Part I

Relativity

2 Relativity

NEARLY all of the mechanical phenomena we observe around us every day have to do with objects moving at speeds rather small compared to the speed of light. The Newtonian mechanics you learned in previous courses handled these cases extraordinarily well. As it turns out, however, Newtonian mechanics breaks down completely when an object's speed is no longer negligible compared to the speed of light. Not only does Newtonian mechanics fail in this situation, it fails *spectacularly*, leading to a variety of paradoxical situations.

The resolution to these paradoxes is given by the theory of relativity, one of the most successful and accurate theories in all of physics, which we will introduce in this chapter. Nature is not always kind, however, and the consequences of relativity seem on their face to flout common sense and our view of the world around us. We are used to the notion that our position changes with time when we are in motion, but relativity implies that *passage of time itself* changes when we are in motion. Nevertheless, we shall see that relativity is an *inescapable* consequence of a few simple principles and experimental facts. Moreover, as it turns out, this new description of nature is critical for properly understanding electricity and magnetism, optics, and nuclear physics ... most of the rest of this course!

2.1 Frames of Reference

Describing motion properly usually requires us to choose a coordinate system, and an origin from which to measure position. Why this is so is more clear when we consider the difference between distance and displacement. For example, we can say that a person moves through a *displacement* of 10 meters, $\Delta x = 10\,\mathrm{m}$, in a particular direction, *e.g.,* to the right. This does not describe the *position* of the person at all, only the *change* in that person's position over some time interval.

Describing position itself requires us to choose first a coordinate system (such as cartesian, spherical, *etc.*), and also an origin for this coordinate system to define our "zero" position. The essential difference is that displacement is independent of the cooordinate system we choose, but position is not. Without choosing a coordinate system, we can only say that the person has run 10 m in a certain time interval, moving from x_i to x_f.

As a concrete example, consider Fig. 2.2a. This illustrates

Figure 2.1: *Albert EInstein (1879 – 1955) in 1921 at age 42.*[3] *Einstein is best known for his theory of relativity and its mass-energy equivalence,* $E = mc^2$.

a person moving 10 m to the right, which perfectly describes a *displacement* Δx. We will choose an $x-y$ cartesian coordinate system, which we will call O, with its origin at the person's starting point. In this system, we can describe the initial and final positions P_i^O and P_f^O in this coordinate system as $P_i^O=(0,0)$ and $P_f^O=(x_f,0)=(\Delta x,0)$. This is shown in Fig. 2.2b. The displacement is the same as it was without a coordinate system. In this chapter we will use the convention that superscripts refer to the coordinate system in which the quantity in question was measured.

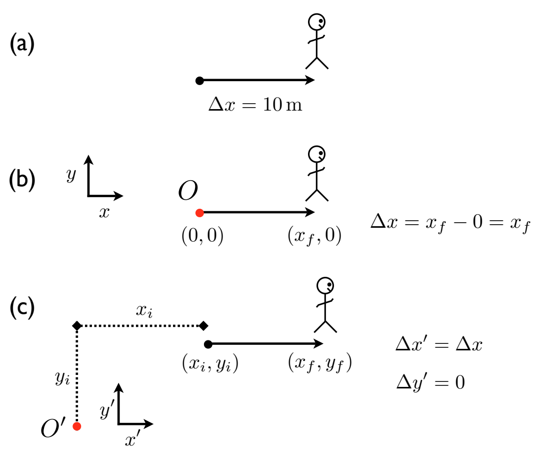


Figure 2.2: *Displacement is independent of the cooordinate system we choose, but position is not.* **(a)** *Without choosing a coordinate system, we can only say that the person has run 10 m in a certain time interval, moving from x_i to x_f.* **(b)** *If we choose an $x-y$ coordinate system O centered with its origin on the person's starting point x_i, we can describe the initial and final positions as $P_i^O=(x_i,0)$ and $P_f^O=(x_f,0)$. The displacement is the same.* **(c)** *If we choose a new coordinate system O', identical to O except shifted downward by y_i and to the left by x_i, now the initial and final positions are $P_i^{O'}=(x_i,y_i)$ and $P_f^{O'}=(x_f,y_i)$. Still, the displacement is the same.*

What happens if we if we instead choose a different coordinate system O', Fig. 2.2c, identical to O except that its origin is shifted downward by y_i and to the left by x_i? Now the initial and final positions of the person are $P_i^{O'}=(x_i,y_i)$ and $P_f^{O'}=(x_f,y_f)$. Still, the displacement Δx is the same, as you can easily verify. No matter whether we observe the person from the O or O' system, we would describe the same *displacement*, even though the actual positions are completely different.

In special relativity, this simple situation no longer holds - observers in different coordinate systems do *not* necessarily describe even the same displacement, much less the same position. Fortunately, the corrections of special relativity to the Newtonian mechanics you have already learned are only appreciable at very high velocities (non-negligible compared to the speed of light), and for most every day situations our usual intuition is still valid.

In any case, particularly those cases where relativistic effects are important, it is crucially important that we specify in which coordinate system quantities have been measured. We will continue to do this with a superscript of some sort to specify the coordinate system, and a subscript of some sort to further describe what is being measured *within* that system. When we only have two frames, like the example above, we will often just use a prime ($'$) to tell them apart. In the previous example, this means we would use P_f' instead of $P_f^{O'}$, and just P_i instead of P_i^O. It seems pedantic now, but careful bookkeeping is the only thing saving us from terrible confusion later!

Coordinate system notation examples:

$x^O_{\text{final}} = x^O_f$ final x position of an object measured in the O coordinate system

$v^{O'}_{\text{car}} \equiv v'_{\text{car}}$ velocity of a car measured in the O' coordinate system

$P^{O'}_f \equiv P'_f = (x'_f, y'_i)$ final position of an object measured in the O' coordinate system

Finally, a word on terminology. In relativity, it is common to use "reference frame" in place of "coordinate system," to make explicit the fact that our coordinate system and origin are the point of reference from which we measure physical quantities. We will use both phrasings interchangeably from here on out.

2.2 Moving Frames of Reference

What about one observer measuring in a coordinate system *moving* at constant velocity relative to another? For example, take Fig. 2.3. A girl holding balloons is standing on the ground, and a bully on a skateboard throws a dart at her balloons. The bully is moving at a velocity v_{bully} relative to the girl, and he throws the dart at a velocity v_{dart} relative to himself. What is the dart's speed relative to the girl?

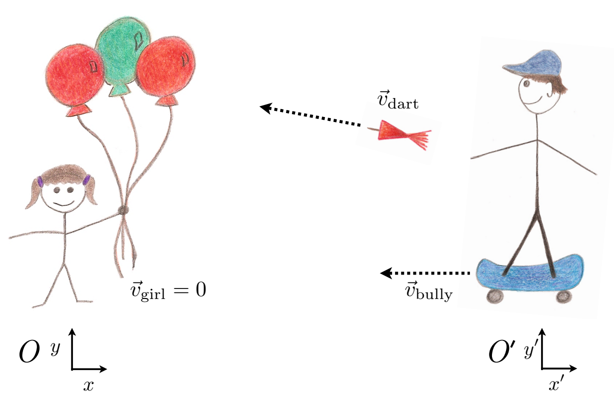


Figure 2.3: *A girl holding balloons is standing on the ground, at rest in reference frame O ($v^O_{girl} = 0$). Meanwhile a bully on a skateboard throws a dart at her balloons. The bully is moving at a velocity v^O_{bully} relative to the girl's reference frame, and he throws the dart at a velocity $v^{O'}_{dart}$ relative to himself (the O' frame). What is the dart's speed as measured by the girl? Drawings by C. LeClair*

First of all, we have to be more explicit about specifying which quantity is measured in which frame. The velocity of the bully on the skateboard is measured relative to the girl standing on the ground, in the O system, so we write v^O_{bully}. When we talk about the dart, however, things are a bit less clear. The bully on the skateboard would say that the velocity of the dart is $v^{O'}_{\text{dart}}$, since he would measure its velocity relative to *himself* in the O' frame. The girl would measure the velocity of the dart relative to *herself* in the O frame, v^O_{dart}. Clearly, $v^{O'}_{\text{dart}} \neq v^O_{\text{dart}}$ – in principle, the two cannot agree on what the velocity of the dart is! Of course, that is a bit of an exaggeration. In this simple everyday case, relative motion is fairly easy to understand, and we can intuitively see exactly what is happening. Our intuition will start to fail us shortly, however, so it is best we proceed carefully.

Explicitly labeling the velocity with the reference frame in which it is measured helps keep everything precise, and helps us find a way out of this conundrum. It may seem like baggage now, but ambiguity would cost us dearly later. Just to summarize, here is how we will keep the velocities straight:

$$\begin{aligned} v^{O}_{\text{bully}} &= \text{velocity of bully measured from the ground} \equiv v_{\text{bully}} \\ v^{O'}_{\text{dart}} &= \text{velocity of dart measured from the skateboard} \equiv v'_{\text{dart}} \\ v^{O}_{\text{dart}} &= \text{velocity of the dart measured by the girl} \equiv v_{\text{dart}} \end{aligned}$$

Whenever we are only dealing with two different coordinate systems, we will trim down the notation a bit. We will just call one system the "primed" system, and add a $'$ superscript to all quantities, and leave the other one as the "unprimed" system, and drop the 'O'. Which one we call "primed" and which one is "unprimed" makes no difference, it is after all just notation and bookkeeping.

What does the girl on the ground, in the O system really observe? Intuitively, we expect this her to see the dart moving at a velocity v_{dart} which is that of the dart relative to the skateboard *plus* that of skateboard relative to the ground:

$$\begin{aligned} v_{\text{dart}} &= v'_{\text{dart}} + v_{\text{bully}} \qquad (2.1) \\ \text{velocity of the dart seen by the girl} &= \text{velocity of dart relative to skateboard} + \text{velocity of skateboard relative to girl} \end{aligned}$$

The bully, in the O' system (who threw the dart in the first place), just sees v'_{dart}. Just to be concrete, let's say that the bully on the skateboard moves with $v_{\text{bully}} = 3\,\text{m/s}$, and he throws the dart with $v'_{\text{dart}} = 2\,\text{m/s}$. Then the girl sees the dart coming at her balloons at $5\,\text{m/s}$.

2.2.1 Lack of a Preferred Reference Frame

Even in the simple example above, velocity depends on your frame of reference. This simple example is completely arbitrary in a sense, though, and implies much more about relative motion. If these two observers can't agree on the velocity of the dart, as measured in their own reference frames, who is to say what the absolute reference frame should be? After all, isn't the ground itself moving due to the rotation of the earth about the sun? And isn't the sun moving relative to the center of the galaxy? Nothing is absolutely at rest, we cannot pick any special frame of reference to define absolute unique velocities.

Still, we might think be tempted to think that there is some sort of reference frame we are forgetting, one that is truly at rest. For instance, what about empty space itself? Can we define absolute coordinates and absolute motion relative to specific points in space? This is a tempting thought, particularly if we make an analogy with sound waves.

As you know from Mechanics, sound is really nothing more than (longitudinal) oscillations of matter, a sort of density wave in a material. We will find out in later Chapters that light is also a wave. If they are both waves, perhaps the nature of sound can help explain the nature of light? Sound can be propagated through matter, or even through air, but it requires a medium to be transmitted – no sound is transmitted in a vacuum. Could we view light as the vibrations of space itself, or of some all-pervasive "fluid" filling all of space? Certainly light waves also need a medium in which to propagate, so the reasoning goes. This all-pervasive fluid would provide a "background" frame of reference, allowing us to measure absolute velocity, somewhat like measuring the velocity of a boat by how fast water moves past its side.

Indeed, this was a very attractive viewpoint through the early 20$^{\text{th}}$ century, and the so-called "luminiferous æther" was the term used to describe the all-pervasive medium for the propagation of light. It fact, is a *testable* idea – this is a crucial point which makes the idea a true scientific theory. How do we test it? If space itself has a background medium within which light propagates, then we should be able to measure the velocity of the earth through this medium as it revolves around the sun. The earth moving through the æther fluid would experience some "drag," again just like a boat moving through water.

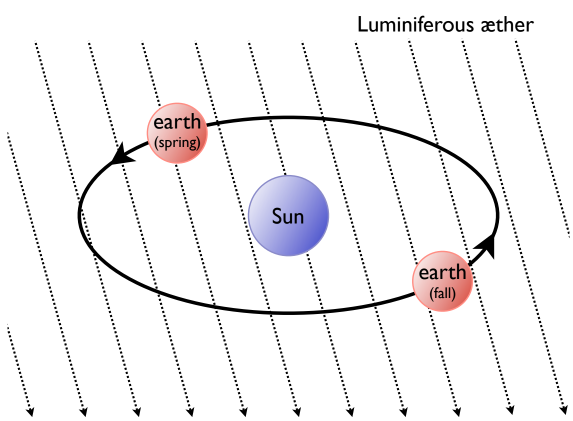


Figure 2.4: *If there were a "luminiferous æther" which light propagates on pervading all space, the earth's revolution around the sun would experience a drag force depending on the season – the "æther wind." At some times, this æther wind would augment the speed of light, and at others it would diminish it. Analyzing the speed of light in different directions at different times of year should allow one to extract the æther velocity. Instead, experiments proved that there is no æther.*

Unfortunately, this idea just isn't right. It has been disproven countless times by experiments, and replaced by the far more successful theory of relativity. Light waves are not like sound waves. Light is in one sense a wave, but a more modern viewpoint treats light as a stream of particles that have a "wave-like nature."[i] Particles do not need a medium to travel, and therefore neither does light. *There is no æther, and there is no preferred frame of reference. All motion is relative.*

[i]We will explore this dual nature of light in Chapter 10.

2.2.2 Relative Motion

Fine. There is no preferred reference frame or coordinate system, and all motion is relative. So what? The example of Fig. 2.3 was plainly understandable. It is disturbingly easy to come up with examples which are *not* so plainly understandable, however, which is one motivation for the theory of relativity in the first place. Consider the two rockets in empty space traveling toward each other in Fig. 2.5, separated by a distance Δx. The pilot of rocket 1 might say he or she is traveling at a speed v_1 in his or her own reference frame (O), and the pilot of rocket 2 may claim he or she is traveling at a speed v_2' in their own O'. Without specifying what point they are measuring their velocity relative to, can we say who is moving at what speed?

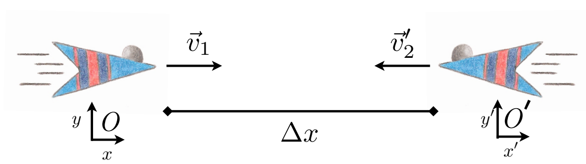


Figure 2.5: *Two identical rockets, separated by a distance Δx, are moving toward each other in empty space. Rocket 1 sees rocket 2 cover a distance $(v_1{+}v_2)t$ in a time t, as if rocket 2 is heading toward it. On the other hand, rocket 2 sees rocket 1 cover the same distance $(v_1{+}v_2)t$ in a time t, as if rocket 1 is heading toward it! Both of them cannot be correct. Without an external reference frame, it is impossible to say who is moving, and at what speed. Drawings by C. LeClair*

We have to imagine that we are deep in empty space, with nothing around either rocket to provide a landmark or point of reference. The occupants of rocket 1 would feel as though they are sitting still, and observe rocket 2 coming toward them, covering a distance $(v_1 + v_2)\Delta t$ in a time interval Δt. The occupants of rocket 2, on the other hand, would think *they* are sitting still, and would observe rocket 1 coming toward *them*, also covering a distance $(v_1 + v_2)\Delta t$ in a time interval Δt.

Without any external reference point, or an absolute frame of reference, *not only can we not say with what speed each rocket is moving, we can't even say* **who** *is moving!* If we decide that rocket 1 is our reference frame, then it is sitting still, and rocket 2 is moving toward it. But we could just as easily pick rocket 2 as our reference frame. Specifying who is moving, and with what speed, is meaningless without a proper origin or frame of reference.

Has anything really changed physically? No. An analogy of sorts is to think about driving along side other cars on the highway, keeping pace with them. You might report your speed as 60 mi/hr. Relative to what? Clearly, in this case it is implied that the ground beneath you provides a reference frame, and you are talking about your velocity relative to the earth. You wouldn't say you are traveling at 60 mi/hr relative to the other cars (we hope) – your speed relative to the other cars is zero if you are staying along side them. Indeed, if you look out your window, the cars next to you appear to be sitting still. This is only true at constant velocity – we can easily detect accelerated motion, or an accelerated frame of reference due to the force experienced. This is the realm of *general* relativity, Sect. 2.5.

