electric charge, and objects carry a net positive or negative charge. On the atomic scale, the carriers of positive charge are the protons. Along with neutrons, which have no electric charge, they comprise the nucleus of an atom (which is about 10^{-15} m across). Electrons are the carriers of negative charge. In a gram of normal matter, there are about 10^{23} protons and an equal number of electrons, so the net charge is zero.

Table 3.1: *Properties of electrons, protons, and neutrons*

Particle	Charge [C]	[e]	Mass [kg]
electron (e^-)	-1.60×10^{-19}	-1	9.11×10^{-31}
proton (p^+)	$+1.60\times10^{-19}$	$+1$	1.67×10^{-27}
neutron (n^0)	0	0	1.67×10^{-27}

Electrons are far lighter than protons, and are more easily accelerated by forces. In addition, they occupy the outer regions of atoms, and are more easily gained or lost. Objects that become charged to so by gaining or losing electrons, not protons. Table 3.1 gives some properties of protons, electrons, and neutrons.

Charge can be transferred from one material to another. Many chemical reactions are, in essence, charge transfer from one species to another (see page 91 for some examples). Rubbing two materials together facilitates this process by increasing the area of contact between the materials – *e.g.,* rubbing a balloon on your hair. Since it is a gain or loss of electrons that give a net charge, this means that when objects become charged, **negative charge is transferred from one object to another**.

Units of charge:
The SI unit of charge is the **Coulomb, [C]**. One unit of charge is $e = 1.6 \times 10^{-19}$ [C]

Charge is never created or destroyed, only transferred from one object to another. **Objects become charged by gaining or losing electrons**, transferring them to other objects. **Charge is also quantized**, meaning it only comes in multiples of the fundamental unit of charge e.

Electrons are transferred, protons stay put!

1. electrons are light, and on the "outside" of the atom.
2. they are more easily moved by electric forces
3. they are more easily removed and transferred to other atoms/objects

An object can have a charge of $\pm e, \pm 2e, \pm 3e$, etc, but not $+0.27e$ or $-0.71e$.[i] Electrons have a negative charge of one unit ($-e$), and protons have a positive charge of one unit ($+e$). The SI unit of charge is the **coulomb [C]**, and e has the value 1.6×10^{-19} C. Since e is so tiny when measured in Coulombs, and since it is the basic fundamental unit of charge, we will sometimes simply measure a small amount of charge in "e's" – how many individual unit charges are present.

[i] Quarks are an exception we will cover at the end of the semester.

Summarizing the properties of charge:

1. Charge is quantized in units of $|e| = 1.6 \times 10^{-19}\,\text{C}$
2. Electrons carry one unit of negative charge, $-e$
3. Protons carry one unit positive charge, $+e$
4. Objects become charged be gaining or losing electrons, not protons
5. Electric charge is always conserved

3.2 Insulators and Conductors

How do materials respond to becoming charged, and how do we charge up a material in the first place? What do we mean by "becoming charged" anyway? This will be more clear shortly, but for now, we will presume that "charging" simply means creating an imbalance of electric charges in a material. A net negative charge can be achieved by adding excess electrons to a material, and a net positive charge can be created by taking away some electrons from a material.

For our purposes, materials respond to becoming charged in one of two ways: the excess charge can move about freely and evenly distribute themselves, or the excess charge can stay localized to the region where it was created. Conductors and insulators are the two broad classes of materials, respectively, which fit these criteria - in **conductors**, excess charges move freely in response to an electric force. All other materials are insulators, and the charges do not move!

In fact, there is nothing particularly special about the excess charge. The excess charge will move in the material in the same way any other charges do - we can't tell the charges apart. In other words, conductors are materials in general where charges move freely, and insulators are materials in general where they do not. There does not need to be *excess* charge for this to be true, charges inside conductors are still in motion even if, over all, they cancel each other out.

Conductors:

1. *e.g.,* metals – silver, gold, aluminum, steel, *etc.*
2. charges are mobile, and move in response to an electric force
3. large number of charges
4. charge distributes evenly over surface

Insulators:

1. *e.g.,* glass, most ceramics, rubbers, and plastics
2. charges are immobile
3. charge deposited on insulators stays localized

Semiconductors:

1. *e.g.,* silicon, gallium arsenide, germanium
2. in between conductors and insulators
3. charges are highly mobile ...
4. ... but the number of charges is small, depends *e.g.,* on temperature and purity
5. conducting properties can be widely varied

Copper and aluminum are typical conductors. **When conductors are charged in some small region, the charge readily distributes itself over the entire surface of the material.** Thus, on a conductor charge is always equally distributed over its entire surface. Charge flows through a conductor readily, and if given a chance, out of it. This is an electric current, as we will see shortly. Glass and rubber are typical insulators. **When insulators are charged (*e.g.,* by rubbing), only the rubbed areas are charged.** There is no tendency for the charge to flow to other regions of the material - charge deposited on insulators will stay localized to a small region.

3.2.1 Charging by Conduction

Conduction is charging through physical contact, which moves e^- from one object to another. One example is charging a balloon by rubbing it on your hair. After doing this, the balloon easily sticks to a wall or picks up little bits of paper, and your hair stands a bit on end. What you have really done is transferred charges from the balloon to your hair, or *vice versa*. Each of your individual hairs becomes charged the same way (either all positive or all negative, depending on what you rubbed on your hair), and the individual strands repel each other. Their repulsion makes them want to maximize the distance between them, which is achieved by standing on end, radiating outward.

As another example, consider rubbing an insulating rod (*e.g.,* rubber, hard plastic glass) against a piece of silk. The act of rubbing these two insulating materials will physically force some charges to move from one object to the other. When charges are transferred to the insulating rod, they do not move – regions of localized charge are created in the rubbed regions. **No charge has been created or destroyed, we simply moved some charges from one place to another - one object ends up with a net positive charge, the other with a net negative charge, equal in magnitude.** One can verify that both objects are charged by trying to pick up bits of paper with them. This is also true when you rub a balloon on your hair - it is clear immediately that *both* the balloon and your hair have become charged! It couldn't be any other way, or we would have had to create charges out of thin air.

Figure 3.2 and its accompanying box illustrates the process of charging a metallic object by conduction. In this example, you take a rubber rod you have already charged (say, with a piece of silk or your hair), and use that to charge a third object.

Charging by conduction: *follow Figure 3.2*

1. Take a charged rod of rubber
2. Bring it near a metal conducting sphere.
3. The charged rod redistributes the charge on the sphere
4. Contact the sphere and the rod!
5. Charges want to neutralize, opposites attract.
6. Negative charge leaves the rod to neutralize the positive charges on the sphere
7. This leaves a net negative region on the sphere

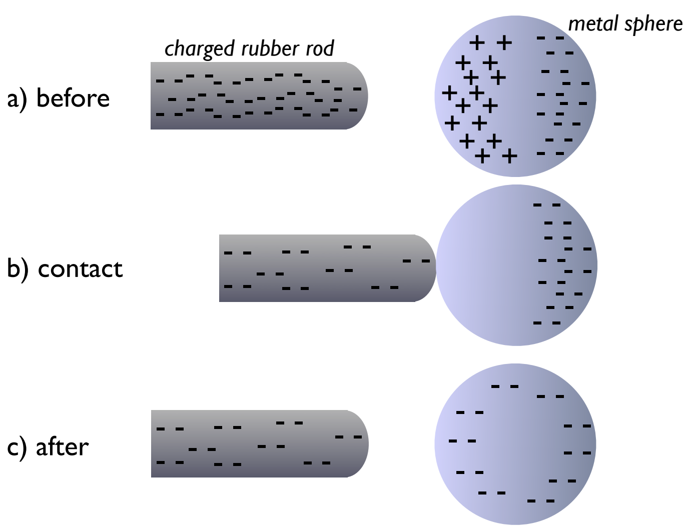


Figure 3.2: *Charging a metallic object by conduction.* **a)** *Just before contact, the negatively charged rod repels the conducting sphere's electrons.* **b)** *After contact, electrons from the rod flow onto the sphere, neutralizing the positive charges.* **c)** *When the rod is removed, the sphere is left with a negative charge.*

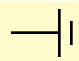
3.2.1.1 Grounding

Sounds simple enough. Why can't we just take a piece of Copper pipe and rub it with a cloth? You can, if you are careful . . . charges flow evenly through a conductor, and if possible, out of the conductor entirely. Only isolated conductors can be charged, electrically contacted conductors cannot. By 'electrically connected,' we mean the conductor we are trying to charge cannot have any sort of conducting path to the earth. The Earth can be considered an (essentially) infinite reservoir for electrons, either sourcing or sinking as many charges as we need. Since charges distribute themselves evenly over a conducting surface, if there were a path to the earth, the mobile charges would follow it to the earth, and keep doing so until none were left on the conductor.

Given a conducting path to the earth, charges from the conductor **will always keep flowing.** If charges can find a way to Earth, they *will* get there (*e.g.,* through pipes, wires, or you!). Another phrase for when *you* are the connection to Earth is "ground fault." This is when you accidentally make *yourself* the connection between a charged (or current-carrying) wire and the Earth ... with potentially disastrous results. A so-called "ground fault interrupter" (GFI) senses when this happens, and very quickly breaks the connection.[ii]

As it turns out, *YOU* are a sufficient conductor to let the charges flow away! Any charges transferred to the Copper rod will flow straight through it, and through you down to the ground. You can make it work, if you wear some insulated rubber or plasticized gloves. The same trick works on a rubber or plastic rod without gloves, because charges deposited on the insulating rods do not flow through the rod and out of it.

The "ground connection" or "ground point" is the place in an electric circuit which is *purposely* connected to the earth, either for safety reasons or just to provide a reference point. The ground point (or just "ground") in a circuit or electrical diagram is usually shown like this:

Circuit diagram symbol for a ground point: 

3.2.2 Charging by Induction

Can we charge without contacting it all? Yes! This is **induction charging.** Now we explicitly need a **ground point** or reference point for this to work though. An object connected to a conducting wire or pipe buried in the Earth is said to be **grounded**, the Earth itself is the *ground point*. As mentioned above, the Earth can be considered an infinite reservoir for electrons, sourcing or sinking an infinite number of charges. Using this idea, we can understand a non-contact charging process known as **induction**.

Figure 3.3 illustrates the process of charging a metallic object by induction. **Charging an object by induction requires no contact with the object inducing the charge.** First, we take an isolated conducting (metal) sphere. From our discussion above, it is crucial that it not be contacted to the ground in any way. Placing it on an insulating stand will do nicely. Next, we bring a negatively charged rod *near*, but not touching, the sphere. We can prepare a negatively charged glass rod by rubbing it with silk (charging the glass by conduction).

Charging by induction: *follow Figure 3.3*

1. Take a neutral conducting sphere
2. Bring a negatively charged rod *near* (but not touching) the sphere.
3. This creates a charge imbalance on the sphere, due to repulsion from the charged rod.
4. Ground the opposite side of the sphere – the charge imbalance forces some e^- to flow to ground!
5. Disconnect the ground wire – this leaves a net + charge on sphere!
6. Remove the charged rod, the net charge has to stay on sphere, and it will distribute itself evenly over the surface.

[ii]Electrical outlets with GFI should be present in your house, usually in bathrooms and kitchens. They typically have little buttons labeled 'test' and 'reset' or something like that.

When the charged rod is near the conducting sphere, the negative charges on the rod will repel the free negative charges (electrons) on the sphere, with the result that the half of the sphere nearest the rod will have a net negative charge (Fig. 3.3b). Now, if we take a conducting wire and connect the *far* end of the sphere to the ground (Fig. 3.3c), the excess negative charge on that side, repelled by the rod, will want to flow down the wire into the earth, effectively draining away a quantity of negative charge from the sphere. Once we have done that, the sphere now has a *net positive charge.*

Removing the ground connection, Fig. 3.3d, will instantaneously leave the side of the sphere near the rod positively charged, and the far side (nearly) uncharged, since we just drained away the negative charges. After a *very* short time, the conducting sphere reaches equilibrium, and we must have a uniform distribution of charge on the surface of the conductor. Thus, the excess positive charge has to be evenly distributed on the surface of the sphere. We are left with a charged conducting sphere!

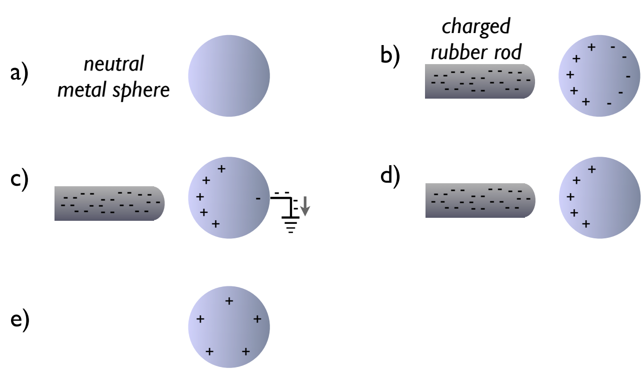


Figure 3.3: *Charging a metallic object by induction.* **a)** *A neutral metallic sphere with equal numbers of positive and negative charges.* **b)** *The charge on a neutral metal sphere is redistributed when a charged rod is brought near it.* **c)** *When the sphere is then grounded, some of the negative charges (electrons) leave it through the ground wire.* **d)** *When the ground connection is removed, excess positive charge is left on the sphere.* **e)** *When the charged rod is removed, the excess positive charge redistributes itself until the sphere's surface is uniformly charged.*

A process similar to charging by induction in conductors takes place in insulators (such as neutral atoms or molecules in particular). The presence of a charged object can result in more positive charge on one side of an insulating body than the other, by realignment of the charges within the individual molecules. This process is known as **polarization**, and we will cover it in more depth in the following chapter.