In the end, one of the fundamental principles of special relativity is that *the description of relative constant velocity does not matter, so far as the laws of physics are concerned.* The laws of physics apply the same way to all objects in uniform (non-accelerated) motion, no matter how we measure the velocity. We *cannot* devise an experiment to measure uniform motion absolutely, only relative to a specific chosen frame of reference. More succinctly:

Principle of relativity:
All laws of nature are the same in all uniformly moving (non-accelerating) frames of reference. No frame is preferred or special.

As another simple example, Fig. 2.6, consider Joe and Moe running at different (constant) speeds in the same direction, initially separated by a distance d_o. Without specifying any particular common frame of reference, we must be able to describe their relative motion, or how the separation between Joe and Moe changes with time, even though we can't speak of their absolute velocities in any sense.

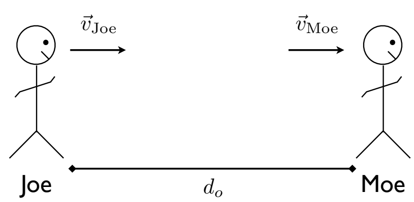


Figure 2.6: *Joe and Moe running at different speeds in the same direction. Both Joe and Moe measure the same relative velocity with respect to each other.*

Let's say we arbitrarily choose Joe's position at $t=0$ as our reference point. It is easy then to write down what Joe and Moe's positions are at any later time interval Δt:

$$x_{\text{Joe}} = v_{\text{Joe}}\Delta t \qquad\qquad x_{\text{Moe}} = d_0 + v_{\text{Moe}}\Delta t \tag{2.2}$$

We can straightforwardly write down the separation between them (their relative displacement) as well:

$$\Delta x_{\text{Moe-Joe}} = x_{\text{Moe}} - x_{\text{Joe}} = d_0 + v_{\text{Moe}}\Delta t - v_{\text{Joe}}\Delta t = d_0 + \left(v_{\text{Moe}} - v_{\text{Joe}}\right)\Delta t \tag{2.3}$$

Sure enough, their relative displacement only depends on their *relative* velocity, $v_{\text{Moe}}-v_{\text{Joe}}$. Further, both Joe and Moe would agree with this, since we could arbitrarily choose *Moe's* position at $t=0$ as our reference point, and *we would end up with the same answer.* Since there is nothing special about either position, we can choose *any* point whatsoever as a reference, and wind up with the same result. We end up with the same physics no matter what reference point we choose, which one we choose is all a matter of convenience in the end.

Choosing a coordinate system:

1. Choose an origin. This may coincide with a special point or object given in the problem - for instance, right at an observer's position, or halfway between two observers. Make it convenient!
2. Choose a set of axes, such as rectangular or polar. The simplest are usually rectangular or *Cartesian* x-y-z, though your choice should fit the symmetry of the problem given - if your problem has circular symmetry, rectangular coordinates may make life difficult.
3. Align the axes. Again, make it convenient - for instance, align your x axis along a line connecting two special points in the problem. Sometimes a thoughtful but less obvious choice may save you a lot of math!
4. Choose which directions are positive and negative. This choice is arbitrary, in the end, so choose the least confusing convention.

This seems simple enough, but if we think about this a bit longer, more problems arise. Who measures the initial separation d_0, Joe or Moe? Who keeps track of the elapsed time Δt? Does it matter at all, can the measurement of distance or time be affected by relative motion? Of course, the answer is an awkward 'yes' or we would not dwell on this point. If we delve deeper on the problem of relative motion, we come to the inescapable conclusion that not only is velocity a relative concept, our notions of distance and time are relative as well, and depend on the relative motion of the observer. In order to properly understand these deeper ramifications, however, we need to perform a few more thought experiments.

2.2.3 Invariance of the Speed of Light

Already, relativity has forced us to accept some rather non-intuitive facts. This is only the beginning! A more fundamental and far-reaching principle of relativity is that *the speed of light is a constant, independent of the observer.* No matter how we measure it, no matter what our motion is relative to the source of the light, we will always measure its velocity to be the same value, c. Light does not obey the principle of relative motion!

Speed of Light in a Vacuum:

$$c = 3 \times 10^8 \text{ m/s}$$

There is a relatively simple way to experimentally demonstrate that this seems to be true, depicted in Fig. 2.7. The earth itself is in constant motion in its orbit around the sun, moving at $\sim 3 \times 10^4$ m/s measured relative to distant stars (this in itself is a measurable quantity). Imagine now that we carefully set up three lasers, each oriented in a different direction relative to earth's orbital velocity – one parallel (A), one antiparallel (B), and one at a right angle (C). We will further set up each laser to emit short pulses of light, and carefully measure the time between pulses. In this way, we can determine the speed of the light coming out of each laser.

Based on simple Newtonian mechanics and velocity addition, we would expect to measure a slightly different velocity for each laser. In case A, we would expect the Earth's velocity to *add* to that of light, $v_{\text{A}}=v_{\text{light}}+v_{\text{orbit}}$, while in case B, it should *subtract*, $v_{\text{B}}=v_{\text{light}}-v_{\text{orbit}}$. In case C, we have to add vectors, $\vec{\mathbf{v}}_{\text{C}}=\vec{\mathbf{v}}_{\text{light}}+\vec{\mathbf{v}}_{\text{orbit}}$, but the idea is the same.

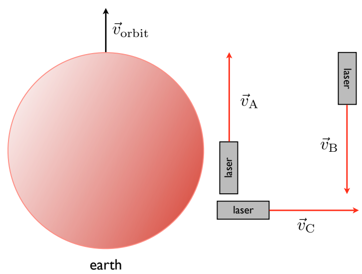


Figure 2.7: *If the velocity of light obeys Newtonian mechanics, then measuring the speed of light from a laser pointed in different directions compared to Earth's orbital velocity should yield different results. In case A, we would expect the Earth's velocity to add to that of light, $v_A = v_{light}+v_{orbit}$, while in case B, it should subtract, $v_B = v_{light}-v_{orbit}$. In case C, we have to add vectors, but the idea is the same. The effect should be small ($\sim 0.01\,\%$), but easily measurable. No effect is observed, the speed of light is always the same value c.*

The effect should be small ($\sim 0.01\,\%$), but easily measurable. No effect is observed, the speed of light is *always* the same value c. This experiment has been performed with increasingly fantastic precision over the last 100 years[4], and no matter what direction we shine the light, we always measure the same speed! (The current best limit[4] on the constancy of the speed of light is about 1 part in 10^{16}.) One straightforward result of this experiment is that the idea of an Æther is clearly not right, as we discussed above. There are much more far-reaching consequences, which we must consider carefully. First, let us re-iterate this idea more formally:

The speed of light is invariant

The speed of light in free space is *independent* of the motion of the source or observer. It is an invariant constant.

This is not just idle speculation or theory, it has been confirmed again and again by careful experiments. These experiments have established, for instance, that the speed of light[ii] does not depend on the wavelength of light, on the motion of the light source, or the motion of the observer. As examples, lack of a wavelength dependence can be strongly ruled out by astronomical observations of gamma ray bursts (to better than 1 part in 10^{15}), while binary pulsars can rule out any dependence on source motion. The lack of a dependence on observer motion was disproved along with the æther (Sec. 2.2.1), which also proved that light requires no medium for propagation.

As an example of this, we turn again to Joe and Moe (Fig. 2.8). Joe is in a rocket (O'), traveling at 90% of the speed of light ($v = 0.9c$), while Moe is on the ground (O) with a flashlight. Moe shines the flashlight parallel to Joe's trajectory in the rocket. On first sight, we would think that Moe would measure the speed of the light leaving the flashlight as c, while Joe would measure $v=c-0.9c=0.1c$.

[ii] Throughout this chapter, we refer to the speed of light in a *vacuum*.

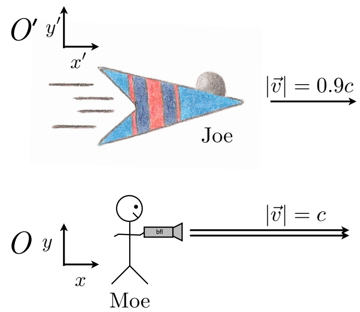


Figure 2.8: *Joe is traveling on a rocket at $|\vec{\mathbf{v}}| = 0.9c$, while Moe on the ground shines a flashlight parallel to Joe's path. Both Joe and Moe observe the light from the flashlight to travel at $|\vec{\mathbf{v}}| = c$ – contrary to our intuition from Newtonian Mechanics.*

Both Joe and Moe measure *the same speed of light c*, despite their relative motion! What if we gave Joe the flashlight inside the rocket? No difference, both Joe and Moe measure the speed of the light to be c. Think back to our example of relative motion in Fig. 2.3. It doesn't seem to make sense that light behaves differently, but that is how it is. As we shall see shortly, our normal intuitions about everyday phenomena at relatively low velocities is no longer valid when velocities approach that of light. The physics is fundamentally different, and our Newtonian instincts are in the end only a low-speed approximation to reality. By the end of this chapter, though, we will be armed with the proper tools to analyze this situation correctly from both viewpoints.

2.2.4 Principles of special relativity

From our discussions so far, relativity when non-accelerating (inertial) reference frames are considered has two basic principles which underpin the entire theory:

Principles of special relativity

1. **Special principle of relativity:** Laws of physics look the same in all inertial (non-accelerating) reference frames. There are no preferred inertial frames of reference.
2. **Invariance of c:** The speed of light in a vacuum is a universal constant, c, independent of the motion of the source or observer.

This theory of relativity restricted to inertial reference frames is known as the *special theory of relativity*, while the more general theory of relativity which also handles accelerated reference frames is simply known as the *general theory of relativity* (which we will touch on in Sect. 2.5).

The second postulate of special relativity - the invariance of the speed of light - can actually be considered as a consequence of the first according to some mathematical formulations of special relativity. That is, the constancy of the speed of light is *required* in order to make the first postulate true. We will continue to hold it up as a second primary postulate of special relativity, however, as some of the more non-intuitive consequences of special relativity are (in our view) more readily apparent when one keeps this fact in mind.

The first principle of relativity essentially states that all physical laws should be exactly the same in any vehicle moving at constant velocity as they are in a vehicle at rest. As a consequence, at constant velocity we are incapable of determining absolute speed or direction of travel, we are only able to describe motion relative to some other object. This idea does not extend to accelerated reference frames, however. When acceleration is present, we feel fictitious forces that betray changes in velocity that would not be present if we were at rest. All experiments to date agree with this first principle: physics is the same in all inertial frames, and no particular inertial frame is special.

The principle of relativity is by itself more general than it appears. The principle of relativity describes a symmetry in the laws of nature, that the laws must look the same to one observer as they do to another. In physics, any symmetry in nature also implies a *conservation law*, such as conservation of energy or conservation of momentum. If the symmetry is in time, such that two observers at different times must observe the same laws of nature, then it is energy that must be conserved. If two observers at different physical locations must observe the same laws of physics (*i.e.,* the laws of physics are independent of spatial translation), it is linear momentum that must be conserved. The relativity principles imply deep conservation laws about space and time that make testable predictions – predictions which must be in accordance with experimental observations in order to be taken seriously. Relativity is not just a principle physicists have proposed, it is a postulate that was in the *required* in order to describe nature as we see it. The consequences of these postulates will be examined presently.

2.3 Consequences of Relativity

We have our principles laid forth, and their rationale clearly provided by our series of thought experiments. All experimental results to date are on the side of these two principles. So, enough already, what are the consequences of these two innocent-looking principles? In some sense you may ask what the big deal is. How often do we deal with objects traveling close to the speed of light? The glib answer is "plenty" – we use light itself pretty much constantly! The more formal answer is that the invariance of the speed of light and the principles of relativity force us to modify our very notions of perception and reality. It is not just fiddling with a few equations to handle special high-velocity cases, we must reevaluate some of our deepest intuitions and physical models. Many books have been written about the implications that relativity has had on philosophy in fact ... however, we will stick to physics.

2.3.1 Lack of Simultaneity

The speed of light is more than just a constant, it is a sort of 'cosmic speed limit' – no object can travel faster than the speed of light, and no information can be transmitted faster than the speed of light. If either were possible, causality would be violated: in some reference frame, information could be received before it had been sent, so the ordering of cause-effect relationships would be reversed. It is a bit much to go into, but the point is this: the speed of light is really a *speed limit*, because if it were not, either cause and effect would not have their usual meaning, or sending information backward in time would be possible. Neither is an easily-stomached possibility. A more readily grasped consequence of all of this is that we must give up on the notion of two events being simultaneous in any absolute sense – whether events are viewed as simultaneous depends on ones reference frame! It should seem odd that a seemingly simple principle like the speed of light being constant would muck things up so much, but in fact we can demonstrate that this *must* be true with a simple thought experiment.

Imagine that Joe is flying in a spaceship at $v = 0.9c$ (we will call his reference frame O'), and Moe is observing him on the ground (in frame O), as shown in Fig. 2.9. Joe, sitting precisely in the middle of the ship, turns on a light at time $t = 0$ also in the middle of the spaceship. A small amount of time Δt later, Joe's superhuman eyes observe the rays of light reach the front and the back of the spaceship simultaneously. So far this makes sense – if the light is exactly in the middle of the ship, light rays from the bulb should reach the front and back at the same time.

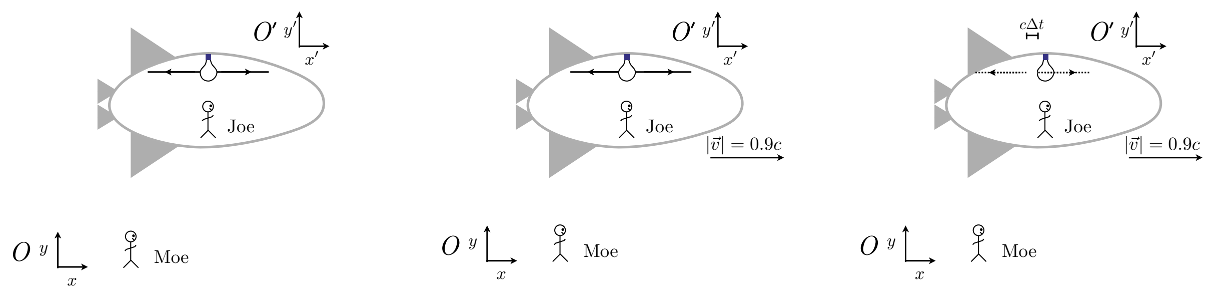


Figure 2.9: left: *Joe is traveling in a (transparent) rocket ship, and turns on a light bulb in the exact center of the rocket.* **middle** *A short time Δt later, in his frame O' Joe sees the light rays hit both sides of the ship at the same time.* **right:** *Moe on the ground observes Joe in his rocket moving at $v = 0.9c$. From his frame O, a time Δt after the light leaves the bulb, the ship moves forward by an amount $c\Delta t$ but the light rays do not. Moe sees the light hit the back of the ship first – Moe and Joe cannot agree on the simultaneity of events.*

Now, what will Moe on the ground see?[iii] From his frame O, Moe sees the light emitted from the bulb at $t = 0$. The ship and the light bulb are both moving relative to Moe at $v = 0.9c$, but we have to be careful. First, Moe observes the same speed of light as Joe, even though the bulb is moving. Once the light bulb is turned on, the first light leaves the bulb at $v = c$ and diverges radially outward from its point of creation. As this first light leaves the bulb, however, *the ship is still moving forward.* The front of the ship moves away from the point of the light's creation, while the back moves *toward* it.

> **Consequence of an invariant speed of light:**
> Events that are simultaneous in one reference frame are **not** simultaneous in another reference frame moving relative to it – and no particular frame is preferred. Simultaneity is not an absolute concept.

In some sense, once the light is created, it isn't really in either reference frame – it is traveling at $v = c$ no matter who observes it. The ship moved forward, but the point at which the light was

[iii]You might think nothing, as we neglected to mention that Joe's ship is transparent.

created did not. We attempt to depict this in Fig. 2.9, where from Moe's point of view, after a time Δt the light rays emitted from the bulb seem to have emanated from a point somewhat behind the rocket – a distance $c\Delta t$ behind it. Thus, after some time, Moe sees the light hit *the back* of the ship first! Joe and Moe seem to observe different events, and they can not agree on whether the light hits the front and the back of the ship simultaneously. Events which are simultaneous in Joe's reference frame are **not** in Moe's reference frame, moving relative to him. Think about how this plays out from Joe and Moe's reference frames carefully. It is strange and non-intuitive, but if we accept the speed of light as invariant, the conclusion is inevitable.