Our discussion of charging allows us to now better appreciate the distinction between conductors and insulators. The difference in the degree of conductivity between conductors and insulators is staggeringly enormous, a factor of 10^{20}. For instance, a charged Copper sphere connected to the ground looses its charge in a millionth of a second, while an otherwise identical glass sphere can hold its charge for years.

3.3 Coulomb's Law

When you charge two objects, such as a balloon and your hair, you invariably end up observing an attraction or repulsion between the charged objects. What is the character of this force? How does it depend on how much the objects have been charged, how far away they are, or anything else? If you continue to experiment with charged objects, you will find that the *force* due to electrically charged objects has the following properties:

An **electric force** has the following properties:

1. It is directed along a line joining the two particles.
2. It is inversely proportional to the square of the distance r_{12} separating them.
3. It is proportional to the product of the magnitudes of the charges, $|q_1|$ and $|q_2|$, of the two particles.
4. It is attractive if the charges are of opposite sign, and repulsive if the charges have the same sign

These properties led Coulomb (Fig. 3.1) to propose a neat mathematical form for the electric force between two charges:[iii]

Coulomb's Law: the force between two charges q_1 and q_2, separated by a distance r_{12} is given by:

$$\vec{\mathbf{F}} = k_e \frac{q_1 q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} \tag{3.1}$$

where k_e is the "Coulomb constant," and $\hat{\mathbf{r}}_{12}$ is a unit vector pointing along a line connecting the two charges.

Equation 3.1 is known as **"Coulomb's law"**. What Coulomb's law states is that the force between two charged objects $\vec{\mathbf{F}}$, depends only on how big the charges are (q_1 and q_2), and how far apart they are (r_{12}). Keep in mind that force is a vector, and the dimensionless *unit vector* $\hat{\mathbf{r}}_{12}$ reminds us that the electric force is directed along a line connecting the two charges q_1 and q_2.

Figure 3.4 schematically shows the electric force between two like and two unlike charges. The distance between the charges r_{12} is given in the SI unit of **meters, [m]**, and the charges q_1 and q_2 are measured in the SI unit of charge, the **Coulomb, [C]**. The charges q_1 and q_2 can be either positive or negative, which makes the resulting force $\vec{\mathbf{F}}$ repulsive when both charges have the same sign, and attractive when they are opposite – just like we expect. The Coulomb constant k_e gives the relative strength of the electric force, just as G gives the relative strength of the gravitational force, and has the SI value and units:

[iii]See Sect. 1 for a summary of units and notation conventions

Coulomb's constant

$$k_e = 8.9875 \times 10^9 \, \frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}} \tag{3.2}$$

In most calculations, k_e can be safely rounded to $\approx 9\times10^9$, which makes it a bit easier to remember. Also, k_e is much, much larger then G[iv], by about *twenty* orders of magnitude, meaning that if we treat Coulombs on equal footing with kilograms for a minute, the electric force is *far, far* stronger than the gravitational force. A pair of 1 Coulomb charges interacting *via* the electric force is the same as two masses of 10^{10} kilograms interacting *via* the gravitational force. Equivalently, one might say gravity is just exceptionally weak, so far as fundamental forces go.

Question: Show that the units of Coulomb's constant above yield a force in Newtons when applied to Equation 3.1.

Note that no matter what the two charges are, Newton's third law[v] still holds, *viz.*, $\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$. The force on charge 1 due to charge 2 is equal and opposite the force on charge 2 due to charge 1, *always.* Even if one charge is a million times larger than the other, this must still be true. Mathematically, this is easy to see from Eq. 3.1 – the force between two charges depends on the *product* of the two charge values q_1q_2, which means it is totally symmetric if we swap 1 for 2 or *vice versa.*

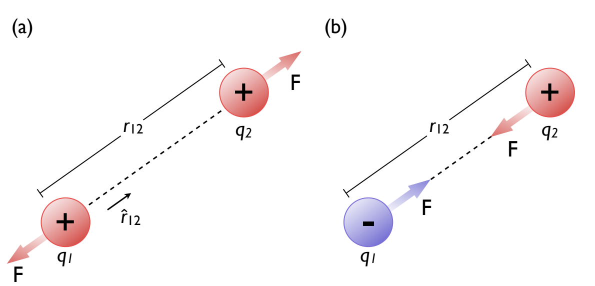


Figure 3.4: *Electrical force between point charges.* **(a)** *Two particles q_1 and q_2 which both have positive charges. The force is repulsive, as it would be for two negative charges, and directed along the dashed line connecting the two charges. The unit vector $\hat{\mathbf{r}}_{12}$ is indicated.* **(b)** *Two particles q_1 and q_2 with charges of opposite sign, separated by a distance r_{12}. The force is now attractive, as we expect.*

When a number of separate charges act on a single charge, each exerts its own electric force. These electric forces can all be computed separately, one at a time, and then added as vectors. This is the powerful **superposition principle**, the same one you used with gravitation. This makes calculating the net force from many charges a lot simpler than you might think. In fact, gravitation and electrostatic forces have a number of similarities, with a few crucial differences, which we list below.

[iv] $G = 6.67 \times 10^{-11}\ \mathrm{m^3/kg{\cdot}s^2}$

[v] When object A exerts a force on object B, B simultaneously exerts a force on A with the same magnitude, in the opposite direction.

The electric force is *similar* to the gravitational force:

1. Both act at a distance without direct contact
2. Both act in a vacuum, without a medium, and propagate at a speed c
3. Both are inversely proportional to the distance squared, with the force directed along a line connecting the two bodies
4. The mathematical form is the same, if one interchanges k_e and G.
5. Both gravitational masses and electric charges obey the superposition principle
6. Both are conservative forces[vi]

The electric force is *different* from the gravitational force:

1. Electric forces can be *attractive* or *repulsive*. Gravity is only attractive.
2. Gravitational forces are independent of the medium, while electric forces depend on the intervening medium
3. The electric force between charged elementary particles is far stronger than the gravitational force between the same particles.

One lingering question is how to relate the *microscopic* charge carriers, the electrons, to the *macroscopic* behavior of charged objects. When we charge a glass rod and pick up bits of paper, how many charges are we dealing with? Referring to Table 3.1, the charge on the proton (p^+) has a magnitude of $e = 1.6 \times 10^{-19}\,\text{C}$, while an electron ($e^-$) has a charge of $-e = -1.6 \times 10^{-19}\,\text{C}$.[vii] This means it takes $1/e \approx 6.3 \times 10^{18}$ protons or electrons to make up a total charge of $\pm 1\,\text{C}$ – so $1\,\text{C}$ is a *seriously* large amount of charge. Typical net charges in electrostatic situations (*i.e.,* static electricity) are of the order of $1\,\mu\text{C}$,[viii] which is still 10^{12} or so electrons – or about one electron for every dollar of our national debt, if that helps bring the magnitude in perspective!

Question: If two charges of $+1\,\mu\text{C}$ are separated by $1\,\text{cm}$ ($= 10^{-2}\,\text{m}$), what is the force between them?

Answer: about 90 N, or roughly 20 lbs!

Technically, Coulomb's law applies in this particular mathematical form only for point charges, or spherical charge distributions (in which case r_{12} is the distance between the centers of the charge distributions, see Sect. 3.8.3). Coulomb's law covers **electrostatic forces**, which are what we call forces between unmoving (stationary) charges. Really, though we only need to take care when we have charges moving at very high velocities, or when charges accelerate. Accelerating charges produce electromagnetic radiation – light – which we will cover in Chapter 9

[vi] A conservative force is one which does no net work on a particle that travels along any *closed* path in an isolated system. For any path, not just a closed one, the work done by a conservative force depends only on the initial and final positions, not on the path taken. Gravity is conservative, friction is not, for example.

[vii] The symbol e will frequently be used to represent the charge of a proton or electron.

[viii] $1\,\mu\text{C} = 10^{-6}\,\text{C}$, see Table 1.7 in Appendix 1

3.4 The Electric Field

Both the gravitational force and the electrostatic force are capable of acting through space, without any physical contact or intervening medium (Sects. 2.2.1, 9.5). That is, electric and gravitational forces can act across an empty vacuum, with no matter to carry them.[ix] These types of forces are known as **field forces**. Corresponding to the electrostatic force, an **electric field** is said to exist in the region of space surrounding a charged object. **The electric field exerts an electric force on any other charged object within the field.**

The field concept partially eliminates the conundrum of "force at a distance", since the force on a charged object is now said to be caused by the *electric field* at that point in space.

The electric field $\vec{\mathbf{E}}$ produced by a charge q at the location of a small "test" charge q_0 is defined as the electric force $\vec{\mathbf{F}}$ exerted by q on q_0, divided by the test charge q_0.

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q_0} \qquad \text{or,} \qquad \vec{\mathbf{F}} = q_0\vec{\mathbf{E}} \tag{3.3}$$

The SI unit for electric field is **Newtons per Coulomb [N/C]**. The direction of $\vec{\mathbf{E}}$ is the direction of the force that acts on a positive test charge q_0 placed in the field.

The test charge q_0 is hypothetical – what *would* the force be on a charge q_0 if we *did* place it at some distance r away? We say that **an electric field exists at a point if a test charge at that point would be subject to an electric force there.**

Using equations 3.1 and 3.3, we can write the *magnitude* of the electric field[x] due to a charge q as:

Magnitude of the electric field at a distance r from a point charge q:

$$|\vec{\mathbf{E}}| = k_e\frac{|q|}{r^2} \tag{3.4}$$

The *direction* of the electric field is the same as the direction of the electric force, since the two are related by a scalar.

The electric field produced by a charge depends only on the magnitude of that charge which sets up the field, and how far away from that charge you are. It does not depend on the presence of a hypothetical test charge.

The principle of superposition also holds for electric fields, just as it did for the electric *force*. In order to calculate the electric field from a group of charges, one may calculate the field from each

[ix] We will find out later that *light* carries electric forces, in fact, and there is no need to invoke "action at a distance."

[x] When it is unambiguous, we will often write the magnitude of a vector, such as the electric field $|\vec{\mathbf{E}}|$, as simply E for convenience. Similarly, $|\vec{\mathbf{B}}|$ becomes B, and $\vec{\mathbf{x}}$ becomes x.

charge individually, and add (as vectors) the individual fields. Symmetry is also very important. For example, if a equal and opposite charges are placed on the x axis at $x=a$ and $x=-a$, the field at the origin is zero – the fields from the positive and negative charges cancel. On page 91 you can find basic instructions on how to approach and solve electric field problems.

Table 3.2: *Approximate electric field values, in [N/C]*

Source	$\lvert\vec{\mathbf{E}}\rvert$	**Source**	$\lvert\vec{\mathbf{E}}\rvert$
Fluorescent lighting tube	10	Atmosphere (fair weather)	10^2
Balloon rubbed on hair	10^3	Atmosphere (under thundercloud)	10^4
Photocopier	10^5	Spark in air	10^6
Across a transistor gate dielectric	10^9	Near electron in hydrogen atom	10^{11}

3.4.1 Electric Field Lines

A convenient way to visualize the electric field is to draw lines pointing in the direction of the electric field vector at any point – **Electric field lines**. Electric field lines have three key properties:

Key properties of electric field lines:

1. The electric field vector $\vec{\mathbf{E}}$ is tangent to the electric field line at any point.
2. The density of the lines (number per unit area) is proportional to the strength of the electric field.
3. Arrows on the lines point in the direction that a hypothetical *positive* test charge would move. Arrows are not always used.

So $\vec{\mathbf{E}}$ is large when the lines are close together, and small when they are far apart. Below are some example, to give you an idea.

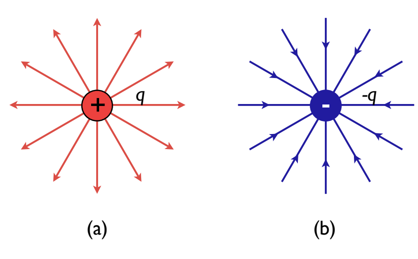


Figure 3.5: *The electric field lines for point charges. (a) For a positive point charge, the lines are directed radially outward. (b) For a negative point charge, the lines are directed radially inward. Note that the figures show only those field lines that lie in the plane of the page. (c) The dark areas are small pieces of thread suspended in oil, which align with the electric field produced by a small charged conductor at the center.*

These 2-D drawings represent field lines for individual point charges. They only contain field lines in the plane of the paper – there are equivalent field lines pointing in all directions. A positive "test charge" placed in the field of the positive charge Fig. 3.5a field would be repelled, hence the lines point outward. On the other hand, for the negative charge in Fig. 3.5b, a positive test charge is attracted and the arrows point in. Note that the lines get more dense as they get closer to the charge, indicating that the field strength is increasing – just what we expect from Equation 3.4.

Rules for drawing field lines:

1. The lines for a group of point charges must **start on positive charges and end on negative charges**. If there is excess charge, some lines will begin or end infinitely far away (or at least off of your page).
2. The number of lines drawn leaving a positive charge or ending on a negative charge is proportional to the magnitude of the charge
3. Field lines cannot cross each other.

3.4.2 What happens when we have two charges together?

3.4.2.1 Two Opposite Charges

Figure 3.6 shows nicely symmetric field lines for two charges of equal magnitude and opposite sign. Here we have omitted the arrows for simplicity, by now you should know how to add them in. This configuration is also known as an **electric dipole**. The number of lines beginning at the positive charge must equal the number of lines ending at the negative charge. Close to each charge, the lines are nearly radial, and the high density of lines between the charges indicates a large electric field in this region. Finally, note that the lines are symmetric about a line connecting the two charges, and to a line perpendicular to that one halfway between the charges.

Figure 3.6: left *Field lines for two equal and opposite charges, an "***electric dipole***." The number of lines leaving the positive charge equals the number terminating at the negative charge* **right** *Field lines for two positive charges of equal magnitude. Can you rank the relative field strengths at points A, B, and C?*[xi]

3.4.2.2 Two like charges

Figure 3.6 also shows the field lines for two positive charges. Again the lines are nearly radial near the charges. The same number of lines leave each charge, since they are of the same magnitude.

Far away from either charge, the field looks nearly the same as it would for a single charge twice as big as either lone charge. In between the charges, the field lines "bulge," representing the

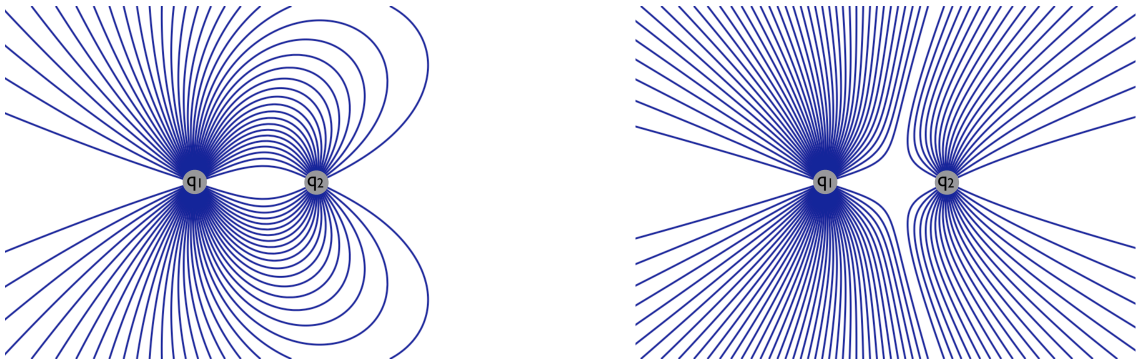


Figure 3.7: left *Field lines for opposite charges of different magnitude. Which is the larger charge?* **right** *Field lines for two charges of the same sign, but of different magnitude. Which is the larger charge?*

repulsive nature of the electric force between like charges. Again, note that the lines are symmetric about a line connecting the two charges, and to a line perpendicular to that one halfway between the charges. The symmetries of the electric field surrounding charge distributions can be very useful in solving electric field problems – for instance, we know without lifting a pencil that the field is precisely zero along the vertical line halfway between the two charges.

Finally, Fig. 3.7 shows the field lines for two charges of different magnitude in two situations. Can you tell which is the larger charge in each case? Can you tell which plot is for charges of the same sign, and which is for charges of opposite signs?

3.5 Conductors in Electrostatic Equilibrium

A good electric conductor like copper, even when electrically neutral, contains electrons which aren't bound to any particular atom, and are free to move about. This is one reason why charge is distributed evenly over the surface of a conductor – the mobile electrons.

Though the individual "free" electrons in the conductor are constantly in motion, in an isolated conductor there is no *net* motion of charge. The random motions of all free electrons cancel out over all. When no net motion of charge occurs, this is called **electrostatic equilibrium**. An isolated conductor is one that is insulated from the ground, and has the following properties:

Properties of isolated conductors:

1. The electric field is zero everywhere inside the conductor.
2. Any excess charge on an isolated conductor must be entirely on its surface.
3. The electric field just outside a charged conductor is perpendicular its surface.
4. On irregularly shaped conductors, charge accumulates at sharp points, where the radius of curvature is smallest.

The first property is most easily understood by thinking about what would happen if it were **not** true, *reductio ad absurdum*. If there were fields inside a conductor, the free charges would move, and "bunch up" at the regions of higher and lower field (depending on whether they are positive or negative). This contradicts the very definition of a conductor – charges are supposed to be mobile, and spread out *evenly* through the conductor. Even if we did create a field inside a conductor, since the charges are mobile they would immediately start to flow to the region where the electric field is, gathering in sufficient number until they cancelled it out. Anyway, if this happened, we would no longer have electrostatic equilibrium in the first place, which is defined by *no net motion of charges.*

The second property is a result of the $1/r^2$ repulsion of like charges in Equation 3.4. If we had excess charge inside a conductor, the repulsive forces between these excess charges would push them as far apart as possible. Since the charges are mobile in a conductor, this happens readily. Every like charge wants to maximize its distance from every other like charge, so excess charge quickly migrates to the surface.

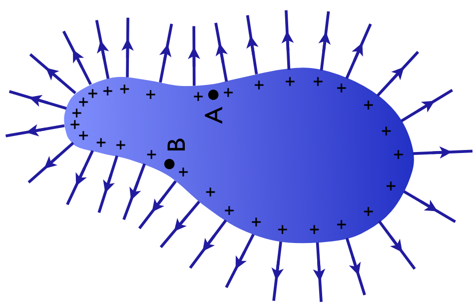


Figure 3.8: *An arbitrarily shaped conductor carrying a positive charge. When the conductor is in electrostatic equilibrium, all of the charge resides at the surface,* $\vec{\mathbf{E}}=0$ *inside the conductor, and the direction of* $\vec{\mathbf{E}}$ *just outside the conductor is perpendicular to the surface. Note from the spacing of the positive signs that the surface charge density is nonuniform due to the varying degree of curvature along the surface.*

This is only true because Coulomb's law (Equation 3.4) is an inverse square law! If it were some other power law, like $1/r^{2+\delta}$, even for *very* tiny δ, excess charges would exist inside the conductor, which we could observe. One of many special facts about inverse square laws, which has been used to test Coulomb's law with fantastic precision.

The third property we also understand by thinking about what would happen if it were not true. If the field was not perpendicular to the conductor's surface, it would have to have a component parallel to the surface. If that were true, free charges on the surface of the conductor would feel this field, and therefore a force (Eq. 3.3) along the surface. Under this force, they would subsequently flow along the surface, and once again, there is a net flow of charge, so we are by definition not in electrostatic equilibrium.

The fourth property is perhaps easiest to understand geometrically, as a consequence of the third property. The requirement that field lines be perpendicular to the surface forces them to "bunch up" wherever the radius of curvature is small, at "sharp" points, see Fig. 3.8. The presence of a sharp point with a high radius of curvature enhances the electric field in that region, and as a result, the mobile surface charges will instantly flow to this region of high curvature. They will do this until the electric field along the surface is cancelled. The sharper the point, the more charges need to flow into the region to ensure that the parallel component of the surface electric field is totally cancelled. This does result in an uneven surface charge density for irregularly shaped conductors, but also an electric field which is uniform and perfectly normal to the surface everywhere.

These rules might be easier to grasp pictorally.[xii] Figure 3.9 shows the field lines between oppositely charged conducting plates – an example of a device known as a *capacitor*, which we will study in Ch. 4. Note that the field in the region between the plates is very uniform, due to the requirement that it be perpendicular to the surface of the conductors. Near the edges of each plate, the field "fringes", and starts to curve slightly outward. Further from the edges of the plates, the field starts to resemble that of a dipole (Fig. 3.6, turned 90°). This is no accident – the excess charges on the very edges of the plates *do* essentially form a dipole, so viewed from far away, the edges of this parallel plate structure look like a long row of dipoles stacked together. Microscopically, this is almost exactly what is happening!

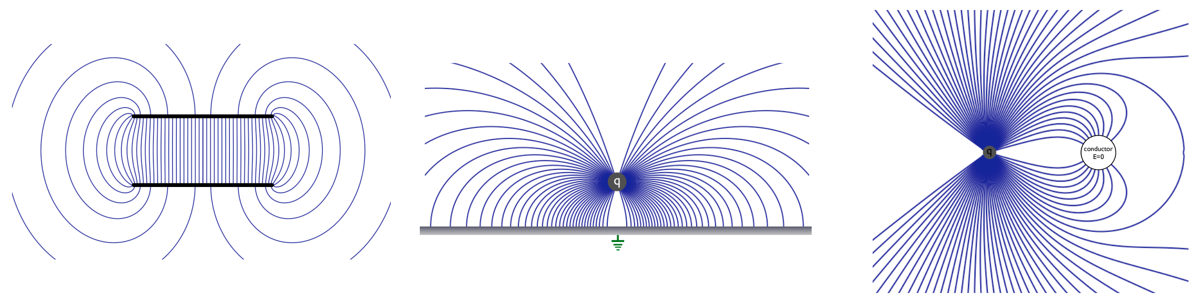


Figure 3.9: **(a)** *Field lines between two oppositely charged plates,* **(b)** *a point charge above a grounded conducting plane, and* **(c)** *a point charge near a conducting sphere. Field lines must be perpendicular to the surface of a conductor at every point, and their density increases near "sharp" points. Note also that there are no field lines inside the sphere, as the field inside a conductor must be zero.*

Figure 3.9 also shows the field lines due to a point charge suspended above a grounded conducting plate. In this case, we again see that field lines *always* intersect the conducting surface at right angles. Again, this resembles Fig. 3.6 – this looks like half of the dipole field, as if there were a mirror halfway between the two charges. This is exactly what is happening – since the field lines have to intersect the plate at right angles, the point charge a distance d from the conducting plate behaves in the same way as if there were an equal and opposite charge a distance $2d$ away. Really, *a conductor is a mirror for electric field lines*! One can use this as a problem-solving trick, known as the "method of images." This is a bit beyond the scope of the current text, but a neat time-saving trick to be aware of. What this also means qualitatively is that *when a charge is present near a conductor, the charge induces an equal and opposite charge spread out on the surface of the conductor.* In this case, a charge q above the conducting plate induces an *overall* charge $-q$ over the whole surface of the conducting plate.

Finally, Fig. 3.9 shows a point charge near a hollow conducting sphere. Note that *everywhere*, the field is perpendicular to the conducting sphere, and the field is zero inside the conductor. Oddly, this figure looks a bit like what we would expect if the conducting sphere were replaced by another

[xii] Appendix B may provide an interesting read for the mathematically inclined.

charge, opposite in sign but smaller than the existing point charge. Can you see why this might be? As a hint, think about conductors being *mirrors* for field lines.

> **Question:** All four properties are exemplified in Fig. 3.9, can you spot where?
>
> Answer: 1 – inside the hollow sphere. 2 – inside the hollow sphere. 3 – true everywhere, check for yourself. 4 – ends of the plates.

3.6 Faraday Cages

A "Faraday Cage" is an enclosed region formed by conducting material – essentially a hollow conductor. Since the electric field inside a conductor is zero, *anything* we enclose inside a hollow conductor will be *completely* shielded from any static electric fields.[xiii] You can see Faraday Cages all around you, if you look carefully - electrical conduits inside the walls are metal boxes, the inside of your cell phone is surrounded by metal foil, and your computer hides inside a metal (or metal-lined) box.

Faraday cages are named for Michael Faraday (Fig. 4.1), who built one in 1836 and explained its operation.[10] Charges in the enclosing conducting shell repel one another, and will always reside on the outside surface of the cage (as discussed above). Any external (static) electrical field will cause the charges on the surface to rearrange until they completely cancel the field's effects in the cage's interior. No matter how large the field outside the cage, the field inside is *precisely zero*, so long as there is no charge inside the box. It seems incredible that the charges on the conductor's surface know just how to arrange themselves to exactly cancel the external field, but this is really what happens.

The most important application of Faraday cages is for this sort of electromagnetic shielding. One example is a shielded coaxial cable (*e.g.,* RCA cables for your stereo, or the coax connecting your cable or satellite box), which has a wire mesh shield surrounding an inner core conductor. The mesh shielding keeps any signal from the core from escaping, and perhaps more importantly, prevents spurious signals from reaching the core.

A more subtle example of a Faraday cages is probably sitting in your kitchen. The door of a microwave oven has a screen built into the glass of the window, with small holes in it. As we will find out in Sect. 9.5.5, this too is a Faraday cage, even though there are holes in the screen. Why does it still work, even though there are holes? How do electric fields relate to microwaves? Before too long, you will know!

3.7 The van de Graaff Generator

In 1929 Robert J. van de Graaff (1901-1967), a Tuscaloosa native and UA graduate (BS '22, MS '23), designed and built an electrostatic generator that has been extensively used in nuclear physics

[xiii] Again, Appendix. B may be insightful.

research. Dr. van de Graaff can be considered the inventor of the first accelerator providing intense particle beams of precisely controllable energy, and one of the pioneers of particle physics.[11,xiv]

The principles of its operation can be understood using the properties of electric fields and charges you have (hopefully) just learned. Figure 3.10 shows the basic construction of Dr. van de Graaff's device, and Fig. 3.11 shows illustrations from Dr. van de Graaff's original patent on the "Electrostatic Generator" from 1931. A motor-driven pulley moves a belt past positively-charged metallic needles at position A. Negative charges are attracted to the needles from the belt, which leaves the left side of the belt with a net positive charge. The moving belt transfers these positive charges up toward the conducting dome.