2.3.2 Time Dilation

Now we have already seen that the constancy of the speed of light has some rather unintuitive and bizarre consequences. For better or worse, it gets stranger! Not only is our comfortable notion of simultaneity sacrificed, our concept of the passage of time itself must be "corrected." Just as the notion of two events being simultaneous or not depend on one's frame of reference, the relative passage of time also depends on the frame of reference in which the measurement of time is made. Again, to illustrate this, we will perform a thought experiment.

First, we need a way to measure the passage of time. The constancy of the speed of light fortunately provides us with a straightforward – if not necessarily experimentally simple – manner in which to do this. We will measure the passage of time by bouncing light pulses between two parallel mirrors, carefully placed a distance d apart. Since we know the speed of light is an immutable constant, so long as the space between the mirrors remains fixed at d, the round-trip time Δt for a pulse of light to start at one mirror, bounce off the second, and return to the first will be a constant. The light pulse travels the distance d between the mirrors, and back again, at velocity c, so the time interval for a round trip is just $\Delta t = 2d/c$.

Now, let's imagine Joe is performing this experiment in a boxcar moving at velocity v relative to the ground, as shown in Fig. 2.10a. We will label Joe's own reference frame inside the boxcar as O', such that the boxcar moves in the x' direction. Both Joe, the mirrors, and the light source are stationary relative to one another, and the mirrors and light source have been carefully positioned a distance d apart such that the light pulses propagate vertically in the y' direction. In Joe's reference frame, he can measure the passage of time by measuring the number of round trips that an individual light pulse makes between the two mirrors. For one round trip, Joe would measure a time interval

$$\Delta t' = \frac{2d}{c} \tag{2.4}$$

So far so good. Since Joe is not moving relative to the mirrors, nothing unusual happens – assuming he has superhuman vision, he just sees the light pulses bouncing back and forth between the mirrors, Fig. 2.10a, straight up and down, and counts the number of round trips. Moe monitors this situation from the ground, in his own reference frame O. Thankfully, the boxcar is transparent, and Moe is able to see the light pulses and mirrors as well as the boxcar, moving at a velocity v from his point of view.

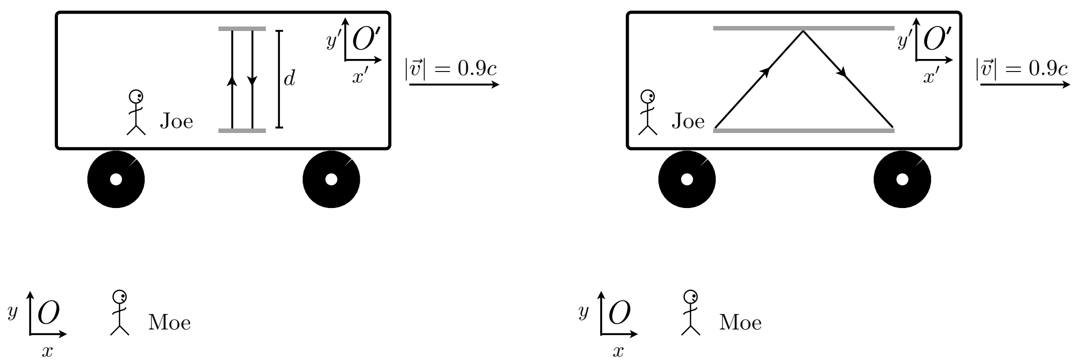


Figure 2.10: left: *Joe is traveling in a (transparent) boxcar, and he bounces laser beams between two mirrors inside the boxcar. Since the distance between the mirrors is known, and the speed of light is constant, he can measure time in this way. Joe measures the round trip time it takes the light to bounce from the bottom mirror, to the top, and back again.* **right:** *Moe observes the mirrors from the ground. From his frame O, the boxcar and mirrors are moving but the light is not. He therefore sees the light bouncing off of the mirrors at an angle. Using geometry and the constant speed of light, Moe also measures a round trip time interval, but since the path he observes for the light is different, he measures a different time interval than Joe.*

What does Moe see inside the boxcar? From his point of view, a light pulse is created at the bottom mirror while the whole assembly moves in the x direction – mirrors, light pulse, and all! Just like in the example of the light being flicked on in a space ship, the boxcar and mirrors have moved, but the point at which the light was created has not – Moe appears to see the light traveling at an angle. A light pulse is created at the bottom mirror, and it travels upward horizontally to reach the top mirror some time later, a bit further along the x axis. Rather than seeing the pulses going straight up and down, from Moe's point of view, they zig-zag sideways along the x axis, as shown in Fig. 2.10b.

So what? We know the speed of light is a constant, so both Joe and Moe must see the light pulses moving at a velocity c, even though they appear to be moving in along a different trajectory. If Moe also uses the light pulses' round trips to measure the passage of time, what time interval does he measure? The speed of light is constant, but the apparent distance covered by the light pulses is larger in Moe's case. Not only has the light traveled in the y direction a distance $2d$, over the course of one round trip it has also moved horizontally due to the motion of the boxcar. If the light has apparently traveled farther from Moe's point of view, and the speed of light is constant, then the apparent passage of time from Moe's point of view must also be greater!

Just how long does Moe observe the pulse round trip to be? Let us examine one half of a round trip, the passage of the light from the bottom mirror to the top. In that interval, from either reference frame, the light travels a vertical distance of d. From Joe's reference frame O', the light does not travel horizontally, so the entire distance covered is just d, and he measures the time interval $\frac{1}{2}\Delta t' = d/c$. From Moe's reference frame, the car has also travelled horizontally. Since he sees the car moving at a velocity v, he would say that in his time interval $\frac{1}{2}\Delta t$ for one half round trip, the car has moved forward by $\frac{1}{2}v\Delta t$. Thus, Moe would see the light cover a horizontal distance of $\frac{1}{2}v\Delta t$ and a vertical distance $\frac{1}{2}c\Delta t$, as shown in Fig. 2.11.

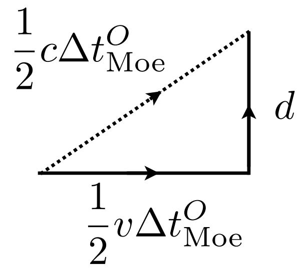


Figure 2.11: *Velocity addition for light pulses leading to time dilation. Within the boxcar (frame O', Joe observes the light pulses traveling purely vertically, covering a distance d. On the ground (frame O), Moe sees the light cover the same vertical distance, but also sees them move horizontally due the motion of the boxcar at velocity v in his reference frame. The total distance the light pulse travels, according to Moe, is then the Pythagorean sum of the horizontal and vertical distances.*

According to Moe, total distance that the light pulse covers in one half of a round trip is the Pythagorean sum of the horizontal and vertical distances:

$$(\text{distance observed by Moe})^2 = d^2 + \left(\frac{1}{2}v\Delta t\right)^2 \tag{2.5}$$

Further, he must also observe the speed of light to be c just as Joe does. If he measures the passage of time by counting the light pulses as Joe does, then he would say that after one half round trip, the light has covered this distance at a speed c, and would equate this with a time interval in his own reference frame Δt. Put another way, he would say that the distance covered by the light in one half round trip is just $\frac{1}{2}c\Delta t$, in which case we can rewrite the equation above:

$$\left(\frac{1}{2}c\Delta t\right)^2 = d^2 + \left(\frac{1}{2}v\Delta t\right)^2 = \left(\frac{1}{2}c\Delta t'\right)^2 + \left(\frac{1}{2}v\Delta t\right)^2 \tag{2.6}$$

Here we made use of the fact that we already know that $d = \frac{1}{2}c\Delta t'$ based on Joe's measeurement. Now we see that if the speed of light is indeed constant, *there is no way that the time intervals measured by Joe and Moe can be the same!* The pulse seems to take longer to make the trip from Moe's perspective, since it also has to travel sideways, not just up and down. Solely due to the constant and invariant speed of light, Joe and Moe must measure different time intervals, and Moe's must be the longer of the two. We can solve the equation above to find out just what time interval Moe measures:

$$\left(\frac{1}{2}c\Delta t\right)^2 = d^2 + \left(\frac{1}{2}v\Delta t\right)^2 \tag{2.7}$$

$$\frac{1}{4}c^2\left(\Delta t\right)^2 = d^2 + \frac{1}{4}v^2\left(\Delta t\right)^2 \tag{2.8}$$

$$\frac{1}{4}\left(\Delta t\right)^2\left(c^2 - v^2\right) = d^2 \tag{2.9}$$

$$\left(\Delta t\right)^2 = \frac{4d^2}{c^2 - v^2} \tag{2.10}$$

$$\Longrightarrow \Delta t = \frac{2d}{\sqrt{c^2 - v^2}} \tag{2.11}$$

The time interval that Joe measures is still just $\Delta t' = 2d/c$. If we factor this out of the expression above, we can relate the time intervals measured by the two observers:

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.12}$$

$$\Delta t_{\text{Moe}} = \Delta t'_{\text{Joe}}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \Delta t'_{\text{Joe}}\gamma \tag{2.13}$$

Here we defined a dimensionless quantity $\gamma = 1/\sqrt{1 - v^2/c^2}$ to simplify things a bit, we'll return to that shortly. So long as $v < c$, the time interval that Moe measures is always *larger* than the one Joe measures, by an amount which increases as the boxcar's velocity increases. This is a general result in fact: the time interval measured by an observer in motion is always longer than that measured by a stationary observer. Typically, we say that the moving observer measures a *dilated* time interval, hence this phenomena is often referred to as *time dilation.* The time dilation phenomena is symmetric – if Moe also had a clock on the ground, Joe would say that Moe's clock runs slow by precisely the same amount. It is only the relative motion that matters.

Time dilation

The time interval Δt between two events *at the same location* measured by an observer moving with respect to a clock is always *larger* than the time interval between the same two events measured by an observer stationary with respect to the clock. The 'proper' time Δt_p is that measured by the stationary observer.

$$\Delta t'_{\text{moving}} = \gamma \Delta t_{\text{stationary}} = \gamma \Delta t_p \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.14}$$

In other words, clocks run slow if you are moving with respect to them.

In the example above, it is Moe who is in a reference frame moving relative to our light 'clock' and Joe is the stationary observer. Therefore, Joe measures the 'proper' time interval, while Moe measures the dilated time interval. Incidentally, for discussions involving relativity, we basically assume that there is always a clock sitting at every possible point in space, constantly measuring time intervals, even though this is clearly absurd. What we really mean is the elapsed time that a clock at a certain position *would* read, if we had one there. For the purpose of illustration, it is just simpler to presume that everyone carries a fantastically accurate clock at all times.

Caveat for time dilation
The analysis above used to derive the time dilation formula relies on both observers measuring the same events taking place at the same physical location at the same time, such as two observers measuring the same light pulses. When timing between spatially separated events or dealing with questions of simultaneity, we must follow the formulas developed in Sect. 2.3.4.

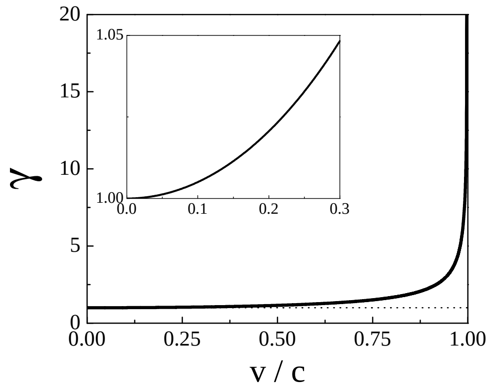
The quantity dimensionless quantity γ is the ratio of the time intervals measured by the observers moving (Moe) and stationary (Joe) relative to the events being timed. This quantity, defined by Eq. 2.15, comes up often in relativity, and it is called the *Lorentz factor*. Since c is the absolute upper limit for the velocity of anything, γ is always greater than 1. So long as the relative velocity of the moving observer is fairly small relative to c, the correction factor is negligible, and we need not worry about relativity (*e.g.,* at a velocity of $0.2c$, the correction is still only about 2%). In some sense, the quantity γ is sort of a gauge for the importance of relativistic effects – if $\gamma \approx 1$, relativity can be neglected, while if γ is much above 1, we must include relativistic effects like time dilation. Figure 2.12 provides a plot and table of γ versus v/c for reference. Note that as v approaches c, γ increases extremely rapidly.

Lorentz factor γ:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1 \tag{2.15}$$

γ is dimensionless, and $\gamma \geq 1$ for $v \leq c$. γ approaches 1 for low velocities, and increases rapidly as v approaches c. If $\gamma \approx 1$, one can safely neglect the effects of relativity.

In the case above, for velocities much less than c, when $\gamma \approx 1$, Eq. 2.14 tells us that both Joe and Moe measure approximately the same time interval, just as our everyday intuition tells us. In fact, for most velocities you might encounter in your everyday life, the correction factor γ is only different from 1 by a miniscule amount, and the effects of time dilation are negligible. They are note, however, *unmeasurable* or *unimportant*, as we will demonstrate in subsequent sections – time dilation has been experimentally verified to an extraordinarily high degree of precision, and does have some everyday consequences.



v [m/s]	$\frac{v}{c}$	γ	$1/\gamma$
0	0	0	∞
3×10^6	0.01	1.00005	0.99995
3×10^7	0.1	1.005	0.995
6×10^7	0.2	1.02	0.980
1.5×10^8	0.5	1.16	0.866
2.25×10^8	0.75	1.51	0.661
2.7×10^8	0.9	2.29	0.436
2.85×10^8	0.95	3.20	0.312
2.97×10^8	0.99	7.09	0.141
2.983×10^8	0.995	10.0	0.0999
2.995×10^8	0.999	22.4	0.0447
2.996×10^8	0.9995	31.6	0.0316
2.998×10^8	0.9999	70.7	0.0141
c	1	∞	0

Figure 2.12: *The Lorentz factor γ and its inverse for various velocities in table and graph form. The inset to the graph shows an expanded view for low velocities.*

2.3.2.1 Example: The Global Positioning System (GPS)

Before we discuss the stranger implications of time dilation, it is worth discussing at least one practical example in which the consequences of time dilation are important: the global positioning system. As you probably know, the Global Positioning System (GPS) is a network of satellites in medium earth orbit that transmit extremely precise microwave signals that can be used by a receiver to determine location, velocity, and timing. Each GPS satellite repeatedly transmits a message containing the current time, as measured by an onboard atomic clock, as well as other parameters necessary to calculate its exact position. Since the microwave signals from the satellites travel at the speed of light (microwaves are just a form of light, Sect. 9.5), knowing time difference between the moment the message was sent and the moment it was received allows an observer to determine their distance from the satellite. A ground-based receiver collects the signals from at least four distinct GPS satellites and uses them to determine its four space and time coordinates - $(x, y, z$ and $t)$.

How does relativity come into play? The 31 GPS satellites currently in orbit are in a medium earth orbit at an altitude of approximately 20, 200 km, which give them a velocity relative to the earth's surface of 3870 m/s.[5][iv] This means that the actual atomic clocks responsible for GPS timing on the satellites are moving at nearly 4000 m/s relative to the receivers on the ground calculating position. Therefore, based on our discussion above, we would expect that the satellite-based GPS clocks would measure longer time intervals that those on the earth – the GPS clocks should run

[iv]You may remember from studying gravitation the the orbital speed can be found from Newton's general law of gravitation and centripetal force, $v = \sqrt{GM/r}$, where G is the universal gravitational constant, M is the mass of the earth, and r is the radius of the orbit, as measured from the earth's center.

slow, a problem for a system whose entire principle is based on precise timing.