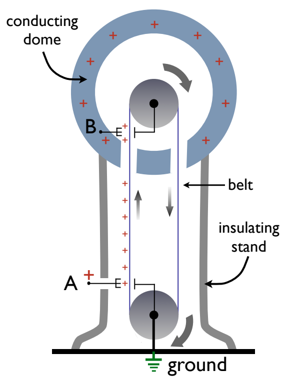


Figure 3.10: *A diagram of a van de Graaff generator. Charge is transferred to the dome by means of a rotating belt. The charge is deposited on the belt at point A and transferred to the dome at point B.*

The positive charges attract electrons on to the belt as it moves past a second set of needles at point B, which increases the excess positive charge on the dome. Because the electric field inside the conducting metal dome is negligible (it would be precisely zero if there were not holes in the dome), the positive charge on it can be easily increased – near zero electric field means near zero repulsive force to add more charge. The result is that extremely large amounts of positive charge can be deposited on the dome.

This charge accumulation cannot occur indefinitely. Eventually, the electric field due to the charges becomes large enough to ionize the surrounding air, increasing the air's conductivity. When sufficiently ionized, the air is nicely conducting, and the charges may rapidly flow off of the dome through the air – a "spark" jumps off of the dome to the nearest ground point. A spectacular example of this can be seen in Figure 3.12.

Since the "sparks" are really charge flowing off of the dome, this eventually limits the highest electric fields obtainable. The easy solution to increase the voltage is to make the domes bigger (decrease their radius of curvature), and put them higher off the ground (the farther a "spark" has to go, the more electric field it takes to create one).

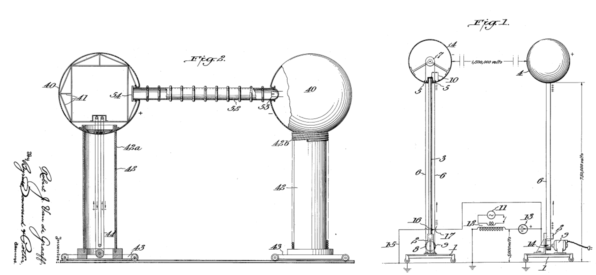


Figure 3.11: *Images from van de Graaff's original patent on the "Electrostatic Generator," filed 16 December, 1931.*[13]

One of the largest Van de Graaff generators in the world, built by Dr. Van de Graaff himself, is now on permanent display at Boston's

[xiv] You might think that is how the Tuscaloosa airport got its name. You would be wrong.[12]

Museum of Science (it is the one shown in Figure 3.12). It uses 15 foot aluminum spheres standing on columns many feet tall, and can reach 2 million volts. The Van de Graaff generator is operated several times per day in the museum's "Theater of Electricity."

Figure 3.12: *The world's largest van de Graaff generator originally built at Round Hill, near South Dartmouth, Massachusetts in 1933.*[14] *In the early 1950's, the giant Van de Graaff generator was donated to the Museum of Science in Cambridge, Massachusetts, where it is now operated at least twice daily for demonstrations. Do you know why the person in this picture is in no danger? Re-read Section 3.6 ... Photo credits: T.L. Carroll*[15]

Van de Graaff information and pictures can be found through the Museum of Science:

`http://www.mos.org/sln/toe/toe.html`

More pictures of the largest van de Graaff generator, including its construction and historical pictures, can be found through the MIT Institute Archives:

`http://libraries.mit.edu/archives/exhibits/van-de-graaff/`

An interesting article from the Tuscaloosa News about Robert van de Graaff:

`http://bama.ua.edu/~jharrell/PH106-S06/vandegraaff.htm`

You can also visit his boyhood home at 1305 Greensnboro Ave.

3.8 Gauss's Law

Gauss's law is a very sneaky technique (based on some basic theorems of vector calculus) for **calculating the average electric field over a closed surface**. What do we mean by a closed surface? **A closed surface has an inside and an outside, it is one that encloses a volume and has no holes in it.** A sphere and a cube are simple examples. If, due to symmetry, **the electric field is constant everywhere on a closed surface, the exact electric field can be found** – in most cases, much more easily than *via* Coulomb's law.

3.8.1 Electric Flux

How do we use this sneaky law? First, we need the concept of **electric flux**, denoted by Φ_E. Electric flux is a measure of how much the electric field vectors penetrate a given surface. If the electric field vectors are tangent to the surface at all points, they don't penetrate at all and the

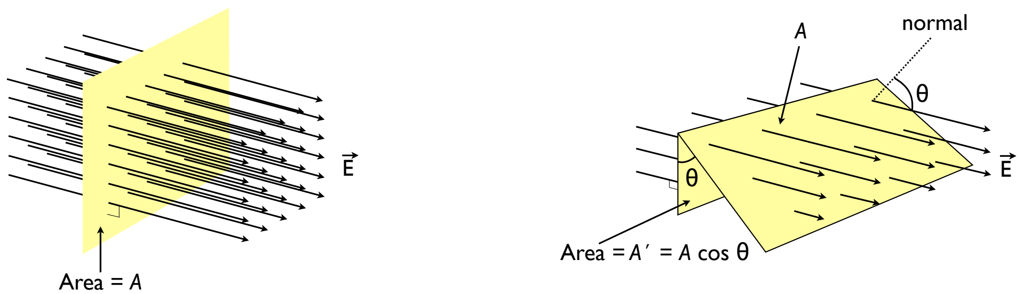


Figure 3.13: **(a)** *Field lines representing a uniform electric field E penetrating a plane of area A perpendicular to the field. The electric flux Φ_E through this area is equal to $|\vec{\mathbf{E}}|A$.* **(b)** *Field lines representing a uniform electric field penetrating an area A that is at an angle θ to the field. Because the number of lines that go through the area A′ is the same as the number that go through A, the flux through A′ is given by $\Phi_E = |\vec{\mathbf{E}}|A\cos\theta$.*

flux is zero. *Basically, we count the number of field lines penetrating the surface per unit area* – lines entering the inside of the surface are positive, those leaving to the outside are negative.

An analogy of electric flux is fluid flux, which is just the volume of liquid flowing through an area per second. *The electric flux due to an electric field $\vec{\mathbf{E}}$ constant in magnitude in direction through a surface of area A is $\Phi = |\vec{\mathbf{E}}|A\cos\theta_{EA}$,* where θ_{EA} is the angle that $\vec{\mathbf{E}}$ makes with the surface *normal*.

Definition of electric flux through a surface

$$\Phi_E = |\vec{\mathbf{E}}|A\cos\theta_{EA} \tag{3.5}$$

where θ_{EA} is the angle between the normal and the electric field.

Consider the surface in Figure 3.13a. The electric field is uniform in magnitude and direction. Field lines penetrate the surface of area A uniformly, and are perpendicular to the surface at every point ($\theta = 0°$). The flux through this surface is just $\Phi = |\vec{\mathbf{E}}|A$.

Now consider the surface A in Figure 3.13b. The uniform electric field penetrates the area A that is at an angle θ to the field, so now the flux is $\Phi_E = |\vec{\mathbf{E}}|A\cos\theta$. For the surface A', the field lines are perpendicular, but the area is reduced by the same amount, so the flux is the same through A and A'.

Just like electric forces and fields, flux also obeys the superposition principle. If we have a number of charges inside a closed surface, the total flux through that surface is just the sum of the fluxes from each individual charge.

Now: on to Gauss's law. What Gauss's law actually relates is the **electric flux** through a closed surface to the total electric charge contained inside that surface – **the electric flux through a closed surface is proportional to the charge contained inside the surface.** To see how this

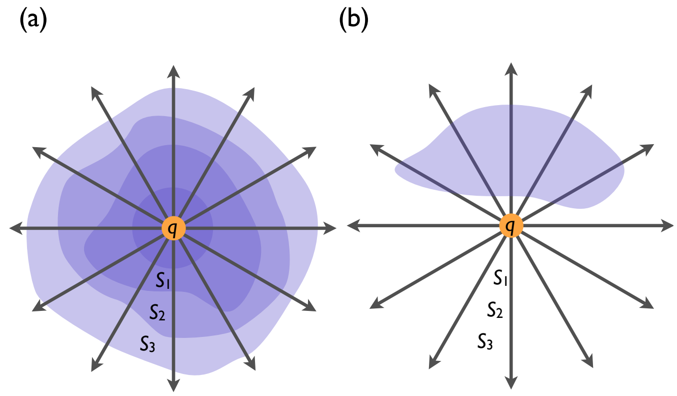


Figure 3.14: **(a)** *Closed surfaces of various shapes surrounding a point charge q. The net electric flux is te same through all surfaces.* **(b)** *If the point charge is outside the surface, the net flux is zero through that surface since the same number of field lines enter and leave. If no charge is enclosed by the surface, there is no net flux.*

works, consider the point charge in Figure 3.14a. The innermost surface is just a sphere, whose radius we will call r. The strength of the electric field everywhere on this sphere is

$$|\vec{\mathbf{E}}| = k_e \frac{q}{r^2} \tag{3.6}$$

since every point on the sphere's surface is a distance r from the charge. We also know that $\vec{\mathbf{E}}$ is perpendicular to the surface everywhere, thanks to the radial symmetry. Finally, we know that the surface area of a sphere is $A = 4\pi r^2$, so the electric flux is

$$\Phi_E = |\vec{\mathbf{E}}|A = k_e \frac{q}{r^2}\left(4\pi r^2\right) = 4\pi k_e q \tag{3.7}$$

If the point charge is *outside* the surface, Fig. 3.14b, the net flux is zero through that surface since the same number of field lines enter and leave. If no charge is *enclosed* by the surface, there is no net flux.

Now the power in Gauss's law is that if we take *any* arbitrarily more complicated surface, so long as it surrounds the point charge q and doesn't have holes in it, we will *always get the same flux!* What this means is that we always choose very convenient surfaces, ones for which the electric field is just a constant over the whole surface. For convenience, we define a new constant $\epsilon_0 = 1/4\pi k_e$, known as the "permittivity of free space:"

Permittivity of free space:

$$\epsilon_0 = \frac{1}{4\pi k_e} = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2} \tag{3.8}$$

Recall k_e is Coulomb's constant from Equation 3.2. (This means of course that we can put all of our other equations, like Eq. 3.1, in terms of ϵ_0 instead of k_e, since $k_e = 1/4\pi\epsilon_0$. You will often see them this way.) This gives Gauss's law a nice simple form:

Gauss's law: the electric flux Φ_E through any *closed* surface is equal to the net charge inside the surface, Q_{inside}, divided by ϵ_0:

$$\Phi_E = \frac{Q_{\text{inside}}}{\epsilon_0} \tag{3.9}$$

We will not *derive* Gauss' law here, but simply state it as fact, and show you a few examples of how to use it.

3.8.2 Gauss' Law as a Conservation Law

Fundamentally, Gauss' law is a manifestations of the *divergence theorem* (*a.k.a.* Green's theorem or the Gauss-Ostrogradsky theorem). Essentially, it states that *the sum of all sources minus the sum of all sinks gives the net flow out of a region.* The same law is applies to fluids. If a fluid is flowing, and we want to know how much fluid flows out of a certain region, then we need to add up the sources inside the region and subtract the sinks. The divergence theorem is basically a conservation law - the volumetric total of all sources minus sinks equals the flow across a volume's boundary.

In the case of electric fields, this gives Gauss' law (Eq. 3.9) – the electric flux through any closed surface must relate to a *net charge* inside the volume bounded by that surface. The net magnitude of the vector components of the electric field pointing outward from a surface must be equal to the net magnitude of the vector components pointing inward, *plus* the amount of free charge inside. This is a manifestation of the fact that electric field lines do have to originate from somewhere – charges. **The difference between the flow of field lines into a surface and the flow out of a surface is just how many charges are within the surface, that is all that Gauss' law says.** This is fundamentally due to the fact that for *all* inverse square laws, like Coulomb's law or Newton's law of gravitation, the strength of the field falls off as $1/r^2$, but the area of an enclosing surface *increases* as r^2. The two dependencies cancel out, and we are left with the result that the flux is only related to difference between the number of enclosed sources and sinks.

Though Gauss's law is very powerful, it is usually used in specially symmetric cases (spheres, cylinders, planes) where it is easy to draw a surface of constant electric field around the charges of interest (like a sphere around a point charge). We will work through a few of these examples presently.

Question: Why would we not want to choose a cube as our surface enclosing the point charge?

Choosing a cube would not give us any nice surfaces with a constant electric field on them.

3.8.3 Example: The Field Around a Spherical Charge Distribution

We can use Gauss' law to calculate the electric field of *any* spherically symmetric distribution of charge, and as a bonus, discover an important fact. A spherically symmetric distribution of charge just means that the number of charges per unit volume (the charge *density*) depends only on the radius from a central point. That doesn't mean that the density doesn't vary with radius, just that it doesn't vary with *angle*. An example of such a distribution is shown in Fig. 3.15a – in this case, the density decreases with radius up to a distance R, beyond which it is zero.

What is the electric field at some arbitrary point P_1 outside the distribution, or at some arbitrary point P_2 inside it (Fig. 3.15b)? Do we have to calculate the field from every tiny bit of charge in the distribution and sum them all together? No, this is the point of Gauss' law – if you have a problem with special symmetries, they can usually be exploited to save a lot of labor.

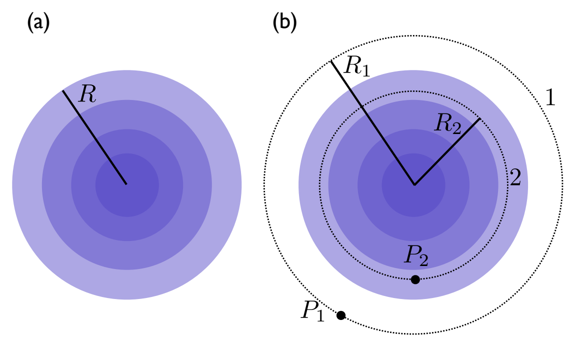


Figure 3.15: (a) *A spherically symmetric charge distribution. The density of charge depends only on the distance from the center point.* **(b)** *Two Gaussian surfaces to determine the field at an arbitrary point outside (P_1) and inside (P_2) the distribution.*

The charge distribution is, by definition, spherically symmetric. As you may have noticed, the electric field must take on the same symmetry as the charge distribution.[xv] That means that the electric field in this case will be spherically symmetric, and will be directed radially from the central point. No other direction is special or unique in this problem, only the radial direction. That means that if we draw a spherical surface of radius $R_1 > R$ completely surrounding the sphere, surface 1 in Fig. 3.15b, the electric field will be constant everywhere on that surface. We can easily calculate the flux through this surface, and hence the electric field:

$$\Phi_E = EA = \frac{Q_{\text{inside1}}}{\epsilon_0} \qquad R_1 > R \tag{3.10}$$

$$= E \times 4\pi R_1^2 = \frac{Q_{\text{inside1}}}{\epsilon_0} \qquad R_1 > R \tag{3.11}$$

$$\Longrightarrow E = \frac{Q_{\text{inside1}}}{4\pi\epsilon_0 R_1^2} = \frac{k_e Q_{\text{inside1}}}{R_1^2} \qquad R_1 > R \tag{3.12}$$

What we now see is that this is the same thing as the field from a point charge – *the field outside a spherically symmetric charge distribution behaves exactly as if all of its charge is concentrated at the center.* This is, in fact, a particular property of $1/r^2$ laws, and you should recall that this principle is true in the gravitational case for spherically symmetric mass distributions. The earth

[xv] Appendix B may help you think about that.

attracts other bodies as if its mass were concentrated at a point in the center. So long as we are dealing with spherically symmetric distributions, it is not even an approximation to deal with infinitesimal point charges!

One thing to keep in mind: this is *not* something like the center of mass. A perfect cube does *not* behave as if it had all its mass concentrated at its center. This all really comes from the nature of $1/r^2$ forces and the divergence theorem.

What about surface 2, radius R_2, drawn inside the charge distribution? From the analysis above, all that matters is how much charge is contained inside the surface. Everything outside the surface contributes an equal contribution, but in all different directions, and the whole thing cancels. What is outside the surface may just as well not exist, so far as the electric field is concerned. Finding the field at point P_2 is then just a matter of figuring out how much charge is inside the second surface. Depending on the distribution, that may not be so easy ... but it would have been a *lot* worse without Gauss' law.

We have actually developed a more important result than we set out to. Using only Gauss' law, we have *derived* that the field from spherically symmetric charge distributions is equivalent to that of a point charge, and follows a $1/r^2$ law. *Actually, we have derived Coulomb's law from Gauss' law. In fact, the two are equivalent.* We could have started from Gauss' law in the first place and arrived at Coulomb's law, instead of *assuming* Coulomb's law to be true and *then* introducing Gauss' law. Gauss' law is in fact far more general in an important way, as we have noted above, since it gives the equivalence relationship for *any* flux (*e.g.,* liquids, electric fields, gravitational fields) flowing out of any closed surface and the enclosed sources and sinks of the flux (*e.g.,* electric charges, masses). We will see in Ch. 7.2.1 that there is also a Gauss' law for magnetism, just as there is a Gauss' law for gravity, *viz.*:

$$\Phi_g = 4\pi GM \tag{3.13}$$

where Φ_g is the flux from the gravitational field through a closed surface, G is the universal gravitational constant, and M is the mass enclosed by the surface. Just as we proved that any spherically symmetric charge distribution behaves as a point charge and follows an inverse square law, one can prove that any spherically symmetric mass distribution is equivalent to a point mass, and follows the familiar inverse square law for gravitation.

3.8.4 Example: The Field Above a Flat Conductor

If we can come up with a clever surface on which to apply Gauss' law, we can solve some otherwise nasty problems. Figure 3.16 shows a large ("infinite") conducting plate, whose surface is charged. What is the field at the surface of this plate due to the charges? We know it is uniform and constant, but that is about it.

Since it is a conductor, the charge distribution on the surface, and hence electric field, are uniform. Since we do not want to restrict ourselves to a plate of any particular size, but rather, solve a *general* problem, we will say that the plate has a certain charge per unit area σ_E, defined as the total charge of the plate divided by its surface area. That way, we can later find the field near *any* plate.

What sort of surface should we take to find the flux? A plain box is a good choice, as it turns out, due to the symmetry of the problem. We will take a box with a top and bottom whose area are A. The area of the sides are not important, as it turns out, but we can call them B just to be complete.

Why would we choose a box in this case, when we just said it is a bad choice for a point charge? We know that the field is perpendicular to the surface of a conductor everywhere, so in this case the field is going to be purely perpendicular to the plate. Therefore, it is only important that we draw a Gaussian surface such that every part of the surface is either perfectly parallel or perfectly perpendicular to the plate. A cylinder would work perfectly fine too, which should be clear from the rest of the discussion.

Along the surface making up the sides of the box, the flux is zero since the field lines are parallel to it everywhere. On the top end cap, the flux is perfectly normal. The bottom end cap is completely inside the conductor, so we know the field has to be zero there! If we call the magnitude of electric field above the plate E, we can readily calculate the flux. Because the plate is supposed to be very large in extent, the field can be assumed to be completely uniform so long as the distances above the plate we consider are small compared to the size of the plate.

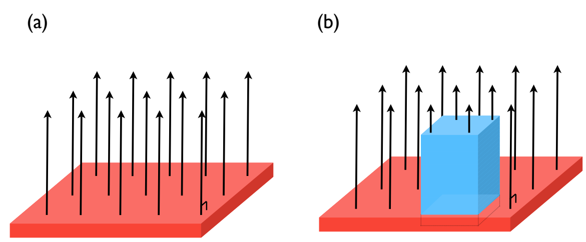


Figure 3.16: **(a)** *A large, flat charged conducting slab. The charge distribution on the surface, and hence electric field, are uniform.* **(b)** *A cylinder is our surface for Gauss' law. Along the sides of the cylinder, the flux is zero since the field lines are parallel – the flux is non-zero only through the end caps.*

The total charge enclosed by this cylinder is just the cross-sectional area of the plate enclosed by the box times the charge per unit area σ_E:

$$Q_{\text{inside}} = \sigma_E A \tag{3.14}$$

Applying Gauss' law is now straightforward, we just have to find the flux through the top end cap:

$$\Phi_E = EA = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{\sigma_E A}{\epsilon_0} \tag{3.15}$$

$$\Longrightarrow E = \frac{\sigma_E}{\epsilon_0} \tag{3.16}$$

No problem! The electric field is indeed constant, as it has to be, and *independent of the distance* *from the plate.* This makes sense too, since the plate is supposed to be very, very large. Strictly, this is true only for an infinite plate, but it is close so long as we consider distances above the plate which are very small compared to the size of the plate. Finally, it should be clear now that it didn't matter what sort of shape we used at all, so long as it has a flat end parallel to the plate, and sides perpendicular to it.

3.8.5 Example: The Field Inside and Outside a Hollow Spherical Conductor

Figure 3.17 shows a point charge Q_1 is inside a thin spherical conducting shell with inner radius R_1 and outer radius R_2. How can we find the electric field inside the shell and outside the shell? Easy, we just have to apply Gauss' law a couple of times.

For any spherical surface *inside* the sphere, say a sphere of radius $r < R_1$ like surface 1 in Fig. 3.17, only the point charge is inside the volume enclosed by the sphere. If we center the sphere exactly on the point charge, since the field of a point charge is spherically symmetric the field is constant everywhere on the sphere's surface. Gauss' law then gives us:

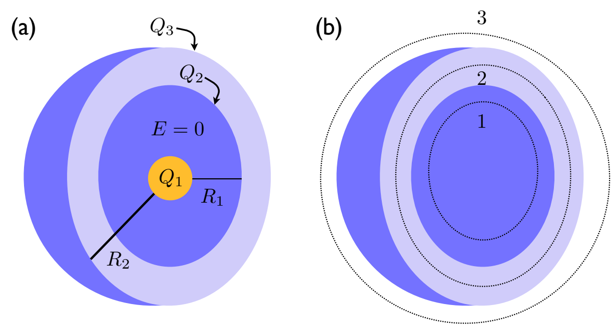


Figure 3.17: **(a)** *A point charge Q_1 is inside a thin spherical conducting shell with inner radius R_1 and outer radius R_2. The presence of the point charge induces an equal but opposite charge on the inner surface of the conductor to satisfy Gauss' law.* **(b)** *Three Gaussian surfaces to find the field inside and outside the conductor.*

$$\Phi_E = EA = \frac{Q_{\text{inside}}}{\epsilon_0} \qquad r < R_1 \tag{3.17}$$

$$= E \times 4\pi r^2 = \frac{Q_1}{\epsilon_0} \qquad r < R_1 \tag{3.18}$$

Now we just need to solve for E, and make use of the fact that $\epsilon_0 = \frac{1}{4\pi k_e}$ (Eq. 3.8):

$$E = \frac{Q_1}{4\pi\epsilon_0 r^2} = \frac{k_e Q_1}{r^2} \quad r < R_1 \tag{3.19}$$

Of course, this makes perfect sense – the field inside is just that of the point charge, as if the conductor were not there at all! As we saw above, electric fields are like gravitational fields in this way – inside a spherical shell, both the gravitational and electrical forces cancel in all directions by symmetry.

Next, we consider surface 2, a surface inside the conductor itself. We know already that everywhere *inside* the conductor, *i.e.,* $R_1 < r < R_2$, we must have $E = 0$. Done! That seemed too easy, didn't it? It was – we missed one little point.

In the end, we also want to find the field *outside* the shell entirely, and for this we have to consider surface 1. Now we have to be careful, and think about what we have missed. For surface 2, drawn inside the conductor, we said $E=0$ as it has to be for a conductor. This is true. But how can that be, with a point charge sitting right inside? Actually, it can't: what happens is that the point charge Q_1 induces a *equal but opposite* charge $Q_2 = -Q_1$ on the inside surface of the conductor. Think of it this way – if this did *not* happen, then the total charge enclosed by surface 2 would not be zero, and by Gauss' law the field inside the conductor could not be zero. The induced charge Q_2 ensures that the total charge enclosed by surface 2 is zero, and thus the field inside the conductor is zero as it has to be. Then we would have a contradiction on our hands, which is not OK. This also is another aspect of conductors looking like mirrors for field lines. Physically, the charge Q_1 attracts opposite mobile charges in the conductor, giving a net negative charge on the inner surface.

Now, what about surface 3? Before we placed the charge Q_1 inside the conductor, it was electrically neutral. This still has to be true after we place the charge – overall, the conductor must have no net charge. Well, if there is a charge $Q_2=-Q_1$ on the inner surface, and overall it is neutral, then there must be a charge $Q_3=Q_1$ induced on the *outer* surface to cancel the induced charge on the inner surface. The net negative charge on the inner surface attracted by the point charge Q_1 leaves a deficit of negative charge on the outer surface, for a net positive surface charge. Now we can run Gauss' law for surface 3:

$$\Phi_E = EA = \frac{Q_{\text{inside}}}{\epsilon_0} \qquad r > R_2 \tag{3.20}$$

$$= E \times 4\pi r^2 = \frac{Q_1 + Q_2 + Q_3}{\epsilon_0} \qquad r > R_2 \tag{3.21}$$

$$= 4\pi r^2 E = \frac{Q_1 - Q_1 + Q_1}{\epsilon_0} \qquad r > R_2 \tag{3.22}$$

$$\Longrightarrow E = \frac{Q_1}{4\pi\epsilon_0 r^2} \qquad r > R_2 \tag{3.23}$$

Lo and behold, the field outside the sphere looks just like that of the original point charge, same as inside the sphere (remembering that $\epsilon_0 = \frac{1}{4\pi k_e}$, Eq. 3.8). Again, what happens physically is that the point charge pulls the mobile charges from the conductor to its inner surface, leaving the inner surface with an equal and opposite charge. This means that the outer surface must be deficient in those same charges, and thus has an equal and like charge to Q_1.