How big is this effect? We already know enough to calculate the timing difference. Let us assume that (somehow) at $t = 0$ we manage to synchronize a GPS clock with a ground-based one. From that moment, we will measure the elapsed time as measured by both clocks until the earth-based clock reads exactly 24 hours. We will call the earth-based clock's reference frame O, and that on the GPS satellite O', and label the time intervals correspondingly. Since are on the ground in the earth's reference frame, obviously we consider the earth-based clock to be the stationary one, measuring the proper time, and the GPS clock is moving relative to us. Applying Eq. 2.14, the elapsed time measured by the GPS clock and an earth-bound clock are related by a factor γ:

$$\Delta t'_{\text{GPS}} = \gamma \Delta t_{\text{Earth}} \tag{2.16}$$

The difference between the two clocks is then straightforward to calculate, given the relative velocity of the satellite of $v = 3870\,\text{m/s} \approx 1.3 \times 10^{-5}c$:

$$\begin{aligned}
\text{time difference} &= \Delta t_{\text{Earth}} - \Delta t_{\text{GPS}} & (2.17)\\
&= \Delta t_{\text{Earth}} - \gamma \Delta t_{\text{Earth}} & (2.18)\\
&= \Delta t_{\text{Earth}} \left(1 - \gamma\right) & (2.19)\\
&= \Delta t_{\text{Earth}} \left[1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right] & (2.20)\\
&= \left[24\,\frac{\text{h}}{\text{day}} \cdot 60\,\frac{\text{min}}{\text{h}} \cdot 60\,\frac{\text{s}}{\text{min}}\right] \left[1 - \frac{1}{\sqrt{1 - \left(1.3 \times 10^{-5}\right)^2}}\right] & (2.21)\\
&\approx \left[86400\,\text{s/day}\right] \left[-8.32 \times 10^{-11}\right] & (2.22)\\
&\approx -7.2 \times 10^{-6}\,\text{s/day} = -7.2\,\mu\text{s/day} & (2.23)
\end{aligned}$$

A grand total of about $7\,\mu$s slow over an entire day (about $0.3\,\mu$s per hour), only about 80 parts per trillion (8×10^{-11}) per day! This may not seem like a lot, until one again considers that the GPS signals are traveling at the speed of light, and even a small error in timing can translate into a relatively large error in position. Remember, it is the travel time of light signals that determines distance in GPS. If time dilation were not accounted for, a receiver using that signal to determine distance would have an error given by the time difference multiplied by the speed of light. If we presume that, conservatively, position measurements are taken only once per hour:

$$\begin{aligned}
\text{position difference in one hour} &= \text{time difference in one hour} \times c & (2.24)\\
&= \left[-3.0 \times 10^{-7}\,\text{s/h}\right]\left[3.0 \times 10^{8}\,\text{m/s}\right] & (2.25)\\
&\approx 90\,\text{m/h} & (2.26)
\end{aligned}$$

In the end, GPS must be far more accurate than this, and the effects of special relativity and time dilation must be accounted for, along with those of general relativity[5] (Sect. 2.5). Both effects together amount to a discrepancy of about $+38\,\mu$s per day. Since the orbital velocity of the satellites is well-known and essentially constant, the solution is simple: the frequency standards for the atomic clocks on the satellites are precisely adjusted to run slower and make up the difference. Though time dilation seems a rather ridiculous notion at first, it has real-world consequences we are familiar with, if unknowingly so.

Time dilation on a 747

The cruising speed of a 747 is about 250 m/s. After a 5 hour flight at cruising speed, by how much would your clock differ from a ground-based clock? How about after a year?

Using the same analysis as above, your clock would differ by about 6×10^{-9} s (6 ns) after five hours, and still only $10\,\mu$s after one year. Definitely not enough to notice, but enough to measure - current atomic clocks are accurate to $\sim 10^{-10}$ s/day (~ 0.1 ns/day). In fact, in 1971 physicists performed precisely this sort of experiment to test the predictions of time dilation in relativity, and found excellent agreement.[6]

2.3.2.2 Example: The Twin 'Paradox'

Now that we have a realistic calculation under our belt, let us consider a more extreme example. We will take identical twins, Joe and Moe, and send Moe on a rocket into deep space while Joe stays home. At the start of Moe's trip, both are 25 years old. Moe boards his rocket, and travels at $v = 0.95c$ to a distant star, and back again at the same speed. According to Joe's clock on earth, this trip takes 40 years, and Joe is 65 years old when Moe returns. Moe, on the other hand, has experienced time dilation, since relative to the earth's reference frame and Joe's clock he has been moving at $0.95c$. Moe's clock, therefore, runs more slowly, registers a smaller delay:

$$\begin{aligned}
\Delta t_{\text{Joe}} &= 40\,\text{yr} & (2.27)\\
\Delta t'_{\text{Moe}} &= \gamma \cdot 40\,\text{yr} & (2.28)\\
&= \frac{40\,\text{yr}}{\sqrt{1-\left(\frac{0.95c}{c}\right)^2}} & (2.29)\\
&\approx 12.5\,\text{yr} & (2.30)
\end{aligned}$$

It would seem, then that while Joe is 65 years old when Moe returns, having aged 40 years, Moe is 37.5 years old, having aged only 12.5 years! On the other hand, one of the principles of relativity is that there is no preferred frame of reference, it should be equally valid to use the clock on Moe's rocket ship as the proper time. From Moe's point of view, the earth is moving away from him at $0.95c$. In his reference frame, Joe's earth-bound clock should run slow, and *Moe* should be older than Joe!

This is the so-called Twin 'Paradox' of special relativity. In fact, it is not a paradox, but a misapplication of the notion of time dilation. The principles of special relativity we have been discussing are only valid for *non-accelerating* reference frames. In order for Moe to move from the earth's reference frame to the moving reference frame of the rocket ship at $0.95c$ and back again, he had to have accelerated during the initial and final portions of the trip, plus at the very least to turn around. The reference frame of the earth is for all intents and purposes not accelerating, but the reference frame on the ship *is*, and our calculation of the time dilation factor is not complete.

While the earth-bound clocks to run slow from the ship's point of view *so long as the velocity of the spaceship is constant*, during the accelerated portions of the trip the earth-bound clocks actually run *fast* and gain time compared to the rocket's clocks. An analysis including accelerated motion is beyond the scope of this text, but the gains of the earth-bound clock during the accelerated portion of the trip more than make up for the losses during the constant velocity portion of the trip, and no matter *who* keeps track, Joe will actually be younger than Moe from any reference frame. In short, there is no 'paradox' so long as the notions of relativity are applied carefully within their limits.

Inertial reference frames:
The principles of special relativity we have been discussing are only valid in *inertial* or non-accelerating reference frames. When accelerated motion occurs, a more complex analysis must be used.

2.3.3 Length Contraction

If the passage of time itself is altered by relative motion, what else must also be different? If the elapsed time interval depends on the relative motion of the clock and observer, then at constant velocity one would also begin to suspect that distance measurements must also be affected. After all, so far we have mostly talked about time in terms of objects or pulses of light traversing specific distances at constant velocity. Naturally, in order to explore this idea we need another thought experiment. Once again, it needs to involve a spaceship.

This time, the experiment is simple: a spaceship departs from earth toward a distant star, Fig. 2.13. In accordance with our discussion above, we stipulate that we *only consider the portion of the ship's journey where it is traveling at constant velocity,* and there is no acceleration to worry about. According to observations on the earth, the star is a distance L away, and the spaceship is traveling at a velocity v. From the earth's reference frame O, the amount of time the trip should

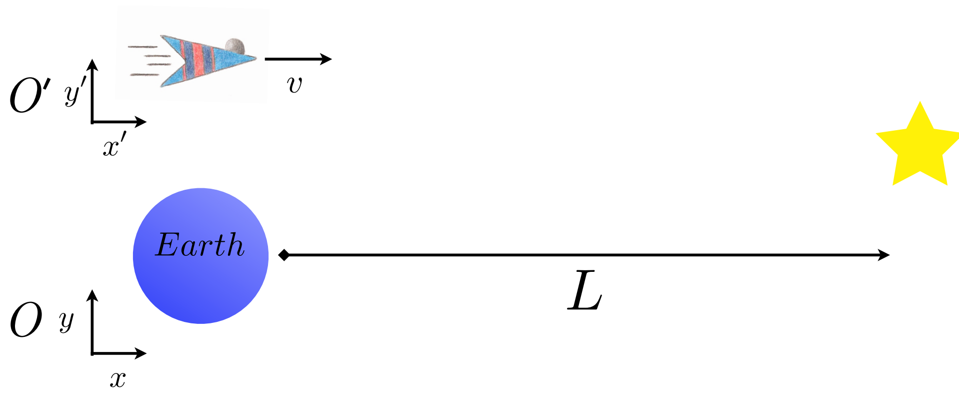


Figure 2.13: *Length contraction and travel to a distant star. A spaceship (frame O') sets out from earth (frame O) at a velocity v toward a distant star. Do the observers in the spaceship and the earth-bound observers agree on the distance to the star?*

take Δt_{E} is easy to calculate:

$$\Delta t_{\text{E}} = \frac{L}{v} \tag{2.31}$$

Fair enough. On the spaceship, however, the passage of time is slowed by a factor γ due to time dilation, and from their point of view, the trip takes less time. Since our spaceship is not accelerating in this example (it doesn't even have to turn around), we can readily apply Eq. 2.14. From the spaceship occupant's point of view, the earth is moving relative to them, so the time interval should be *divided* by γ to reflect their shorter elapsed time interval.

$$\Delta t'_{\text{ship}} = \frac{\Delta t_{\text{E}}}{\gamma} \tag{2.32}$$

Keep in mind, by *clock*, we mean the passage of time itself, this includes biological processes. We already know what Δt_{E} must be from Eq. 2.31, so we can plug that in to Eq. 2.32 above:

$$\Delta t'_{\text{ship}} = \frac{L}{v\gamma} \tag{2.33}$$

Do I divide or multiply by γ?
The Lorentz factor γ is always greater or equal to 1, $\gamma \geq 1$. If you are unsure about whether to divide or multiply by γ, think qualitatively about which quantity should be larger or smaller. In the example above, Eq. 2.32, we know the spaceship's time interval should be larger than that measured on earth, so we know we have to *divide* the earth's time interval by γ.

If the occupants of the ship also measure their velocity relative to the earth (we will pretend they even communicate with earth to make sure all observers agree on the relative velocity, $v' = v$), then they will presume that upon arrival at the distant star, the distance covered must be their velocity times their measured time interval. From the ship occupant's point of view, then, the distance to the star measured in their reference frame, L' is

$$L' = v\Delta t'_{\text{ship}} = \frac{v\Delta t_{\text{E}}}{\gamma} = \frac{L}{\gamma} \neq L \tag{2.34}$$

If you ask the people on the ship, the distance to the star is shorter, because their apparent time interval is! As we might have guessed, the relativity of time measurement also manifests itself in measurements of length, a phenomena known as *length contraction* or *Lorentz contraction*.

Length Contraction The length of an object or the distance to an object as measured by an observer in relative motion is *shorter* than that measured by an observer at rest by a factor $1/\gamma$. The stationary observer measures the proper length, L_p

$$L'_{\text{moving}} = \frac{L_{\text{stationary}}}{\gamma} = \frac{L_p}{\gamma} \tag{2.35}$$

That is, objects and distances appear shorter by $1/\gamma$ if you are moving.

There are a few caveats, however: the length contraction appears *only along the direction of relative motion*. For example, a baseball moving past you at very high velocity would be shortened only along one axis parallel to the direction of motion, and would appear as an ellipsoid, not as a smaller sphere. It would be "squashed" along the direction of the baseball's motion only, as shown in Fig. 2.14.

Just like time dilation, the length contraction effect is negligibly small at everyday velocities. Unlike time dilation, there is as yet no everyday application of time dilation, and no simple and straightforward experimental proof. We have no practical way of measuring the length of an object at extremely high velocities with sufficient precision at the moment. Collisions of elementary particles at very high velocities in particle accelerators provides some strong but indirect evidence for length contraction, and in some sense, since length contraction follows directly from time dilation, the experimental verifications of time dilation all but verify length contraction.

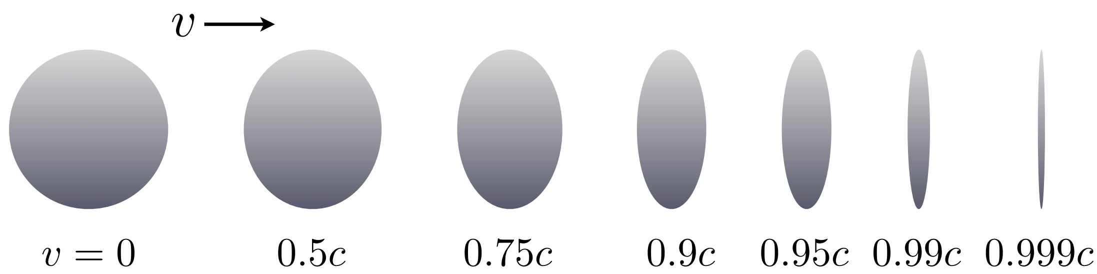


Figure 2.14: *Length contraction of a sphere traveling at various speeds, viewed side-on. The length contraction occurs only along the direction of motion. Hence, to a stationary observer, the moving sphere appears 'flattened' along the direction of motion into an ellipsoid.*

A summary of sorts:

1. objects and distances in relative motion appear shorter by $1/\gamma$
2. the length contraction is only along the direction of motion
3. the objects to not actually get shorter in their own reference frame, it is only apparent to the moving observer

2.3.4 Time and position in different reference frames

Now that we have a good grasp of time dilation and length contraction, we can start to answer the more general question of how we translate between time and position of events seen by observers in different reference frames. For example, consider Fig. 2.15. A girl in frame O is stationary relative to a star at point P, known to be a distance x away, which suddenly undergoes a supernova explosion. At precisely this instant, a boy travels past her in on a skateboard at constant relative velocity v along the x axis (frame O'). For convenience, we will assume that at the moment the explosion occurs, he is exactly the same distance away as the girl. When and where does the supernova occur, according to their own observations? How can we relate the distances and times measured by the girl to that measured by the boy, and *vice versa?*

All we need to do is apply what we know of relativity thusfar, and compare what each observer would measure in their own frame with what the *other* would measure. In the girl's case, the situation is fairly straightforward. She is a distance x from the star, and the first light from the explosion travels that distance at a velocity c. Therefore, according to her observations, the first light from the supernova arrives after:

$$t_{\text{arrival}} = \frac{x}{c} \tag{2.36}$$

What about the boy on the skateboard, in frame O? Since he is moving relative to the star, the distance to the star appears length contracted from his point of view. At the instant of the supernova, he measures a distance shorter by a factor γ compared to that measured by the girl. Furthermore, from his point of view in his own reference frame, he is siting still, and *the supernova*

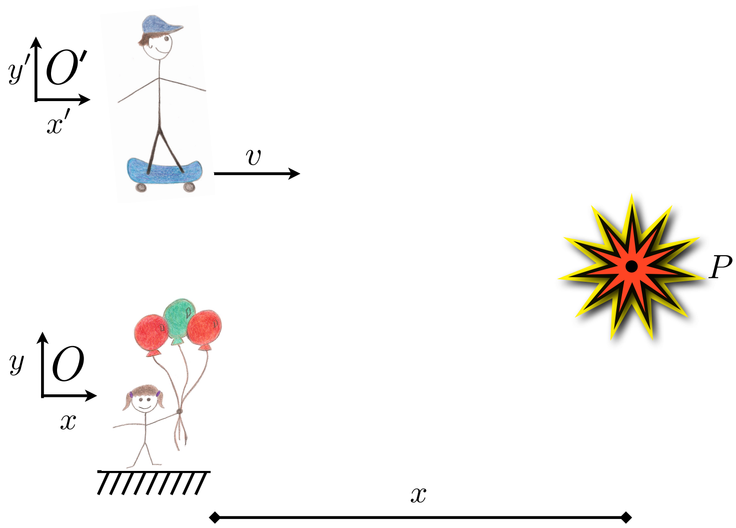


Figure 2.15: *A stationary and moving observer watch a supernova explosion. A girl in frame O is stationary relative to the supernova, a distance x away. A boy on a skateboard in the O' frame is traveling at v relative to frame O. How long does it take before the first light of the supernova reaches each of them?*

is moving toward him at velocity v. Therefore, from his point of view the supernova is getting closer to him. After t' seconds by his clock, the supernova is a distance vt' closer. Putting these two bits together, the distance x' the boy would measure to the supernova is:

$$x' = \frac{x}{\gamma} - vt' \tag{2.37}$$

So the distance to the supernova he claims is the original distance, length contracted due to his motion relative to the supernova, minus the rate at which he gets closer to the supernova.