Now we can combine our results, and we have the electric field in all three regions:

$$E = \frac{k_e Q_1}{r^2} \qquad r > R_2 \tag{3.24}$$

$$E = 0 \qquad R_1 < r < R_2 \tag{3.25}$$

$$E = \frac{k_e Q_1}{r^2} \qquad r < R_1 \tag{3.26}$$

3.8.6 Example: The Due to a Line of Charge

As one last example, we will use Gauss' law to find the electric field due to an infinite line of charge, or equivalently, a conducting wire with a net surface charge, as shown in Fig. 3.18a. What does the electric field look like? If the line of charge is infinite (or at least very long compared to the distance we are away from it), all of the transverse components of the field will cancel each other, and by symmetry, the field must be radially symmetric about the wire. That is, the field must point perpendicularly away from the wire axis.

With the symmetry of the wire being cylindrical, it makes most sense to use a cylinder drawn concentrically around the wire as our Gaussian surface, Fig. 3.18b. We will choose a cylinder of radius r, and length l. The field is parallel to the end caps of the cylinder, so they contribute no flux at all. Being radially symmetric, the field is perfectly *perpendicular* to the round surface of the cylinder, and we can easily calculate the flux and find the electric field. First, we remember that the surface area of a cylinder (without the end caps) is $2\pi rl$. Second, the cylinder of length l encloses a length l of the wire, which must contain λl charges since λ is the charge per unit length. Putting that all together:

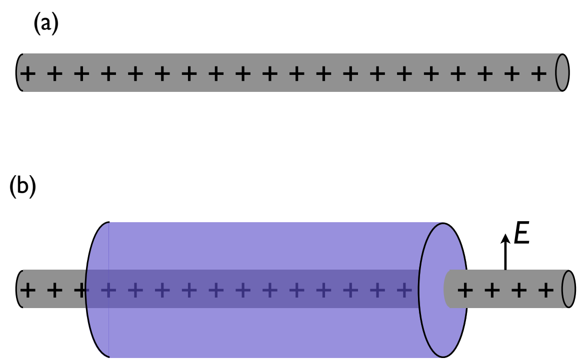


Figure 3.18: **(a)** *An "infinite" line charge, with λ charges per unit length.* **(b)** *A cylindrical Gaussian surface. On the caps of the cylinder, the field is parallel, and the flux is zero.*

$$\Phi_E = E \cdot 2\pi rl = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0} \tag{3.27}$$

$$\Longrightarrow E = \frac{\lambda}{2\pi r\epsilon_0} = \frac{2k_e\lambda}{r} \tag{3.28}$$

In this case, the field falls off as $1/r$, far slower than a point charge, but *not independent of distance* like we found for the sheet of charge. It *is* independent of the length of the cylinder we chose, as it *must* be: the wire is supposed to be infinite, and the value of l was chosen arbitrarily!

File this result away. We will need it again in Chapter 7!

3.9 Miscellanea

Solving electric field problems

1. **Convert all units** to SI – charges in Coulombs, distances in meters.
2. **Draw** a diagram of the charges in the problem.
3. **Identify** the charge of interest, and what you want to know about it.
4. **Choose** your coördinate system and origin – pick the most convenient one based on the symmetry of the problem. Usually, this is an $x-y$ Cartesian system, with the origin at some special point (*e.g.,* on one charge or between two charges)
5. **Apply Coulomb's law** For each charge Q, find the electric force on the charge of interest q. The vector direction of the force is along the line of the two charges, directed away from Q if it has the same sign as q and toward Q if it has the opposite sign as q. Find the angle θ this vector makes with the positive x axis – the x component of the electric force will be $F\cos\theta$, the y component will be $F\sin\theta$.
6. **Sum the x components** from each charge Q to get the resultant x component of the electric force.
7. **Sum the y components** from each charge Q to get the resultant y component of the electric force.
8. **Find the total resultant force** from the total x and y components, using the Pythagorean theorem and trigonometry to find the magnitude and direction: $|F_{\text{tot}}|=\sqrt{|F_x|^2+|F_y|^2}$ and $\tan\theta=\frac{F_y}{F_x}$

Charge transfer and chemical reactions: batteries

reaction: $PbO_2(s) + SO_4^{2-}(aq) + 4H^+ + 2e^- \longrightarrow PbSO_4(s) + 2H_2O(l)$ (*reduction*)

From one point of view, this reaction is nothing more than charges being transferred from one species to another. If we only write down the charges involved, we would have this:

charge transfer: $0 + (\text{-}2e) + 4e + (\text{-}2e) \longrightarrow 0 + 0$

This is consistent with only electrons being transferred from one object to another – four electrons are being transferred to the four H^+, including two from the SO_4^{2-} and two 'free' electrons. Charge is conserved in this reaction as well. Another example:

reaction: $Pb(s) + SO_4^{2-}(aq) \longrightarrow PbSO_4(s) + 2e^-$ (*oxidation*)

In terms of charges,

charge transfer: $0 + (\text{-}2e) \longrightarrow 0 + (\text{-}2e)$

The above reactions are essentially what take place in a normal lead-acid car battery. Plates of lead (Pb) and lead oxide (PbO_2) immersed in a sulfuric acid (H_2SO_4) electrolyte. The Pb plate is oxidized, releasing two electrons per Pb atom, while the PbO_2 plate is reduced, accepting two electrons per molecule. Connecting the two plates together through a circuit lets electrons released from the Pb plate travel to the PbO_2 plate, which makes an electric current.

3.10 Quick Questions

1. Two charges of +1 μC each are separated by 1 cm. What is the force between them?

- ◯ 0.89 N
- ◯ 90 N
- ◯ 173 N
- ◯ 15 N

2. The electric field *inside* an isolated conductor is

- ◯ determined by the size of the conductor
- ◯ determined by the electric field outside the conductor
- ◯ always zero
- ◯ always larger than an otherwise identical insulator

3. Which statement is false?

- ◯ Charge deposited on conductors stays localized
- ◯ Charge distributes itself evenly over a conductor
- ◯ Charge deposited on insulators stays localized
- ◯ Charges in a conductor are mobile, and move in response to an electric force

4. Which of the following is true for the electric force, but not the gravitational force?

- ◯ The force propagates at a speed of c
- ◯ The force acts at a distance without any intervening medium
- ◯ The force between two bodies depends on the square of the distance between them
- ◯ The force between two bodies can be repulsive as well as attractive.

5. Two charges of +1 μC are separated by 1 cm. What is the magnitude of the electric field halfway between them?

- ◯ 9×10^7 N/C
- ◯ 4.5×10^7 N/C
- ◯ 0
- ◯ 1.8×10^8 N/C

6. A circular ring of charge of radius b has a total charge of q uniformly distributed around it. The magnitude of the electric field at the center of the ring is:

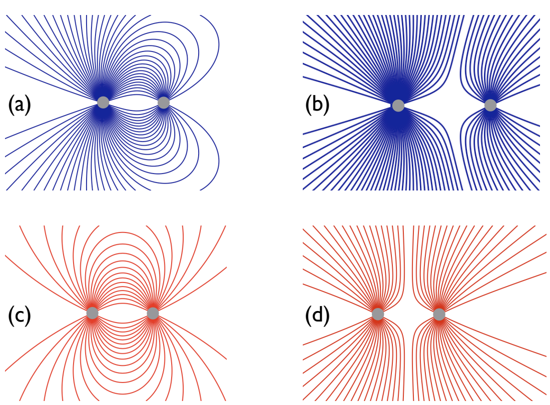
◯ 0
◯ $k_e q/b^2$
◯ $k_e q^2/b^2$
◯ $k_e q^2/b$
◯ none of these.

7. Two isolated identical conducting spheres have a charge of q and $-3q$, respectively. They are connected by a conducting wire, and after equilibrium is reached, the wire is removed (such that both spheres are again isolated). What is the charge on each sphere?

◯ $q,\ -3q$
◯ $-q,\ -q$
◯ $0,\ -2q$
◯ $2q,\ -2q$

8. A single point charge $+q$ is placed exactly at the center of a hollow conducting sphere of radius R. Before placing the point charge, the conducting sphere had zero net charge. What is the magnitude of the electric field *outside* the conducting sphere at a distance r from the center of the conducting sphere? *I.e., the electric field for* $r > R$.

◯ $|\vec{\mathbf{E}}| = -\frac{k_e q}{r^2}$
◯ $|\vec{\mathbf{E}}| = \frac{k_e q}{(R+r)^2}$
◯ $|\vec{\mathbf{E}}| = \frac{k_e q}{R^2}$
◯ $|\vec{\mathbf{E}}| = \frac{k_e q}{r^2}$



9. Which set of electric field lines could represent the electric field near two charges of the *same sign*, but *different magnitudes*?

◯ a
◯ b
◯ c
◯ d

10. Referring again to the figure above, which set of electric field lines could represent the electric field near two charges of *opposite sign* and *different magnitudes*?

◯ a
◯ b
◯ c
◯ d

11. A "free" electron and a "free" proton are placed in an identical electric field. Which of the following statements are true? *Check all that apply.*

◯ Each particle is acted on by the same electric force and has the same acceleration.
◯ The electric force on the proton is greater in magnitude than the force on the electron, but in the opposite direction.
◯ The electric force on the proton is equal in magnitude to the force on the electron, but in the opposite direction.
◯ The magnitude of the acceleration of the electron is greater than that of the proton.
◯ Both particles have the same acceleration.

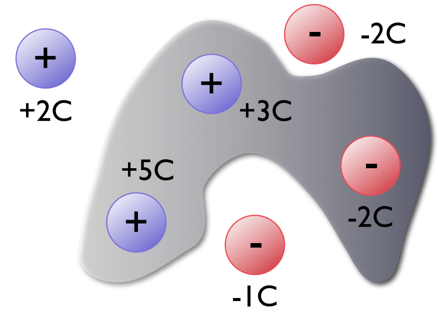
12. A point charge q is located at the center of a (non-conducting) spherical shell of radius a that has a charge $-q$ uniformly distributed on its surface. What is the electric field *for all points outside the spherical shell?*

◯ none of these
◯ $E=0$
◯ $E=q/4\pi r^2$
◯ $E=kq/r^2$
◯ $E=kq^2/r^2$

13. What is the electric field inside the same shell a distance $r<a$ from the center (*i.e.,* a point inside the spherical shell)?

◯ $E=kq/r^2$
◯ $E=kq^2/r^2$
◯ none of these
◯ $E=0$
◯ $E=q/4\pi r^2$

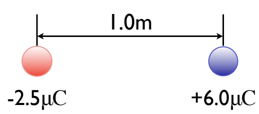
14. What is the electric flux through the surface at right?



- ◯ $+5\,\mathrm{C}/\epsilon_0$
- ◯ $-3\,\mathrm{C}/\epsilon_0$
- ◯ 0
- ◯ $+6\,\mathrm{C}/\epsilon_0$

15. A spherical conducting object A with a charge of $+Q$ is lowered through a hole into a metal (conducting) container B that is initially uncharged (and is not grounded). When A is at the center of B, but not touching it, the charge on the inner surface of B is:

- ◯ $+Q$
- ◯ $-Q$
- ◯ 0
- ◯ $+Q/2$
- ◯ $-Q/2$



16. Determine the point (other than infinity) at which the total electric field is zero. This point is not between the two charges.

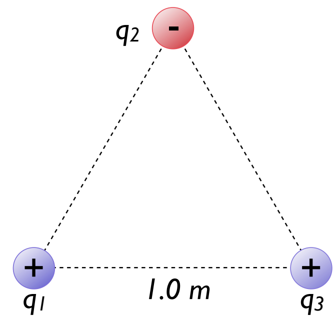
- ◯ 3.5 m to the left of the negative charge
- ◯ 2.1 m to the right of the positive charge
- ◯ 1.3 m to the right of the positive charge
- ◯ 1.8 m to the left of the negative charge

17. A flat surface having an area of $3.2\,\mathrm{m}^2$ is rotated in a uniform electric field of magnitude $E = 5.7 \times 10^5\,\mathrm{N/C}$. What is the electric flux when the electric field is parallel to the surface?

- ◯ $1.82 \times 10^6\,\mathrm{N \cdot m^2/C}$
- ◯ $0\,\mathrm{N \cdot m^2/C}$
- ◯ $3.64\,\mathrm{N \cdot m^2/C}$
- ◯ $0.91\,\mathrm{N \cdot m^2/C}$

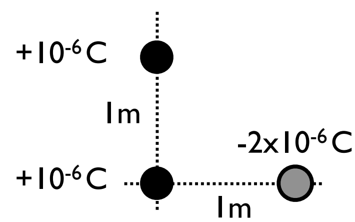
18. Three charges are arranged in an equilateral triangle, as shown at left. All three charges have the same *magnitude* of charge, $|q_1| = |q_2| = |q_3| = 10^{-9}\,\mathrm{C}$ (note that q_2 is negative though). What is the **force** on q_2, magnitude and direction?

- ◯ $9.0\,\mu\mathrm{N}$, up (90°);
- ◯ $16\,\mu\mathrm{N}$, down (-90°);
- ◯ $18\,\mu\mathrm{N}$, down and left (225°);
- ◯ $8.0\,\mu\mathrm{N}$, up and right (-45°)

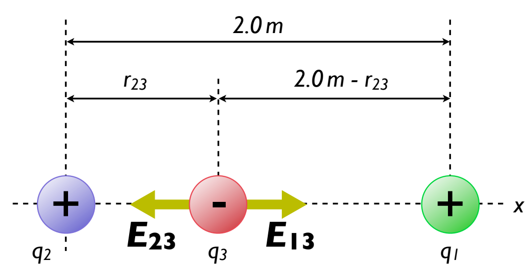


3.11 Problems

1. Two charges of $+10^{-6}\,\mathrm{C}$ are separated by 1 m along the vertical axis. What is the net **horizontal** force on a charge of $-2\times10^{-6}\,\mathrm{C}$ placed one meter to the right of the lower charge?



2. Three point charges lie along the x axis, as shown at left. A positive charge $q_1 = 15\,\mu\mathrm{C}$ is at $x = 2\,\mathrm{m}$, and a positive charge of $q_2 = 6\,\mu\mathrm{C}$ is at the origin. Where must a *negative* charge q_3 be placed on the x-axis **between the two positive charges** such that the resulting electric force on it is zero?



3.12 Solutions to Quick Questions

1. 90 N. We just need to use Eq. 3.1 and plug in the numbers ... remembering that μ means 10^{-6}:

$$\begin{aligned}
\vec{\mathbf{F}} &= k_e \frac{q_1 q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} \\
|\vec{\mathbf{F}}| &= k_e \frac{q_1 q_2}{r_{12}^2} \\
&= 8.9875 \times 10^9 \,\frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}} \left[\frac{\left(1\times 10^{-6}\,\mathrm{C}\right)\left(1\times 10^{-6}\,\mathrm{C}\right)}{\left(1\times 10^{-2}\,\mathrm{m}\right)^2}\right] \\
&\approx 9\times 10^9 \,\frac{\mathrm{N \cdot \cancel{m^2}}}{\cancel{\mathrm{C^2}}} \left[\frac{1\times 10^{-12}\,\cancel{\mathrm{C^2}}}{1\times 10^{-4}\,\cancel{\mathrm{m^2}}}\right] \\
&= 9\times 10^1\,\mathrm{N} \\
|\vec{\mathbf{F}}| &= 90\,\mathrm{N}
\end{aligned}$$

2. Always zero. Re-read Sect. 3.5 to remind yourself *why* this must be true.

3. Charge deposited on conductors stays localized. See Sect. 3.2.

4. The force between two bodies can be repulsive as well as attractive. Both the electric and gravitational forces propagate at the speed of light, both act through empty space, and both are inverse-square laws. The only difference is that gravity can only be attractive, since there is no such thing as negative mass.

5. 0. Halfway between, the magnitude of the field from each individual charge is the same, but *they act in opposite directions.* Therefore, exactly in the middle, they cancel, and the field is zero. This is the same as the field exactly at the midpoint of an electric dipole. It might be easier to convince yourself the field is zero if you draw a picture including the electric field lines.

6. 0. The field at the center from a point on the ring is always canceled by the field from another point 180° away.

7. $-q$, $-q$. The thing to remember is that any charge on a conductor spreads out evenly over its surface. When we have the conducting spheres isolated, they have q and $-3q$ respectively, and this charge is spread evenly over each sphere. When we connect them with a conducting wire, suddenly charges are free to move from one conductor, across the wire, into the other conductor. Its just the same as if we had one big conductor, and all the *total net charge of the two conductors combined* will spread out evenly over *both* spheres and the wire.

If the charge from each sphere is allowed to spread out evenly over both spheres, then the $-3q$ and $+q$ will both be spread out evenly everywhere. The $+q$ will cancel part of the $-3q$, leaving a total net charge of $-2q$ spread over evenly over both spheres, or $-q$ on each sphere. Once we disconnect the two spheres again, the charge remains equally distributed between the two.

8. $|\vec{\mathbf{E}}|=\frac{k_eq}{r^2}$**.** The easiest way out of this one is Gauss' law. First, Gauss' law told us that any spherically symmetric charge distribution behaves as a point charge. Second, Gauss' law tells us that the electric flux out of some surface depends only on the enclosed charge. If we draw a spherical surface of radius r and area A around the shell and point charge, centered on the center of the conducting sphere, Gauss' law gives:

$$\begin{aligned}\Phi_E &= \frac{q_{encl}}{\epsilon_0} = 4\pi k_e q_{encl}\\ EA &= 4\pi k_e q_{encl}\\ E &= \frac{4\pi k_e q_{encl}}{A}\end{aligned}$$

The surface area of a sphere is $A=4\pi r^2$. In this case, the enclosed charge is just q, since the hollow conducting sphere itself has no charge of its own. Gauss' law only cares about the *total net charge* inside the surface of interest. This gives us:

$$E = \frac{4\pi k_e q}{4\pi r^2} = \frac{\cancel{4\pi} k_e q}{\cancel{4\pi} r^2} = \frac{k_e q}{r^2}$$

There we have it, it is just the field of a point charge q at a distance r.

If we want to get formal, we should point out that the point charge q induces a negative charge $-q$ on the inner surface of the hollow conducting sphere. Since the sphere is overall neutral, the outer surface must therefore have a net positive charge $+q$ on it. This makes no difference in the result – the total *enclosed* charge, for radii larger than that of the hollow conducting sphere $(r>R)$, is still just q. If we start with an uncharged conducting sphere, and keep it physically isolated, any induced charges have to cancel each other over all.

If this is still a bit confusing, go back and think about induction charging again. A charged rod was used to induce a positive charge on one side of a conductor, and a negative charge on the other. *Overall*, the 'induced charge' was just a rearrangement of existing charges, so if the conductor started out neutral, no amount of 'inducing' will change that. We only ended up with a *net charge* on the conductor when we used a ground connection to 'drain away' some of the induced charges. Or, if you like, when we used a charged rod to repel some of the conductor's charges through the ground connection, leaving it with a net imbalance.

9. (b). If the charges are of the *opposite* sign, then the field lines would have to run from one charge directly to the other. Field lines start on a positive charge and end on a negative one, and there should be many lines which run from one charge to the other. Since opposite charges attract, the field between them is extremely strong, the lines should be densest right between the charges. This is the case in (a) and (b), so they are not the right ones.

By the same token, for charges of the *same* sign, the force is repulsive, and the electric field midway between them cancels. The field lines should "push away" from each other, and no field line from a given charge should reach the other charge – field lines cannot start and end on the same sign charge. This means that only (b) and (d) could possibly correspond to two charges of the same sign.

Next, the field lines leaving or entering a charge has to be proportional to the magnitude of the charge. In (d) there are the same number of lines entering and leaving each charge, so the charges are of the same magnitude. One can also see this from the fact that the lines are symmetric about a vertical line drawn midway between the charges. In (b) there are clearly many more lines near the left-most charge.

Or, right off the bat, you could notice that only (a) and (b) are asymmetric, and only (b) and (d) look like two like charges. No sense in over-thinking this one.

10. (a). By similar reasoning as above, only figure **a** could represent two opposite charges of different magnitude.

11. The electric force on the proton is equal in magnitude to the force on the electron, but in the opposite direction. The magnitude of the acceleration of the electron is greater than that of the proton.

12. $E=0$. The simplest way to solve this one is with Gauss' law. First, Gauss law told us that any spherically symmetric charge distribution behaves as a point charge. Second, Gauss law tells us that the electric flux out of some surface depends only on the enclosed charge. If we draw a spherical surface of radius r and area A enclosing the shell *and* the point charge, centered on the center of the conducting sphere, the total enclosed charge is that of the shell plus that of the point charge: $q_{\text{encl}}=q+(-q)=0$. If the enclosed charge is zero for any sphere drawn outside of and enclosing the spherical shell, then the electric field for all points outside the spherical shell.

13. $E=k_eq/r^2$. Just like the last question, we need Gauss' law. This time, we have to draw a sphere surrounding the point charge, but *inside* of the spherical shell. Gauss' law tells us that the electric field depends only on the *enclosed* charge within our sphere. The only charge enclosed is the point charge at the center of the shell, q – the charge on the spherical shell is outside of our spherical surface, so it is not enclosed and does not contribute to the electric field inside. Now we just apply Gauss' law, knowing that the enclosed charge is q, and the surface area of the sphere is $4\pi r^2$:

$$\Phi_E = \frac{q_{\text{encl}}}{\epsilon_0} = 4\pi k_e q \tag{3.29}$$

$$EA = 4\pi k_e q \tag{3.30}$$

$$E = \frac{4\pi k_e q}{4\pi r^2} = \frac{k_e q}{r^2} \tag{3.31}$$

14. $+6\,\text{C}/\epsilon_0$. Again, this question requires Gauss' law. We know that the electric flux through this surface only depends on the total amount of enclosed charge. All we need to do is add up the *net* charge inside the surface, since any charges outside the surface do not contribute to the flux. There are only three charges enclosed by the surface ... so:

$$\text{net charge} = 3\,\text{C} + 5\,\text{C} - 2\,\text{C} = 6\,\text{C} \tag{3.32}$$

The electric flux Φ_E is then just the enclosed charge divided by ϵ_0, or $+6\,\text{C}/\epsilon_0$.

15. $-Q$. The charge $+Q$ on object A induces a negative charge $-Q$ on the inner surface of the conducting container B.

16. 1.8 **m to the left of the negative charge.** By symmetry, we can figure out on which side the field should be zero. In between the two charges, the field from the positive and negative charges *add together*. The force on a fictitious positive test charge placed in between the two would experience a force to the left due to the positive charge, and another force to the left due to the negative charge. There is no way the fields can cancel here.

If we place a positive charge to the *right of the positive charge*, it will feel a force to the right from the positive charge, and a force to the left from the negative charge. The directions are opposite, but the fields still cannot cancel because the test charge is closest to the larger charge.

This leaves us with points to the left of the negative charge. The forces on a positive test charge will be in opposite directions here, and we are closer to the smaller charge. What position gives zero field? First, we will call the position of the negative charge $x = 0$, which means the positive charge is at $x = 1\,\text{m}$. We will call the position where electric field is zero x. The distance from this point to the negative charge is just x, and the distance to the positive charge is $1+x$. Now write down the electric field due to each charge:

$$\begin{aligned} E_{\text{neg}} &= \frac{k_e(-2.5\,\mu\text{C})}{x^2} \\ E_{\text{pos}} &= \frac{k_e(6\,\mu\text{C})}{(1+x)^2} \end{aligned}$$

The field will be zero when $E_{\text{neg}} + E_{\text{pos}} = 0$:

$$\begin{aligned} E_{\text{neg}} + E_{\text{pos}} &= 0 \\ \frac{k_e(-2.5\,\mu\text{C})}{x^2} + \frac{k_e(6\,\mu\text{C})}{(1+x)^2} &= 0 \\ \frac{\cancel{k_e}(-2.5\,\cancel{\mu\text{C}})}{x^2} + \frac{\cancel{k_e}(6\,\cancel{\mu\text{C}})}{(1+x)^2} &= 0 \\ \frac{-2.5}{x^2} + \frac{6}{(1+x)^2} &= 0 \\ \Rightarrow \qquad \frac{2.5}{x^2} &= \frac{6}{(1+x)^2} \end{aligned}$$

Cross multiply, apply the quadratic formula:

$$\begin{aligned}
2.5(1+x)^2 &= 6x^2 \\
2.5+5x+2.5x^2 &= 6x^2 \\
3.5x^2-5x-2.5 &= 0 \\
\Rightarrow \qquad x &= \frac{-(-5)\pm\sqrt{5^2-4(-2.5)(3.5)}}{2(3.5)} \\
x &= \frac{5\pm\sqrt{25+35}}{7} \\
x &= \frac{5\pm 7.75}{7} = 1.82, -0.39
\end{aligned}$$

Which root do we want? We wrote down the distance x the distance to the *left* of the negative charge. A negative value of x is then in the wrong direction, in between the two charges, which we already ruled out. The positive root, $x = 1.82$, means a distance 1.82 m to the *left* of the negative charge. This is what we want.

17. $0\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}$**.** Remember that electric flux is $\Phi_E = EA\cos\theta$, where θ is the angle between a line perpendicular to the surface and the electric field. If E is *parallel* to the surface, then $\theta = 90$ and $\Phi_E = 0$.

Put more simply, there is only an electric flux if field lines penetrate the surface. If the field is parallel to the surface, no field lines penetrate, and there is no flux.

18. $16\,\mu$**N, down (-90°).** The easiest way to solve this one is by symmetry and elimination. The negative charge q_2 feels an attractive force from both q_1 and q_2. Since both charges are the same vertical distance away and below q_2, both will give a force in the vertical downward direction of equal magnitude and direction. Since both charges are horizontally the same direction away but *on opposite sides*, the horizontal forces will be equal in magnitude but *opposite* in direction – the horizontal forces will cancel. Therefore, the net force has to be purely in the vertical direction and downward, so the second choice is the only option! Of course, you can calculate all of the forces by components and add them up ... you will arrive at the same answer.

3.13 Solutions to Problems

1. $-0.0244\,\mathbf{N}$**.** We are only interested in the x component of the force, which makes things easier. First, we are trying to find the force on a negative charge due to two positive charges. Both positive charges are to the left of the negative charge, and both forces will be attractive. We will adopt the usual convention that the positive horizontal direction is to the right and called $+x$, and the negative horizontal direction is to the left and called $-x$.