What would the girl say about all this? The distance between the boy and the supernova, from her point of view, would have to be contracted to x'/γ since the boy is in motion relative to her. Additionally, from her point of view, since the boy is moving away from her at v, the distance between the two is *increasing* by vt after t seconds. We can express her perceived distance to the supernova as the sum of two distances: the distance from her to the boy, and the distance from the boy to the supernova:

$$x = vt + \frac{x'}{\gamma} \tag{2.38}$$

Now we have consistent expressions relating the distance measured by one observer to that measured by the other. If we rearrange Eqs. 2.37 and 2.38 a bit, and put primed quantities on one side and unprimed on the other, we arrive the transformations between positions measured by moving observers in their usual form:

Transformation of distance between reference frames:

$$x' = \gamma\,(x - vt) \tag{2.39}$$
$$x = \gamma\left(x' + vt'\right) \tag{2.40}$$

Here (x, t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v, measures the same event to be at position and time (x', t').

These equations include the effects of length contraction and time dilation we have already developed, as well as including the relative motion between the observers. If we use Eqs. 2.37 and 2.38 together, we can also arrive at a more direct expression to transform the measurement times as well. To start, we'll take Eq. 2.39 as written, and substitute it into Eq. 2.40:

$$\begin{aligned} x &= \gamma\left(x' + vt'\right) &(2.41)\\ &= \gamma\left(\gamma\,(x - vt) + vt'\right) &(2.42)\\ &= \gamma^2 x - \gamma^2 vt + \gamma vt' &(2.43) \end{aligned}$$

So far its a bit messy, but it will get better. Now let's solve that for t'. A handy relationship we will make use of is $\left(1-\gamma^2\right)/\gamma^2 = -v^2/c^2$, which you should verify for yourself.

$$\begin{aligned} \gamma vt' &= \left(1-\gamma^2\right)x + \gamma^2 vt &(2.44)\\ \implies \quad t' &= \gamma t + \frac{\left(1-\gamma^2\right)x}{\gamma v} &(2.45)\\ &= \gamma\left[t + \frac{1-\gamma^2}{\gamma^2}\left(\frac{x}{v}\right)\right] &(2.46)\\ &= \gamma\left[t - \frac{vx}{c^2}\right] &(2.47) \end{aligned}$$

And there we have it, the transformation between time measured in different reference frames. A similar procedure gives us the reverse transformation for t in terms of x' and t'.

Time measurements in different non-accelerating reference frames:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.48}$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{2.49}$$

Here (x,t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v, measures the same event to be at position and time (x',t').

The first term in this equation is just the time it takes light to travel across the distance x from point P, corrected for the effects of time dilation we now expect. The second term is new, and it represents an additional *offset* between the clock on the ground and the one in the car, not just one running slower than the other. What it means is that events seen by the girl in frame O do *not* happen at the same time as viewed by the boy in O'!

This is perhaps more clear to see if we make two different measurements, and try to find the elapsed time between two events. If our girl in frame O sees one even take place at position x_1 and time t_1, labeled as (x_1,t_1), and a second event at x_2 and t_2, labeled as (x_2,t_2), then she would say that the two events were spatially separated by $\Delta x = x_1 - x_2$, and the time interval between them was $\Delta t = t_1 - t_2$. If we follow the transformation to find the corresponding times that the boy observes, t_1' and t_2', we can also calculate the boy's perceived time interval between the events, $\Delta t'$:

Elapsed times between events in non-accelerating reference frames:

$$\Delta t' = t_1' - t_2' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \tag{2.50}$$

If observer in O stationary relative to the events (x_1,t_1) and (x_2,t_2) measures a time difference between them of $\Delta t = t_1 - t_2$ and a spatial separation $\Delta x = x_1 - x_2$, an observer in O' measures a time interval for the same events $\Delta t'$. Events simultaneous in one frame $(\Delta t = 0)$ are only simultaneous in the other $(\Delta t' = 0)$ when there is no spatial separation between the two events $(\Delta x = 0)$.

For two events to be simultaneous, there has to be no time delay between them. For the girl to say the events are simultaneous requires that she measure $\Delta t = 0$, while for the boy to say the same requires $\Delta t' = 0$. We cannot satisfy both of these conditions based on Eq. 2.50 unless there is no relative velocity between observers $(v = 0)$, or the events being measured are not spatially separated $(\Delta x = 0)$. This means *two observers in relative will only find the same events simultaneous if the events are not spatially separated!* **Put simply, events are only simultaneous in both reference frames if they happen at the same spot.** At a given velocity, the larger the separation between the two events, the greater the degree of non-simultaneity. Similarly, for a given separation, the larger the velocity, the greater the discrepancy between the two frames. This is sometimes called **failure if simultaneity at a distance.**

In the end, this is our *general* formula for time dilation, including events which are spatially separated. If we plough still deeper into the consequences of special relativity and simultaneity, we will find that our principles of relativity have indeed preserved causality - cause always precedes effect - it is just that what one means by "precede" depends on which observer you ask. What relativity says is that cause must precede its effect according to all observers in inertial frames, which equivalently prevents both faster than light travel or communication and influencing the past.

2.3.4.1 Summary of sorts: the Lorentz Transformations

We are now ready to make a summary of the relativistic transformations of time and space. Let us consider two reference frames, O and O', moving at a **constant** velocity v relative to one another. For simplicity, we will consider the motion to be along the x and x' axes in each reference frame, so the problem is still one-dimensional. The observer in frame O measures an event to occur at time t and position (x, y, z). The event is *at rest with respect to the O frame.* Meanwhile, the observer in frame O' measures the *same event* to take place at time t' and position (x', y', z'). Based on what we have learned so far, we can write down the general relations between space and time coordinates in each frame, known as the *Lorentz transformations:*

Lorentz transformations between coordinate systems:

$$x' = \gamma\,(x - vt) \quad \text{or} \quad x = \gamma\left(x' + vt'\right) \tag{2.51}$$

$$y' = y \tag{2.52}$$

$$z' = z \tag{2.53}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{or} \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{2.54}$$

Here (x, y, z, t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v along the x axis, measures the same event to be at position and time (x', y', z', t').

Here we have provided both the 'forward' and 'reverse' forms of the transformations for convenience. Again, the distance is only contracted along the direction of motion, the x and x' directions – the y and z coordinates are thus unaffected. When the velocity is small compared to c ($v \ll c$), the first equation gives us our normal Newtonian result, the position in one frame relative to the other is just offset by their relative velocity times the time interval, and the time is the same. These compact equations encompass all we know of relativity so far - length contraction, time dilation, and lack of simultaneity.

Why are the transformations the way they are?

They take this form because they are the ones that leave the velocity of light constant at c in *every* reference frame.

Relativity for observers in relative motion at constant velocity:

1. Moving observers see lengths contracted along the direction of motion.
2. Moving observers' clocks 'run slow', less time passes for them.
3. Events simultaneous in one frame are not simultaneous in another unless they occur at the same position
4. All observers measure the same speed of light c

2.3.5 Addition of Velocities in Relativity

The invariance of the speed of light has another interesting consequence, namely, that one can no longer simply add velocities together to compute relative velocities in different reference frames in the way we did at the beginning of this chapter. Think about one of our original questions regarding relative motion, Fig. 2.3, in which a bully threw a dart off of a moving skateboard at a little girl's balloon. In that case, we said that the girl observed the dart to move at a velocity which was the *sum* of the velocities of the skateboard relative to the girl and the dart relative to the skateboard. When velocities are an appreciable fraction of the speed of light, this simple velocity addition breaks down.

In the end, it *has to*, or the speed of light could not be an absolute cosmic speed limit. Think about this: if you are driving in your car at 60 mi/hr down the freeway and turn on your headlights, do the light beams travel at c, or c plus 60 mi/hr? We know already that the answer must be c, but that is not at all consistent with our usual ideas of relative motion. If we can't just add the velocities together, what do we do? Is there a way to combine relative velocities such that the speed of light remains a constant and an upper limit? There is a relatively simple mathematical way to accomplish this. Once again, we will derive the result in the context of yet another thought experiment and try to show you how to use it.

The present thought experiment is just a variation the dart thrown from the skateboard, and is shown in Fig. 2.16. An observer on the ground (frame O) sees a person on a cart (frame O') moving at velocity v_a, as measured in the ground-based reference frame O. The person on the cart throws a ball at a velocity v'_b relative to the cart, which is measured as v_b in the ground-based frame. The ground-based observer measures v_a and v_b, while the observer on the cart measures the cart's velocity as v'_a and the ball's velocity as v'_b. How do we relate the velocities measured in the different frames O and O', without violating the principles of relativity we have investigated so far?

We can't simply add and subtract the velocities like we want to, our thought experiment of Sect. 2.2.3 involving a flashlight and a rocket ruled this out already, since this does not keep the speed of light invariant. So how *do* we properly add the velocities? Velocity is just displacement

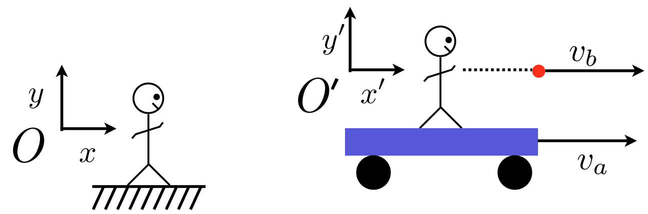


Figure 2.16: *Relativistic addition of velocities. An observer on the ground (frame O) sees a person on a cart (frame O') traveling at velocity v_a throw a ball off of the car at a velocity v_b relative to the ground. How do we relate the velocities as measured in the different reference frames?*

per unit time. If we calculate the displacement and time in one reference frame, then transform *both* to the other reference frame, we can divide them to *correctly* find velocity.

Let's start with the velocity of the ball as measured by the observer on the cart, v'_b. The displacement of the ball relative to the cart at some time t' after it was thrown, also measured in the cart's frame O', is just $x'_b = v'_b t'$. This is just how far ahead of the car the ball is after some time t'. We can substitute this into Eq. 2.51 to find out what displacement the observer on the ground in O should measure, remembering that v_a is the relative velocity of the observers:

$$x_b = \gamma\left(x'_b + v_a t'\right) = \gamma\left(v'_b t' + v_a t'\right) \tag{2.55}$$

But now we have x, the displacement of the ball seen from O, in terms of t', the time measured in O'. If we want to find the velocity of the ball as measured by an observer in O, *we have to divide the distance measured in O by the time measured in O!* We can't divide one person's position by another person's time, we have to transform *both*. So we should use Eq. 2.54 to find out what t is from t' too:

$$t = \gamma\left(t' + \frac{v_a x'}{c^2}\right) = \gamma\left(t' + \frac{v_a v'_b t'}{c^2}\right) \tag{2.56}$$

Now we have the displacement of the ball x and the time t as measured by the observer on the ground in O. The velocity in O is just the ratio of x to t:

$$v_b = \frac{x}{t} \tag{2.57}$$

$$= \frac{\gamma\left(v_b' t' + v_a t'\right)}{\gamma\left(t' + \frac{v_a v_b' t'}{c^2}\right)} \tag{2.58}$$

$$= \frac{v_b' + v_a'}{1 + \frac{v_a v_b}{c^2}} \tag{2.59}$$

For the last step, we divided out $\gamma t'$ from everything, by the way. So, this is the proper way to compute relative velocity of the ball observed from the ground, consistent with our framework of relativity.

$$\text{velocity of ball observed from the ground} = v_b = \frac{v_a + v_b'}{1 + \frac{v_a v_b'}{c^2}} \tag{2.60}$$

In the limiting case that the velocities are very small compared to c, then it is easy to see that the expression above reduces to $v_b = v_a + v_b'$ – the velocity of the ball measured from the ground is the velocity of the car relative to the ground plus the velocity of the ball relative to the car. But, this is *only* true when the velocities are small compared to c.[v] Similarly, we could solve this equation for v_b' instead and relate the velocity of the ball as measured from the car to the velocities measured from the ground:

$$\text{velocity of ball observed from the cart} = v_b' = \frac{v_b - v_a}{1 - \frac{v_a v_b}{c^2}} \tag{2.61}$$

The equation above allows us to calculate the velocity of the ball as observed from the car if we only had ground-based measurements. Again, for low velocities, we recover the expected result $v_b' = v_b - v_a$. What about the velocity of the cart? We don't need to transform it, since it is already the *relative* velocity between the frames O and O', and hence between the ground-based observer and the car. **We only need the velocity addition formula when a third party is involved.** Out of the three relevant velocities, we only ever need to know two of them.

So this is it. This simple formula is all that is needed to properly add velocities and obey the principles of relativity we have put forward. Below, we put this in a slightly more general formula.

[v] Or, more precisely, when the *product* of the velocities is small compared to c^2.

Relativistic velocity addition:
We have an observer in a frame O, and a second observer in another frame O' who are moving relative to each other at a velocity v. Both observers measure the velocity of another object in their own frames (v_{obj} and v'_{obj}). We can relate the velocities measured in the different frames as follows:

$$v_{\text{obj}} = \frac{v + v'_{\text{obj}}}{1 + \frac{vv'_{\text{obj}}}{c^2}} \qquad v'_{\text{obj}} = \frac{v_{\text{obj}} - v}{1 - \frac{vv_{\text{obj}}}{c^2}} \tag{2.62}$$

Again, v_{obj} is the object's velocity as measured from the O reference frame, and v_{obj} is its velocity as measured from the O' reference frame.

Velocities greater than c?
The velocity addition formula shows that one cannot accelerate something past the speed of light. No matter what subluminal velocities you add together, the result is *always* less than c. Try it! Our relativistic equations for momentum and energy will further support this.

Remember, c isn't just the speed of light, it is a limiting speed for *everything*!

2.3.5.1 Example: throwing a ball out of a car

Just to be clear, let us make our previous example more concrete. Let's say we have Joe in reference frame O, sitting on the ground, while Moe is in a car (frame O') moving at $v_{\text{car}} = \frac{3}{4}c$. Moe throws a ball *very hard* out of the car window, such that he measures its velocity to be $v'_{\text{ball}} = \frac{1}{2}c$ in his reference frame. What would Joe say that the velocity of the ball is, relative to his reference frame on the ground?

Basically, Joe wants to know the velocity of the ball relative to the ground, not relative to the car. What we need to do is relativistically combine the velocity of the car relative to the ground and the velocity of the ball relative to the car. Classically, we would just add them together:

$$v_{\text{ball}} = v_{\text{car}} + v'_{\text{ball}} = \frac{3}{4}c + \frac{1}{2}c = \frac{5}{4}c = 1.25c \qquad \text{WRONG!}$$

Clearly this is an absurdity - the ball cannot be traveling faster than the speed of light in *anyone's* reference frame. We need to use the proper relativistic velocity addition formula, Eq. 2.62. We know the velocity of the ball relative to the car in frame O', v'_{ball} and the velocity of the car relative to the ground in the O frame, v_{car}, so we just substitute and simplify:

$$v_{\text{ball}} = \frac{v_{\text{car}} + v'_{\text{ball}}}{1 + \frac{v_{\text{car}} v'_{\text{ball}}}{c^2}} \tag{2.63}$$

$$= \frac{\frac{3}{4}c + \frac{1}{2}c}{1 + \frac{\left(\frac{3}{4}c\right)\left(\frac{1}{2}c\right)}{c^2}} \tag{2.64}$$

$$= \frac{\frac{5}{4}c}{1 + \frac{3}{8}c} = \frac{10}{11}c \approx 0.91c \tag{2.65}$$

So, in relativity, three quarters plus one half is only about 0.9! But this is the result we are looking for - no matter what velocities $v < c$ we add together, we always get an answer less than c. Put another way, no matter what reference frame we consider, the velocity of an object will *always* observed to be less than c. So our relativistic velocity addition works so far. But what about applying it to light, which is actually traveling right at c. Does everything still come out ok?

2.3.5.2 Example: shining a flashlight out of a rocket

What if, instead of throwing a ball out of the window, Moe uses a flashlight to send out a light pulse? In that case, we have to find that the velocity of light is c no matter which frame we use. Remember our problem in Sect. 2.2.3? We had Joe traveling on a rocket at $0.99c$, while Moe on the ground shines a flashlight parallel to his path, shown again in Fig: 2.17. Our claim at the time was that both Moe and Joe should measure the same speed of light. Does our new velocity addition formula work for this case?