First, we will find the force on the negative charge due to the positive charge in the lower left, which we will call "1" to keep things straight. We will call the negative charge "2." This is easy, since the force is purely in the $-x$ direction:

$$\begin{aligned}
F_{x,1} &= k_e \frac{q_1 q_2}{r_{12}^2} \\
&= \left(9\times 10^9\,\mathrm{N\cdot m^2/C^2}\right)\frac{\left(10^{-6}\,\mathrm{C}\right)\cdot\left(-2\times 10^{-6}\,\mathrm{C}\right)}{\left(1\,\mathrm{m}\right)^2} \\
&= \left(9\times 10^9\,\mathrm{N}\cdot \cancel{\mathrm{m}^2}/\cancel{\mathrm{C}^2}\right)\left(-2\times 10^{-12}\,\cancel{\mathrm{C}^2}/\cancel{\mathrm{m}^2}\right) \\
&= -18\times 10^{-3}
\end{aligned}$$

So far so good, but now we have to include the force from the upper left-hand positive charge, which we'll call "3." We calculate the force in exactly the same way, with two little difference: the separation distance is slightly larger, and now the force has both a horizontal and vertical component. First, let's calculate the magnitude of the net force, we'll find the horizontal component after that.

Plane geometry tells us that the separation between charges 3 and 2 has to be $\sqrt{2}\cdot 1\,\mathrm{m}$, or $\sqrt{2}\,\mathrm{m}$ – connecting the charges with straight lines forms a 1-1-$\sqrt{2}$ right triangle, with 45° angles.

$$\begin{aligned}
F_{net,3} &= k_e \frac{q_2 q_3}{r_{23}^2} \\
&= \left(9\times 10^9\,\mathrm{N\cdot m^2/C^2}\right)\frac{\left(10^{-6}\,\mathrm{C}\right)\cdot\left(-2\times 10^{-6}\,\mathrm{C}\right)}{\left(\sqrt{2}\,\mathrm{m}\right)^2} \\
&= \left(9\times 10^9\,\mathrm{N}\cdot \cancel{\mathrm{m}^2}/\cancel{\mathrm{C}^2}\right)\frac{-2\times 10^{-12}\,\cancel{\mathrm{C}^2}}{2\,\cancel{\mathrm{m}^2}} \\
&= -9\times 10^{-3}\,\mathrm{N}
\end{aligned}$$

So the *net* force from the upper left charge is just half as much, since it is a factor $\sqrt{2}$ farther away. We only want the horizontal component though! Since we are dealing with a 45-45-90 triangle here, the horizontal component is just the net force times $\cos 45°$:

$$\begin{aligned}
F_{x,3} &= F_{net,3}\cos 45° \\
&= -9\times 10^{-3}\cdot\frac{\sqrt{2}}{2}\,\mathrm{N} = -9\times 10^{-3}\cdot 0.707\,\mathrm{N} \\
&\approx -6.4\times 10^{-3}\,\mathrm{N}
\end{aligned}$$

The total horizontal force is just the sum of the horizontal forces from the two positive charges:

$$\begin{aligned}
F_{x,\text{total}} &= F_{x,1} + F_{x,3} \\
&= \left(-18 \times 10^{-3}\right) + \left(-6.4 \times 10^{-3}\right)\,\text{N} \\
&= -24.4 \times 10^{-3}\,\text{N} = -0.0244\,\text{N}
\end{aligned}$$

2. $x = 0.77\,\mathbf{m}$**.** We have one negative charge (q_3) sitting between two positive charges (q_2 and q_1). The force from each positive charge will act in the opposite direction, and we want to find the position r_{23} such that both forces are equal in magnitude. All charges are on the x axis, so the problem is one-dimensional and does not require vectors.

Let F_{32} be the force on q_3 due to q_2, and F_{31} be the force on q_3 due to q_1, and we will take the positive x direction to be to the right. Since both forces are repulsive, F_{32} acts in the $-x$ direction *and must therefore be negative*, while F_{31} acts in the $+x$ direction and is positive. We are not told about any other forces acting, so our force balance is this:

$$-F_{32} + F_{31} = 0 \qquad \Longrightarrow \qquad F_{32} = F_{31}$$

It didn't really matter which one we called negative and which one we called positive, just that they have different signs. The separation between q_2 and q_3 is r_{23}, and the separation between q_1 and q_3 is then $2-r_{23}$. Now we just need to down the electric forces. We will keep everything perfectly general, and plug in actual numbers at the end ... this is always safer.

$$\begin{aligned}
F_{32} &= F_{31} \\
\frac{k_e q_3 q_2}{r_{23}^2} &= \frac{k_3 q_3 q_2}{\left(2 - r_{23}\right)^2} \\
\frac{\cancel{k_e}\cancel{q_3} q_2}{r_{23}^2} &= \frac{\cancel{k_3}\cancel{q_3} q_1}{\left(2 - r_{23}\right)^2} \\
\frac{q_2}{r_{23}^2} &= \frac{q_1}{\left(2 - r_{23}\right)^2}
\end{aligned}$$

Note how this doesn't depend at all on the actual magnitude *or sign* of the charge in the middle! From here, there are two ways to proceed. We could cross-multiply, use the quadratic formula, and that would be that. On the other hand, since we know that q_3 is supposed to be between the other two charges, then r_{23} must be positive, and less than 2. That means that we can just take the square root of both sides of the equation above without problem, since neither side would be negative afterward.[xvi] Using this approach first:

$$\begin{aligned}
\frac{q_2}{r_{23}^2} &= \frac{q_1}{\left(2 - r_{23}\right)^2} \\
\Longrightarrow \frac{\sqrt{q_2}}{r_{23}} &= \frac{\sqrt{q_1}}{2 - r_{23}}
\end{aligned}$$

Now we can cross-multiply, and solve the resulting linear equation:

[xvi] This would not work if we wanted the point to the left of q_2.

$$\begin{aligned}
\sqrt{q_2}\,(2-r_{23}) &= \sqrt{q_1}r_{23}\\
2\sqrt{q_2}-\sqrt{q_2}r_{23} &= \sqrt{q_1}r_{23}\\
2\sqrt{q_2} &= \left(\sqrt{q_2}+\sqrt{q_1}\right)r_{23}\\
r_{23} &= \frac{2\sqrt{q_2}}{\sqrt{q_2}+\sqrt{q_1}}
\end{aligned}$$

Plugging in the numbers we were given (and noting that all the units cancel):

$$r_{23}=\frac{2\sqrt{q_2}}{\sqrt{q_2}+\sqrt{q_1}}=\frac{2\sqrt{6\,\mu\text{C}}}{\sqrt{6\,\mu\text{C}}+\sqrt{15\,\mu\text{C}}}=\frac{2\sqrt{6}}{\sqrt{6}+\sqrt{15}}=\frac{2\sqrt{2}}{\sqrt{2}+\sqrt{5}}\approx 0.77\,\text{m}$$

For that very last step, we factored out $\sqrt{3}$ from the top and the bottom. An unnecessary step if you are using a calculator anyway, but we prefer to stay in practice.

The more general solution is to go back before we took the square root of both sides of the equation and solve it completely:

$$\begin{aligned}
\frac{q_2}{r_{23}^2} &= \frac{q_1}{(2-r_{23})^2}\\
q_2\,(2-r_{23})^2 &= q_1r_{23}^2\\
q_2\left(4-4r_{23}+r_{23}^2\right) &= q_1r_{23}^2\\
(q_2-q_1)\,r_{23}^2-4q_2r_{23}+4q_2 &= 0
\end{aligned}$$

Now we just have to solve the quadratic ...

$$\begin{aligned}
r_{23} &= \frac{4q_2\pm\sqrt{(-4q_2)^2-4\,(q_2-q_1)\cdot 4q_2}}{2\,(q_1-q_2)}\,\text{m}\\
&= \frac{4\cdot 6\,\mu\text{C}\pm\sqrt{(-4\cdot 6\,\mu\text{C})^2-4\,(6\,\mu\text{C}-15\,\mu\text{C})\cdot 4\cdot 6\,\mu\text{C}}}{2\,(6\,\mu\text{C}-15\,\mu\text{C})}\,\text{m}
\end{aligned}$$

We can cancel all of the μC ...

$$\begin{aligned}
r_{23} &= \frac{24\pm\sqrt{24^2-4(-9)(4)(6)}}{2(-9)}\,\text{m}\\
&= \frac{24\pm\sqrt{24^2+36(24)}}{-18}\,\text{m}\\
&= \frac{-24\mp\sqrt{1440}}{18}\,\text{m}\\
&= (0.775,-3.44)\ \text{m}
\end{aligned}$$

So there is one solution where q_3 is right between the two positive charges, at $r_{23}=0.77\,\text{m}$, and one solution where q_3 is to the left of q_2 by $3.44\,\text{m}$. We were asked to find the point between the two charges where the force is zero, so we discard the negative solution.

4 Electrical Energy and Capacitance

POTENTIAL energy and the principle of conservation of energy often let us solve difficult problems without dealing with the forces involved directly. More to the point, using an energy-based approach to problem solving let us work with *scalars* instead of *vectors*. This way we get to deal with just plain numbers, which is nice.

In this chapter, we will learn that, as with the gravitational field, the electric field has an associated **potential** and **potential energy**. The electric potential will, in many cases, let us solve problems more easily than with the electric field and, as it turns out, electric potential is what we normally identify with 'voltage' in everyday life.

Figure 4.1: *Michael Faraday (1791 - 1867), an English physicist and chemist who contributed significantly to the field of electromagnetism.*[16]

4.1 Electrical Potential Energy

The work done on an object by a conservative force, such as the electric force, depends only on the initial and final positions of the object, *not on the path taken between initial and final states.* For example, the work done by gravity depends only on the change in height. When a force is conservative, it means that there exists a **potential energy** function, PE, which gives the potential energy of an object subject to this conservative force which depends *only on the object's position. Potential energy* is sometimes called the "energy of configuration" since it only depends on the position of objects in a system. Thus, for the conservative electric force, we can find a change in electrical potential energy just by knowing the starting and final configurations of the system we are studying – nothing in between matters.

As you know, **potential energy is a scalar quantity**, and the *change* in potential energy is equal to the work done by a conservative force.

Potential energy difference, ΔPE

$$\Delta PE = PE_f - PE_i = -W_F \tag{4.1}$$

where the subscripts $f(i)$ refer to the final (initial position), and W_F is the work done by the conservative force $\vec{\mathbf{F}}$.

This is just how you dealt with gravity – moving an object of mass m through a vertical displacement h gives a changes in potential energy $\Delta PE = mgh$. Electrical forces and gravitational forces have a number of useful similarities, as you now know, and the same is true for their respective potential energies.

The Electric Force is Conservative:

1. We can define an *electrical potential*
2. There is potential energy associated with the presence of an electric field
3. Electric potential is *potential energy per unit charge*

Consider a small positive test charge q in a uniform electric field $\vec{\mathbf{E}}$, as shown in Figure 4.2. As the charge moves from point A to point B, covering a displacement $\Delta x = x_f - x_i$, the work done on the charge by the electric field is the component of the force $\vec{\mathbf{F}}_e = q\vec{\mathbf{E}}$ parallel to the displacement Δx:

[i]**Work done moving a charge q in a constant electric field $\vec{\mathbf{E}}$:**

$$\Delta W_{AB} = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = |\vec{\mathbf{F}}|\,|\Delta x| \cos\theta = qE_x\,(x_f - x_i) = qE_x\Delta x \tag{4.2}$$

where q is the charge, E_x is the component of the electric field $\vec{\mathbf{E}}$ along the direction of displacement, and θ is the angle between the force $\vec{\mathbf{F}}$ and the displacement $\Delta\vec{\mathbf{x}}$ (of length Δx).

Note that q, E_x, and Δx can all be either positive or negative. Also recall that E_x is the x-*component* of the electric field $\vec{\mathbf{E}}$, *not* the magnitude! Equation 4.2 is valid for the work done on a charge by *any constant electric field*, no matter what the direction of the field, or sign of the charge. Just remember that the angle between the field and displacement does matter!

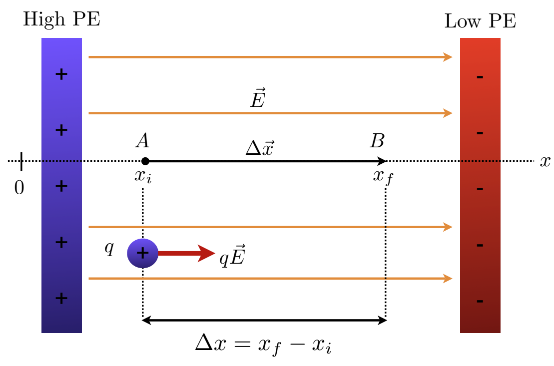


Figure 4.2: *When a charge q moves in a uniform electric field $\vec{\mathbf{E}}$ from point A to point B, covering a distance Δx, the work done on the charge by the electric force is $qE_x\Delta x$.*

Now that we have found the work done by the electric field, the work-energy theorem gives us the potential energy change:

[i]At this point you may want to remind yourself about the scalar or "dot" product, $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |A||B| \cos\theta_{AB}$, where θ_{AB} is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.