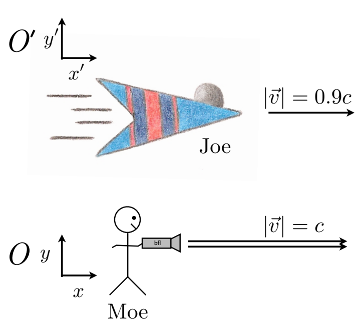


Figure 2.17: *Joe is traveling on a rocket at $|\vec{\mathbf{v}}| = 0.99c$, while Moe on the ground shines a flashlight parallel to Joe's path. Both Joe and Moe observe the light from the flashlight to travel at $|\vec{\mathbf{v}}| = c$, consistent with our relativistic velocity addition formula!*

In this case, Joe is on a rocket (frame O') moving at $v_{\text{rocket}} = 0.99c$ relative to Moe on the ground. Moe knows that in his frame O, the light from the flashlight travels away from him at velocity $v_{\text{light}} = c$. What is the velocity of light observed by Joe in the rocket, $v_{\text{light}'}$, if we use the velocity addition formula? All we have to do is subtract the speed of light as measured by Moe from Joe's speed on the rocket ship, according to the second equation in 2.62:

$$v'_{\text{light}} = \frac{v_{\text{light}} - v_{\text{rocket}}}{1 - \frac{v_{\text{rocket}} v_{\text{light}}}{c^2}} \tag{2.66}$$

$$= \frac{c - 0.99c}{1 - \frac{(0.99c)(c)}{c^2}} \tag{2.67}$$

$$= \frac{0.01c}{1 - 0.99} = c \tag{2.68}$$

Lo and behold, the thing works! Our velocity addition formula correctly calculates that both Joe and Moe have to measure the same speed of light, since the speed of light is the same when observed from any reference frame. We shouldn't be too surprised, however: the velocity addition formula was *constructed* to behave in exactly this way. How about if Joe holds the flashlight while in the rocket, what is the speed of light as measured by Moe on the ground? Now we have to add the velocities of the light coming out of the rocket and the velocity of the rocket itself, according the first equation in 2.62. Still no problem:

$$v_{\text{light}} = \frac{v_{\text{rocket}} + v'_{\text{light}}}{1 + \frac{v_{\text{rocket}} v'_{\text{light}}}{c^2}} \tag{2.69}$$

$$= \frac{0.99c + c}{1 + \frac{(0.99c)(c)}{c^2}} \tag{2.70}$$

$$= \frac{1.99c}{1 + 0.99} = c \tag{2.71}$$

In the end, we have succeeded in constructing a framework of mechanics that keeps the speed of light invariant in all reference frames, and answers (nearly) all the questions raised at the beginning of the chapter.

Is everything relative then?

Not quite!

- All observers will agree on an objects *rest* length
- All observers will agree on the proper time
- All observers will agree on an objects rest mass
- The speed of light is an upper limit to physically attainable speeds

2.4 Mass, Momentum, and Energy

So far, the simple principles of relativity have had enormous consequences. Our basic notions of time, position, and even simultaneity all needed to be modified. If position and time must be altered, then it stands to reason that *velocity* - the change of position with time - must also be altered. Sure enough, the velocity addition formula was also a required change. What next? If our notions of relative velocity need to be altered, then the next thing must surely be momentum and kinetic energy. As it turns out, even our concept of *mass* needs to be tweaked a bit.

2.4.1 Relativistic Momentum

First, let's consider momentum. Classically, we define momentum in terms of mass and velocity, $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. A basic principle of classical mechanics you have learned is that momentum must be conserved, no matter what. What about in relativity? In relativity, exactly what $\vec{\mathbf{v}}$ is depends on the reference frame in which it is measured. That means that our usual definition of momentum above depends on the reference frame as well. It gets worse. Using our simple $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, not only would the total amount of momentum depend on the choice of reference frame, conservation of momentum in one frame would not necessarily be true in another. How can a fundamental conservation law depend on the frame of reference?

It cannot - this is one of our basic principles of relativity, *viz.*, the laws of physics are the same for all non-accelerating frames of reference. We *must* have conservation of momentum, independent of what frame in which the momentum us measured. How do we construct a new equation for momentum, one for which conservation of momentum is always valid, but at low velocities reduces to our familiar $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$? The result is not surprising: we only need to transform velocity the same way we transformed position:

Relativistic momentum:

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}} \tag{2.72}$$

Here $\vec{\mathbf{p}}$ is the momentum vector for an object of mass m moving with velocity $\vec{\mathbf{v}}$.

The derivation is a bit beyond the scope of our discussion, but defining momentum in this way makes it independent of the choice of reference frame, and restores conservation of momentum as a fundamental physical law. For low velocities ($v \ll c$), $\gamma \approx 1$, and this reduces to the familiar result. For velocities approaching c, the momentum grows much more quickly than we would expect. In fact, an object traveling at c would require *infinite* momentum (and therefore infinite kinetic energy), clearly an absurdity. This is one good reason why nothing with finite mass can ever travel at the speed of light! Only light itself, with no mass, can travel at the speed of light.

2.4.2 Relativistic Energy

The relativistic correction to momentum is straightforward. Given that kinetic energy depends on the momentum of an object (one can write $KE = p^2/2m$), one would expect a necessary revision for kinetic energy as well. This one is not so straightforward, however. First, we need to think about what we mean by energy in the first place.

In classical mechanics, for a single point mass in linear motion (*i.e.,* not rotating), the kinetic energy simply goes to zero when the body stops, $KE = \frac{1}{2}mv^2 = p^2/2m$. For an arbitrary body, however, the result is not so simple. If a composite object contains multiple, independently moving bodies (such the individual atoms making up matter, for instance), the individual entities may interact among themselves and move about, and the object possesses *internal energy* E_i as well as the kinetic energy due to the motion of the whole mass. Overall, classically the kinetic energy of such a body is the sum of these two energies – the energy due to the motion of the object as a whole, and the energy due to the motion of the constituents of the object, $KE = \frac{1}{2}mv^2 + E_i$. Any moving body more complex than a single point mass has a contribution due to its internal energy.

In relativity, the kinetic energy does still depend on the motion of a body as a whole as well as its internal energy content. As with momentum, conservation of energy requires that *the energy of a body is independent of the choice of reference frame*, the *total energy* of a body cannot depend on the frame in which it is measured. The total energy – kinetic plus internal – must be the same in all reference frames. A derivation requires somewhat more math than we would like, but the result is simple:

Relativistic energy of a moving body:

$$E = \gamma mc^2 \tag{2.73}$$

This equation already tells us that the energy content of a body grows rapidly as v approaches c, and reaching the speed of light would require a body to have infinite energy. What is more interesting, however, is when the velocity of the body is *zero*, *i.e.*, $\gamma = 1$. In this case, $E = mc^2$ - the body has finite energy even when not in motion! This is Einstein's most famous equation, and it represents the fundamental equivalence of mass and energy. Any object has an *intrinsic, internal energy* associated with it by virtue of having mass. This constant energy is called the *rest energy*:

Rest Energy:

$$E_R = mc^2 \tag{2.74}$$

As Einstein himself put it, "Mass and energy are therefore essentially alike; they are only different expressions for the same thing."[7] Matter is basically an extremely dense form of energy – is convertible into energy, and *vice versa*. In fact, the rest energy content of matter is enormous, owing to the enormity of c^2 - one gram of normal matter corresponds to about 9×10^{13} J, the same energy content as 21 ktons of TNT! It is the conversion of matter to energy that is responsible for the enormous energy output of nuclear reactions, such as those that power the sun, a subject we will return to.

The equivalence of matter and energy, or, if you like, the presence of an internal energy due solely to a body's matter content, is an unexpected consequence of relativity. But we still have not determined the actual kinetic energy of a relativistic object! Again, the derivation is somewhat laborious, but the result is easy enough to understand. If we take the total energy of an object, Eq. 2.73, and subtract off the velocity-independent rest energy, Eq. 2.74, what we are left with is the part of a body's energy that depends solely on velocity. This is the kinetic energy we are looking for, and it means the *total energy of a body is the sum of its rest and kinetic energies:*

Relativistic kinetic energy:

$$KE = (\gamma - 1)\, mc^2 \tag{2.75}$$

Total energy:

$$E_{\text{total}} = KE + E_R \tag{2.76}$$

Since $\gamma = 1$ when $v = 0$, the kinetic energy of a stationary body is zero, as we expect. At low velocities ($v \ll c$), one can show that this expression correctly reduces to $\frac{1}{2}mv^2$. As with the total energy, for a body to actually acquire a velocity of c it would need an infinite kinetic energy, again, a primary reason why no object with mass can travel at the speed of light.

For completion, we should note that it is still possible to relate relativistic energy and momentum, just like it was possible to relate classical kinetic energy and momentum, though we will not derive the expressions here:

Relativistic energy-momentum equations:

$$E^2 - (pc)^2 = \left(mc^2\right)^2 \tag{2.77}$$

$$Ev = pc^2 \tag{2.78}$$

here p is the momentum of a body, m its mass, v its velocity, E its energy, and c is the speed of light. We can use this to write the relativistic kinetic energy and momentum equations in a different form:

$$KE = \sqrt{p^2c^2 + m^2c^4} - mc^2 \qquad \text{and} \qquad p = \sqrt{\frac{E^2}{c^2} - m^2c^4} \tag{2.79}$$

The energy content of a body still scales with its momentum, and for a body at rest ($p = 0$), the energy content is purely the rest energy mc^2. Once again we have an unexpected result, however: *objects with no mass must also have momentum, so long as they have energy.* For massless particles – such as the photons that make up a beam of light – we have the result $E = pc$, or $p = E/c$. This is truly another odd result of relativity, completely unexpected from classical physics! How can objects with no mass still have momentum? Since matter and energy are equivalent according to relativity, having energy is just as good as having mass, and still leads to a net momentum. This will become an important consideration when we begin to study optics and modern physics.

Momentum of massless objects:

$$p = \frac{E}{c} \tag{2.80}$$

If you combine Eqs. 2.78 and 2.80, you come to an even wilder conclusion. If the particle has zero mass, but *some* energy greater than zero, then we can write

$$v = \frac{pc^2}{E} = \frac{\frac{E}{c}c^2}{E} = c \tag{2.81}$$

A particle with zero mass always moves at the speed of light, and can never stop moving! It doesn't matter what the energy of the particle is, anything with finite energy but zero mass has to travel at the speed of light. The converse is true as well – anything moving at the speed of light must be massless. Just to drive the point home one last time: *the speed of light is an upper limit to physically attainable speeds for material bodies.*

2.4.3 Relativistic Mass

About the only thing left we have not modified with relativity is mass. Most modern interpretations of relativity consider mass to be an *invariant* quantity, properly measured when the body is at rest (or measured within its own reference frame). This rest mass of an object in its own reference frame is called the *invariant mass* or *rest mass*, and is an observer-independent quantity synonymous with our usual definition of "mass."

These days, we say that while the *momentum* of a body must be the same in all reference frames, and hence must be transformed, the *mass* of a body is just a constant, and is measured in the body's own reference frame. Rest mass is in some sense just counting the number of atoms in an object, something we really only do in the object's reference frame anyway. If we are measuring an object from another reference frame, we will typically be measuring its *momentum*, or kinetic energy, not counting how many atoms it contains. Thus it is momentum and kinetic energy we transform to be invariant in all reference frames and mass we simply say is a property of an object measured in its own reference frame.

2.5 General Relativity

2.6 Quick Questions

1. An astronaut traveling at $v=0.80c$ taps her foot 3.0 times per second. What is the frequency of taps determined by an observer on earth? (*Hint: be careful about the difference between time and frequency!*)

- ◯ 5.0 taps/sec
- ◯ 6.7 taps/sec
- ◯ 1.8 taps/sec
- ◯ 3.0 taps/sec

2. A spaceship moves away from earth at high speed. How do experimenters on earth measure a clock in the spaceship to be running? How do those in the spaceship measure a clock on earth to be running?

- ◯ slow; fast
- ◯ slow; slow
- ◯ fast; slow
- ◯ fast; fast

3. If you are moving in a spaceship at high speed relative to the earth, would you notice a difference in your pulse rate? In the pulse rate of the people back on earth?

- ◯ no; yes
- ◯ no; no
- ◯ yes; no
- ◯ yes; yes

4. The period of a pendulum is measured to be 3.00 in its own reference frame. What is the period as measured by an observer moving at a speed of $0.950c$ with respect to the pendulum?

- ◯ 6.00 sec
- ◯ 13.4 sec
- ◯ 0.938 sec
- ◯ 9.61 sec

5. The Stanford Linear Accelerator (SLAC) can accelerate electrons to velocities very close to the speed of light (up to about $0.99999999995c$ or so). If an electron travels the 3 km length of the accelerator at $v=0.999c$, how long is the accelerator from the *electron's* reference frame?

- ◯ 134 m
- ◯ 67.1 km
- ◯ 94.9 m
- ◯ 300 m

6. A spacecraft with the shape of a sphere of diameter D moves past an observer on Earth with a speed $0.5\,c$. What shape does the observer measure for the spacecraft as it moves past?

◯ streak
◯ ellipsoid
◯ sphere
◯ cube

7. Suppose you're an astronaut being paid according to the time you spend traveling in space. You take a long voyage traveling at a speed near that of light. Upon your return to earth, you're asked how you would like to be paid: according to the time elapsed by a clock on earth, or according to the ship's clock. Which do you choose to maximize your paycheck?

◯ The earth clock.
◯ The ship's clock.
◯ It doesn't matter.

2.7 Problems

1. In the 1996 movie *Eraser*,[8] a corrupt business Cyrez is manufacturing a handheld rail gun which fires aluminum bullets at nearly the speed of light. Let us be optimistic and assume the actual velocity is $0.75\,c$. We will also assume that the bullets are tiny, about the mass of a paper clip, or $m = 5 \times 10^{-4}\,\text{kg}$.

(a) What is the relativistic kinetic energy of such a bullet?
(b) Let us further assume that Cyrez has managed to power the rail guns by matter-energy conversion. What amount of mass would have to be converted to energy to fire a single bullet? (For comparison, note that 1 kg of TNT has an equivalent energy content of about 4×10^9 J.)

2. Show that the kinetic energy of a (non-relativistic) particle can be written as $KE = p^2/2m$, where p is the momentum of a particle of mass m.

3. A pion at rest ($m_\pi = 273\,m_{e^-}$) decays to a muon ($m_\mu = 207\,m_{e^-}$) and an antineutrino ($m_{\overline{\nu}} \approx 0$). This reaction is written as $\pi^- \rightarrow \mu^- + \overline{\nu}$. Find the kinetic energy of the muon and the energy of the antineutrino in electron volts. *Hint: relativistic momentum is conserved.*

4. An alarm clock is set to sound in 15 h. At $t = 0$ the clock is placed in a spaceship moving with a speed of $0.77c$ (relative to Earth). What distance, as determined by an Earth observer, does the spaceship travel before the alarm clock sounds?

5. The average lifetime of a pi (π) meson in its own frame of reference (*i.e.*, the proper lifetime) is 2.6×10^{-8} s

(a) If the meson moves at $v = 0.98c$, what is its mean lifetime as measured by an observer on earth?

(b) What is the average distance it travels before decay, measured by an observer on Earth?
(c) What distance would it travel if time dilation did not occur?

6. You are packing for a trip to another star, and on your journey you will travel at $0.99c$. Can you sleep in a smaller cabin than usual, because you will be shorter when you lie down? Explain your answer.

7. A deep-space probe moves away from Earth with a speed of $0.88c$. An antenna on the probe requires 4.0 s, in probe time, to rotate through 1.0 rev. How much time is required for 1.0 rev according to an observer on Earth?

8. A friend in a spaceship travels past you at a high speed. He tells you that his ship is 24 m long and that the identical ship you are sitting in is 18 m long.

(a) According to your observations, how long is your ship?
(b) According to your observations, how long is his ship?
(c) According to your observations, what is the speed of your friend's ship?

9. A Klingon space ship moves away from Earth at a speed of $0.700c$. The starship Enterprise pursues at a speed of $0.900c$ relative to Earth. Observers on Earth see the Enterprise overtaking the Klingon ship at a relative speed of $0.200c$. With what speed is the Enterprise overtaking the Klingon ship as seen by the crew of the Enterprise?

10. An observer sees two particles traveling in opposite directions, each with a speed of $0.99000c$. What is the speed of one particle with respect to the other?

2.8 Solutions to Quick Questions

1. 1.8 **taps/sec.** The 'proper time' Δt_p is that measured by the astronaut herself, which is 1/3 of a second between taps (so that there are 3 taps per second). The time interval *between taps* measured on earth is dilated (longer), so there are *less* taps per second. For the astronaut:

$$\Delta t_p = \frac{1\,\text{s}}{3\,\text{taps}}$$

On earth, we measure the dilated time:

$$\Delta t' = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{0.8^2c^2}{c^2}}} \cdot \left(\frac{1\,\text{s}}{3\,\text{taps}}\right) = \frac{1}{\sqrt{1-0.8^2}} \cdot \left(\frac{1\,\text{s}}{3\,\text{taps}}\right) \approx \frac{0.56\,\text{s}}{\text{tap}} = \frac{1\,\text{s}}{1.8\,\text{taps}}$$

2. slow; slow. The time-dilation effect is symmetric, so observers in each frame measure a clock in the other to be running slow. Put another way, the *relative* velocity of the earth and the ship is the same no matter who you ask – each says the other is moving with some speed v, and they are sitting still. Therefore, the dilation effect is the same in both cases.

3. no; yes. There is no relative speed between you and your own pulse, since you are in the same reference frame, so there is no difference in your pulse rate (possible space-travel-related anxieties aside). There is a relative velocity between you and the people back on earth, however, so you would find their pulse rate *slower* than normal. Similarly, they would find *your* pulse rate slower than normal, since you are moving relative to them. Relativistic effects are always attributed to the other party – you are always at rest in your own reference frame.

4. 9.61 **sec.** The proper time is that measured by in the reference frame of the pendulum itself, $\Delta t_p = 3.00\,\text{sec}$. The moving observer has to observe a *longer* period for the pendulum, since from the observer's point of view, the pendulum is moving relative to it. Observers always perceive clocks moving relative to them as running slow. The factor between the two times is just γ:

$$\Delta t' = \gamma \Delta t_p = \frac{3.0\,\text{sec}}{\sqrt{1-\frac{0.95^2c^2}{c^2}}} = \frac{3.0\,\text{sec}}{\sqrt{1-0.95^2}} \approx 9.61\,\text{sec}$$

5. 134 **m.** The electron in its own reference frame sees the *accelerator* moving toward it at $0.999c$, and sees a contracted length:

$$L = \frac{L_p}{\gamma} = 3\,\text{km} \cdot \sqrt{1-\frac{0.999^2c^2}{c^2}} = 3\,\text{km} \cdot \sqrt{1-0.999^2} = 0.134\,\text{km} = 134\,\text{m}$$

6. ellipsoid. The sphere is length contracted only along its direction of motion, *i.e.,* only along one axis. Squishing a sphere along one axis makes an ellipsoid.

7. The earth's clock. Less time will have passed in your reference frame, since you are moving relative to the earth. The earth's clock will have registered more time elapsed than yours.

2.9 Solutions to Problems

1. 2.3×10^{13} **J,** 2.56×10^{-4} **kg**. First part: relativistic kinetic energy is given by:

$$\text{KE} = (\gamma - 1)\, mc^2$$

First, we'll calculate γ based on the given velocity:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.75c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.75^2}} = 1.51$$

Next, we'll calculate the mc^2 bit:

$$mc^2 = \left(5\times10^{-4}\,\text{kg}\right)\left(3\times10^{8}\,\text{m/s}\right)^2 = 4.5\times10^{13}\,\text{kg}\cdot\text{m}^2/2^2 = 4.5\times10^{13}\,\text{J}$$

Putting it all together:

$$\text{KE} = (\gamma-1)\, mc^2 = (1.51-1)\left(4.5\times10^{13}\,\text{J}\right) = 2.30\times10^{13}\,\text{J} = 23.0\,\text{TJ}$$

Second part: what rest mass is equivalent to this amount of kinetic energy? We just need to use the mass-energy equivalence formula:

$$\begin{aligned} E_R &= mc^2 = \text{KE} \\ \Longrightarrow m &= \frac{\text{KE}}{c^2} = \frac{(\gamma-1)\, mc^2}{c^2} \\ &= (\gamma-1)\, m = 0.51m \\ &= 2.56\times10^{-4}\,\text{kg} \end{aligned}$$

In other words, it takes fully half the mass of the bullet itself, completely converted to pure energy, to fire one round. Using more conventional propellants, that would mean 5760 kg ($\sim$ 6 tons) of TNT per round.

2. We'll run it both forwards and backwards:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{mv\cdot v}{2} = \frac{mv\cdot v}{2}\frac{m}{m} = \frac{mv\cdot mv}{2m} = \frac{p\cdot p}{2m} = \frac{p^2}{2m}$$

Or, since you know the answer you want ...

$$\frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{m^2v^2}{2m} = \frac{mv^2}{2} = \frac{1}{2}mv^2$$

3. 4.08 **MeV for the muon,** 29.6 **MeV for the antineutrino.** This one is a bit lengthier than most of the others! Before the collision, we have only the pion, and since it is at rest, it has zero momentum and zero kinetic energy. After it decays, we have a muon and an antineutrino created and speed off in opposite directions (to conserve momentum). Both total energy - including rest energy - and momentum must be conserved before and after the collision.

First, conservation of momentum. Before the decay, since the pion is at rest, we have zero momentum. Therefore, afterward, the muon and antineutrino must have equal and opposite momenta. This means we can essentially treat this as a one-dimensional problem, and not bother with vectors. A consolation prize of sorts.

$$\begin{aligned}
\text{initial momentum} &= \text{final momentum} \\
p_\pi &= p_\mu + p_\nu \\
0 &= p_\mu + p_\nu \\
\Longrightarrow p_\nu &= -p_\mu = -\gamma_\mu m_\mu v_\mu
\end{aligned}$$

For the last step, we made use of the fact that relativistic momentum is $p = \gamma m v$. Now we can also write down conservation of energy. Before the decay, we have only the rest energy of the pion. Afterward, we have the energy of both the muon and antineutrino. The muon has both kinetic energy and rest energy, and we can write its total kinetic energy in terms of γ and its rest mass, $E = \gamma m c^2$. The antineutrino has negligible mass, and therefore no kinetic energy, but we can still assign it a total energy based on its momentum, $E = pc$.

$$\begin{aligned}
\text{initial energy} &= \text{final energy} \\
E_\pi &= E_\mu + E_\nu \\
m_\pi c^2 &= \gamma_\mu m_\mu c^2 + p_\nu c \\
m_\pi &= \gamma_\mu m_\mu + \frac{p_\nu}{c}
\end{aligned}$$

Now we can combine these two conservation results and try to solve for the velocity of the muon:

$$\begin{aligned}
m_\pi &= \gamma_\mu m_\mu + \frac{p_\nu}{c} \\
m_\pi &= \gamma_\mu m_\mu - \gamma_\mu m_\mu \frac{v_\mu}{c} \\
\frac{m_\pi}{m_\mu} &= \gamma_\mu - \gamma_\mu \frac{v_\mu}{c} \\
\frac{m_\pi}{m_\mu} &= \gamma \left[1 - \frac{v_\mu}{c}\right]
\end{aligned}$$

We will need to massage this quite a bit more to solve for v_μ ...

$$
\begin{aligned}
\frac{m_\pi}{m_\mu} &= \gamma\left[1-\frac{v_\mu}{c}\right] = \frac{1-\frac{v_\mu}{c}}{\sqrt{1-\frac{v_\mu^2}{c^2}}} \\
\left(\frac{m_\pi}{m_\mu}\right)^2 &= \frac{\left(1-\frac{v_\mu}{c}\right)^2}{1-\frac{v_\mu^2}{c^2}} \\
&= \frac{\left(1-\frac{v_\mu}{c}\right)^2}{\left(1-\frac{v_\mu}{c}\right)\left(1+\frac{v_\mu}{c}\right)} \\
&= \frac{\left(1-\frac{v_\mu}{c}\right)^{\not 2}}{\cancel{\left(1-\frac{v_\mu}{c}\right)}\left(1+\frac{v_\mu}{c}\right)} \\
&= \frac{1-\frac{v_\mu}{c}}{1+\frac{v_\mu}{c}}
\end{aligned}
$$

Now we're getting somewhere. Take what we have left, and solve it for v_μ ... we will leave that as an exercise to the reader, and quote only the result, using the given masses of the pion and muon:

$$
\frac{v_\mu}{c} = \frac{1-\left(\frac{m_\pi}{m_\mu}\right)^2}{1+\left(\frac{m_\pi}{m_\mu}\right)^2} \approx -0.270
$$

From here, we are home free. We can calculate γ_μ and the muon's kinetic energy first. It is convenient to remember that the electron mass is $511\,\text{keV/c}^2$.

$$
\begin{aligned}
\gamma_\mu &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.27c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.27^2}} \approx 1.0386 \\
\text{KE}_\mu &= (\gamma_\mu - 1)\, m_\mu c^2 = (1.0386-1)\,(207 m_{e^-})\, c^2 \\
&= 0.0386\left(207\cdot 511\,\text{keV}/c^2\right)c^2 \approx 4.08\times 10^6\,\text{eV} = 4.08\,\text{MeV}
\end{aligned}
$$

Finally, we can calculate the energy of the antineutrino as well:

$$
\begin{aligned}
E_\nu &= p_\nu c = -p_\mu c \\
&= -\gamma_\mu m_\mu v_\mu \\
&= -1.0386\cdot\left(207\cdot 5.11\,\text{keV}/c^2\right)\cdot(-0.270c) \\
&\approx 2.96\times 10^7\,\text{eV} = 29.6\,\text{eV}
\end{aligned}
$$

4. 1.96×10^{13} **m.** The 15 h set on the alarm clock in the spaceship is the proper time interval, Δt_p. Since the space ship is moving away from the earth at $v=0.77c$, an earthbound observer observes a longer dilated time interval, $\Delta t'$. Based on this longer time interval, the earthbound observer will measure that the space ship has covered a distance of $v\Delta t'$. So, first: we need to calculate γ, then the dilated time interval, then finally the distance measured by the earthbound observer.

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.77c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.77^2}} = 1.57 \\
\Delta t' &= \gamma \Delta t_p \\
&= 1.57 \cdot 15\,\text{h} = 1.57 \cdot 5.4 \times 10^4\,\text{s} \approx 8.48 \times 10^4\,\text{s} \\
d' &= v\Delta t' \\
&= 0.77c \cdot 8.48 \times 10^4\,\text{s} = 0.77 \cdot 3 \times 10^8\,\text{m/s} \cdot 8.48 \times 10^4\,\text{s} \\
d' &\approx 1.96 \times 10^{13}\,\text{m}
\end{aligned}$$

5. 1.31×10^{-7} **s, 38.4 m, 7.64 m** The π meson's lifetime in its own frame is the proper time interval, $\Delta t_p = 2.6 \times 10^{-8}$ s. An earthbound observer measures a longer dilated time interval $\Delta t'$. To calculate it, we need only calculate γ for the velocity given, $v_\pi = 0.98c$.

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.98c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.98^2}} = 5.03 \\
\Delta t' &= \gamma \Delta t_p \\
&= 5.03\left(2.6 \times 10^{-8}\,\text{s}\right) \\
&\approx 1.31 \times 10^{-7}\,\text{s}
\end{aligned}$$

The distance the π meson travels in the earthbound observer's reference frame, d' is the π meson's velocity multiplied by the time interval measured by the earthbound observer. We don't need to worry about whether the velocity is measured in the π meson's or the observer's frame - since it is a relative velocity, it is the same either way.

$$d' = \gamma v_\pi \Delta t_p = v_\pi \Delta t' = (0.98c)\cdot\left(1.31 \times 10^{-7}\,\text{s}\right) = \left(0.98 \cdot 3 \times 10^8\,\text{m/s}\right)\cdot\left(1.31 \times 10^{-7}\,\text{s}\right) \approx 38.4\,\text{m}$$

Without time dilation, the distance traveled would just be the proper lifetime multiplied by the meson's velocity:

$$d = v_\pi \Delta t_p = (0.98c)\cdot\left(2.6 \times 10^{-8}\,\text{s}\right) = \left(0.98 \cdot 3 \times 10^8\,\text{m/s}\right)\cdot\left(2.6 \times 10^{-8}\,\text{s}\right) \approx 7.64\,\text{m}$$

6. No. There is no relative speed between you and your cabin, since you are in the same reference frame. You and your bed will remain at the same lengths relative to each other.

7. 8.42 s. The time interval in the probe's reference frame is the proper one Δt_p ... which makes sense, since the antenna is part of the probe itself! The probe and antenna are moving relative to the earth, and therefore the earthbound observer measures a longer, dilated time interval $\Delta t'$:

$$\begin{aligned}
\text{probe} &= \Delta t_p \\
\text{earth} &= \Delta t' \\
\Delta t' &= \gamma \Delta t_p
\end{aligned}$$

As usual, we first need to calculate γ. No problem, given the probe's velocity of $0.88c$ relative to earth:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.88c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.88^2}} = 2.11$$

The proper time interval for one revolution Δt_p in the probe's reference frame is 4.0 s, so we can readily calculate the time interval observed by the earthbound observer:

$$\Delta t' = \gamma \Delta t_p = 2.11 \cdot (4.0\,\mathrm{s}) = 8.42\,\mathrm{s}$$

8. 24 m; 18 m; $0.661c$. Once again: if you are observing something in your own reference frame, there is no length contraction or time dilation. You always observe your own ship to be the same length. If your friend's ship is 24 m long, and yours is identical, you will measure it to be 24 m.

On the other hand, you are moving relative to his ship, so you would observe his ship to be length contracted, and measure a shorter length. Your friend, on the other hand, will observe *exactly the same thing* - he will see *your* ship contracted, by precisely the same amount. Your observation of his ship has to be the same as his observation of his ship - since you are only the two observers, and you both have the same *relative* velocity, you must observe the same length contraction. If he sees your ship as 18 m long, then you would also see his (identical) ship as 18 m long.

Given the relationship between the contracted and proper length, we can find the relative velocity easily. Your measurement of your own ship is the proper length L_p, while your measurement of your friend's ship is the contracted length L':

$$\begin{aligned}
L_p &= \gamma L' \\
\Longrightarrow \gamma &= \frac{L_p}{L'} = \frac{24}{18} = \frac{4}{3} \\
\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} &= \frac{4}{3} \\
1-\frac{v^2}{c^2} &= \frac{3^2}{4^2} = \frac{9}{16} \\
\frac{v^2}{c^2} &= 1-\frac{9}{16} = \frac{7}{16} \\
v &= \sqrt{\frac{7}{16}}c = \frac{\sqrt{7}}{4}c \approx 0.661c
\end{aligned}$$

9. $0.541c$. This is just a problem of relativistically adding velocities, if we can keep them all straight. Let the unprimed system denote velocities measured relative to the earth, and the primed system those measured relative to the enterprise. We have, then:

$$
\begin{aligned}
v_e = 0.900c &= \text{Enterprise relative to earth} \\
v_k = 0.700c &= \text{Klingon ship relative to earth} \\
v_k' = ? &= \text{Klingon ship, relative to Enterprise}
\end{aligned}
$$

Since the Enterprise is moving faster relative to the earth than the Klingon ship, that means that from the Enterprise's point of view, the Klingons are actually moving backwards toward them. If we plug what we know into the velocity addition formula ...

$$v_k = \frac{v_e + v_k'}{1 + \frac{v_e v_k'}{c^2}}$$

It takes a bit of algebra, but we can readily solve this for v_k':

$$v_k' = \frac{v_e - v_k}{1 - \frac{v_e v_k}{c^2}}$$

Not so surprisingly, what we have just done is to re-write the 'velocity addition formula' as a 'velocity subtraction formula.' It is just rearranging same formula (you can verify that both equations above are equivalent ...), but the second form is far more convenient for our present purposes.

We can find the velocity of the Klingon ship relative to the enterprise in terms of both ships' velocities relative to the earth. In the limit that both velocities are much smaller than c, we see that $v_k' \approx v_e - v_k = 0.200c$, just as we would expect from normal Newtonian physics. Since in this case, neither velocity is negligible compared to c, the actual v_k' will be significantly larger. At this point, we can just plug in the numbers we have and see:

$$
\begin{aligned}
v_k' &= \frac{v_e - v_k}{1 - \frac{v_e v_k}{c^2}} \\
&= \frac{0.900c - 0.700c}{1 - \frac{(0.900c)(0.700c)}{c^2}} \\
&= \frac{0.200c}{1 - (0.900)(0.700)} = \frac{0.200c}{0.37} \\
v_k' &\approx 0.541c
\end{aligned}
$$

So, as far as the crew of the Enterprise is concerned, they are overtaking the Klingon ship at a rate of $0.541c$.

10. $0.99995c$. Let the observer be in frame O'. In the reference frame of one of the particles, labeled O, the observer is traveling at $v = 0.99c$, and the second particle is traveling at $v_2' = 0.99c$ relative to the observer. We can then find the velocity of the second particle relative to the first, v_2, through velocity addition:

$$v_2 = \frac{v + v_1'}{1 + \frac{vv_1'}{c^2}} \tag{2.82}$$

$$= \frac{0.99c + 0.99c}{1 + \frac{(0.99c)(0.99c)}{c^2}} \tag{2.83}$$

$$= \frac{1.98c}{1 + 0.9801} \approx 0.99995c \tag{2.84}$$

This is an example of a problem where you need to make sure to use enough significant digits!

Part I

Relativity

2

Relativity

NEARLY all of the mechanical phenomena we observe around us every day have to do with objects moving at speeds rather small compared to the speed of light. The Newtonian mechanics you learned in previous courses handled these cases extraordinarily well. As it turns out, however, Newtonian mechanics breaks down completely when an object's speed is no longer negligible compared to the speed of light. Not only does Newtonian mechanics fail in this situation, it fails *spectacularly*, leading to a variety of paradoxical situations.

The resolution to these paradoxes is given by the theory of relativity, one of the most successful and accurate theories in all of physics, which we will introduce in this chapter. Nature is not always kind, however, and the consequences of relativity seem on their face to flout common sense and our view of the world around us. We are used to the notion that our position changes with time when we are in motion, but relativity implies that *passage of time itself* changes when we are in motion. Nevertheless, we shall see that relativity is an *inescapable* consequence of a few simple principles and experimental facts. Moreover, as it turns out, this new description of nature is critical for properly understanding electricity and magnetism, optics, and nuclear physics ... most of the rest of this course!

2.1 Frames of Reference

Describing motion properly usually requires us to choose a coordinate system, and an origin from which to measure position. Why this is so is more clear when we consider the difference between distance and displacement. For example, we can say that a person moves through a *displacement* of 10 meters, $\Delta x = 10\,\text{m}$, in a particular direction, *e.g.,* to the right. This does not describe the *position* of the person at all, only the *change* in that person's position over some time interval.

Describing position itself requires us to choose first a coordinate system (such as cartesian, spherical, *etc.*), and also an origin for this coordinate system to define our "zero" position. The essential difference is that displacement is independent of the cooordinate system we choose, but position is not. Without choosing a coordinate system, we can only say that the person has run 10 m in a certain time interval, moving from x_i to x_f.

Figure 2.1: *Albert EInstein (1879 – 1955) in 1921 at age 42.*[3] *Einstein is best known for his theory of relativity and its mass-energy equivalence,* $E = mc^2$.

As a concrete example, consider Fig. 2.2a. This illustrates a person moving 10 m to the right, which perfectly describes a *displacement* Δx. We will choose an $x-y$ cartesian coordinate system, which we will call O, with its origin at the person's starting point. In this system, we can describe the initial and final positions P_i^O and P_f^O in this coordinate system as $P_i^O = (0,0)$ and $P_f^O = (x_f, 0) = (\Delta x, 0)$. This is shown in Fig. 2.2b. The displacement is the same as it was without a coordinate system. In this chapter we will use the convention that superscripts refer to the coordinate system in which the quantity in question was measured.

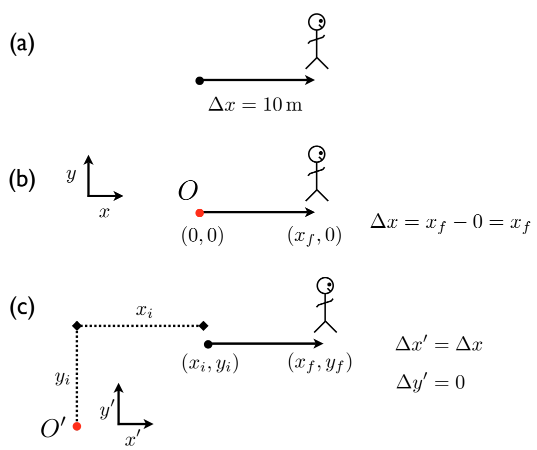


Figure 2.2: *Displacement is independent of the cooordinate system we choose, but position is not.* **(a)** *Without choosing a coordinate system, we can only say that the person has run 10 m in a certain time interval, moving from x_i to x_f.* **(b)** *If we choose an $x-y$ coordinate system O centered with its origin on the person's starting point x_i, we can describe the initial and final positions as $P_i^O = (x_i, 0)$ and $P_f^O = (x_f, 0)$. The displacement is the same.* **(c)** *If we choose a new coordinate system O', identical to O except shifted downward by y_i and to the left by x_i, now the initial and final positions are $P_i^{O'} = (x_i, y_i)$ and $P_f^{O'} = (x_f, y_i)$. Still, the displacement is the same.*

What happens if we if we instead choose a different coordinate system O', Fig. 2.2c, identical to O except that its origin is shifted downward by y_i and to the left by x_i? Now the initial and final positions of the person are $P_i^{O'} = (x_i, y_i)$ and $P_f^{O'} = (x_f, y_f)$. Still, the displacement Δx is the same, as you can easily verify. No matter whether we observe the person from the O or O' system, we would describe the same *displacement*, even though the actual positions are completely different.

In special relativity, this simple situation no longer holds - observers in different coordinate systems do *not* necessarily describe even the same displacement, much less the same position. Fortunately, the corrections of special relativity to the Newtonian mechanics you have already learned are only appreciable at very high velocities (non-negligible compared to the speed of light), and for most every day situations our usual intuition is still valid.

In any case, particularly those cases where relativistic effects are important, it is crucially important that we specify in which coordinate system quantities have been measured. We will continue to do this with a superscript of some sort to specify the coordinate system, and a subscript of some sort to further describe what is being measured *within* that system. When we only have two frames, like the example above, we will often just use a prime ($'$) to tell them apart. In the previous example, this means we would use P_f' instead of $P_f^{O'}$, and just P_i instead of P_i^O. It seems pedantic now, but careful bookkeeping is the only thing saving us from terrible confusion later!

Coordinate system notation examples:

$x^O_{\text{final}} = x^O_f$ final x position of an object measured in the O coordinate system

$v^{O'}_{\text{car}} \equiv v'_{\text{car}}$ velocity of a car measured in the O' coordinate system

$P^{O'}_f \equiv P'_f = (x'_f, y'_i)$ final position of an object measured in the O' coordinate system

Finally, a word on terminology. In relativity, it is common to use "reference frame" in place of "coordinate system," to make explicit the fact that our coordinate system and origin are the point of reference from which we measure physical quantities. We will use both phrasings interchangeably from here on out.

2.2 Moving Frames of Reference

What about one observer measuring in a coordinate system *moving* at constant velocity relative to another? For example, take Fig. 2.3. A girl holding balloons is standing on the ground, and a bully on a skateboard throws a dart at her balloons. The bully is moving at a velocity v_{bully} relative to the girl, and he throws the dart at a velocity v_{dart} relative to himself. What is the dart's speed relative to the girl?

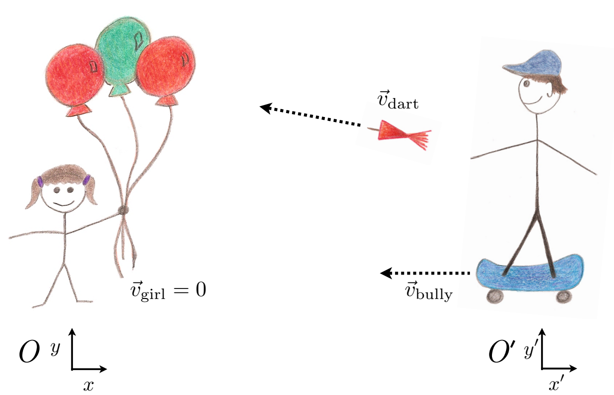


Figure 2.3: *A girl holding balloons is standing on the ground, at rest in reference frame O ($v^O_{girl} = 0$). Meanwhile a bully on a skateboard throws a dart at her balloons. The bully is moving at a velocity v^O_{bully} relative to the girl's reference frame, and he throws the dart at a velocity $v^{O'}_{dart}$ relative to himself (the O' frame). What is the dart's speed as measured by the girl? Drawings by C. LeClair*

First of all, we have to be more explicit about specifying which quantity is measured in which frame. The velocity of the bully on the skateboard is measured relative to the girl standing on the ground, in the O system, so we write v^O_{bully}. When we talk about the dart, however, things are a bit less clear. The bully on the skateboard would say that the velocity of the dart is $v^{O'}_{\text{dart}}$, since he would measure its velocity relative to *himself* in the O' frame. The girl would measure the velocity of the dart relative to *herself* in the O frame, v^O_{dart}. Clearly, $v^{O'}_{\text{dart}} \neq v^O_{\text{dart}}$ – in principle, the two cannot agree on what the velocity of the dart is! Of course, that is a bit of an exaggeration. In this simple everyday case, relative motion is fairly easy to understand, and we can intuitively see exactly what is happening. Our intuition will start to fail us shortly, however, so it is best we proceed carefully.

Explicitly labeling the velocity with the reference frame in which it is measured helps keep everything precise, and helps us find a way out of this conundrum. It may seem like baggage now, but ambiguity would cost us dearly later. Just to summarize, here is how we will keep the velocities straight:

$$\begin{aligned}
v^{O}_{\text{bully}} &= \text{velocity of bully measured from the ground} \equiv v_{\text{bully}} \\
v^{O'}_{\text{dart}} &= \text{velocity of dart measured from the skateboard} \equiv v'_{\text{dart}} \\
v^{O}_{\text{dart}} &= \text{velocity of the dart measured by the girl} \equiv v_{\text{dart}}
\end{aligned}$$

Whenever we are only dealing with two different coordinate systems, we will trim down the notation a bit. We will just call one system the "primed" system, and add a $'$ superscript to all quantities, and leave the other one as the "unprimed" system, and drop the 'O'. Which one we call "primed" and which one is "unprimed" makes no difference, it is after all just notation and bookkeeping.

What does the girl on the ground, in the O system really observe? Intuitively, we expect this her to see the dart moving at a velocity v_{dart} which is that of the dart relative to the skateboard *plus* that of skateboard relative to the ground:

$$\begin{aligned}
v_{\text{dart}} &= v'_{\text{dart}} + v_{\text{bully}} \qquad (2.1)\\
\text{velocity of the dart seen by the girl} &= \text{velocity of dart relative to skateboard} + \text{velocity of skateboard relative to girl}
\end{aligned}$$

The bully, in the O' system (who threw the dart in the first place), just sees v'_{dart}. Just to be concrete, let's say that the bully on the skateboard moves with $v_{\text{bully}} = 3\,\text{m/s}$, and he throws the dart with $v'_{\text{dart}} = 2\,\text{m/s}$. Then the girl sees the dart coming at her balloons at $5\,\text{m/s}$.

2.2.1 Lack of a Preferred Reference Frame

Even in the simple example above, velocity depends on your frame of reference. This simple example is completely arbitrary in a sense, though, and implies much more about relative motion. If these two observers can't agree on the velocity of the dart, as measured in their own reference frames, who is to say what the absolute reference frame should be? After all, isn't the ground itself moving due to the rotation of the earth about the sun? And isn't the sun moving relative to the center of the galaxy? Nothing is absolutely at rest, we cannot pick any special frame of reference to define absolute unique velocities.

Still, we might think be tempted to think that there is some sort of reference frame we are forgetting, one that is truly at rest. For instance, what about empty space itself? Can we define absolute coordinates and absolute motion relative to specific points in space? This is a tempting thought, particularly if we make an analogy with sound waves.

As you know from Mechanics, sound is really nothing more than (longitudinal) oscillations of matter, a sort of density wave in a material. We will find out in later Chapters that light is also a wave. If they are both waves, perhaps the nature of sound can help explain the nature of light? Sound can be propagated through matter, or even through air, but it requires a medium to be transmitted – no sound is transmitted in a vacuum. Could we view light as the vibrations of space itself, or of some all-pervasive "fluid" filling all of space? Certainly light waves also need a medium in which to propagate, so the reasoning goes. This all-pervasive fluid would provide a "background" frame of reference, allowing us to measure absolute velocity, somewhat like measuring the velocity of a boat by how fast water moves past its side.

Indeed, this was a very attractive viewpoint through the early 20$^{\text{th}}$ century, and the so-called "luminiferous æther" was the term used to describe the all-pervasive medium for the propagation of light. It fact, is a *testable* idea – this is a crucial point which makes the idea a true scientific theory. How do we test it? If space itself has a background medium within which light propagates, then we should be able to measure the velocity of the earth through this medium as it revolves around the sun. The earth moving through the æther fluid would experience some "drag," again just like a boat moving through water.

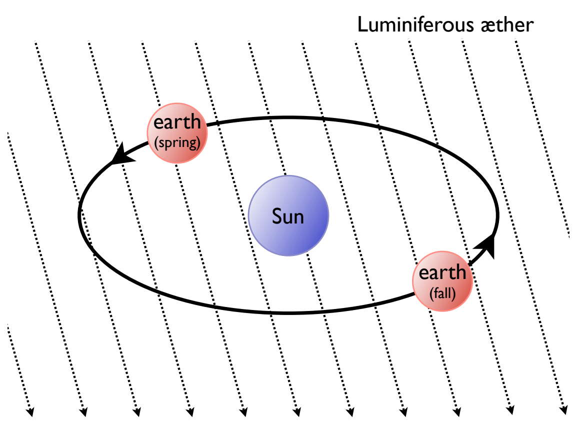


Figure 2.4: *If there were a "luminiferous æther" which light propagates on pervading all space, the earth's revolution around the sun would experience a drag force depending on the season – the "æther wind." At some times, this æther wind would augment the speed of light, and at others it would diminish it. Analyzing the speed of light in different directions at different times of year should allow one to extract the æther velocity. Instead, experiments proved that there is no æther.*

Unfortunately, this idea just isn't right. It has been disproven countless times by experiments, and replaced by the far more successful theory of relativity. Light waves are not like sound waves. Light is in one sense a wave, but a more modern viewpoint treats light as a stream of particles that have a "wave-like nature."[i] Particles do not need a medium to travel, and therefore neither does light. *There is no æther, and there is no preferred frame of reference. All motion is relative.*

[i] We will explore this dual nature of light in Chapter 10.

2.2.2 Relative Motion

Fine. There is no preferred reference frame or coordinate system, and all motion is relative. So what? The example of Fig. 2.3 was plainly understandable. It is disturbingly easy to come up with examples which are *not* so plainly understandable, however, which is one motivation for the theory of relativity in the first place. Consider the two rockets in empty space traveling toward each other in Fig. 2.5, separated by a distance Δx. The pilot of rocket 1 might say he or she is traveling at a speed v_1 in his or her own reference frame (O), and the pilot of rocket 2 may claim he or she is traveling at a speed v_2' in their own O'. Without specifying what point they are measuring their velocity relative to, can we say who is moving at what speed?

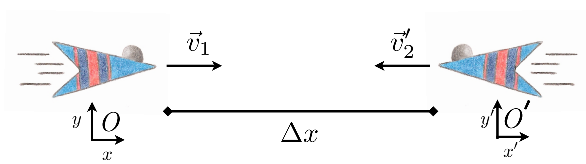


Figure 2.5: *Two identical rockets, separated by a distance Δx, are moving toward each other in empty space. Rocket 1 sees rocket 2 cover a distance $(v_1+v_2)t$ in a time t, as if rocket 2 is heading toward it. On the other hand, rocket 2 sees rocket 1 cover the same distance $(v_1+v_2)t$ in a time t, as if rocket 1 is heading toward it! Both of them cannot be correct. Without an external reference frame, it is impossible to say who is moving, and at what speed. Drawings by C. LeClair*

We have to imagine that we are deep in empty space, with nothing around either rocket to provide a landmark or point of reference. The occupants of rocket 1 would feel as though they are sitting still, and observe rocket 2 coming toward them, covering a distance $(v_1+v_2)\Delta t$ in a time interval Δt. The occupants of rocket 2, on the other hand, would think *they* are sitting still, and would observe rocket 1 coming toward *them*, also covering a distance $(v_1+v_2)\Delta t$ in a time interval Δt.

Without any external reference point, or an absolute frame of reference, *not only can we not say with what speed each rocket is moving, we can't even say* **who** *is moving!* If we decide that rocket 1 is our reference frame, then it is sitting still, and rocket 2 is moving toward it. But we could just as easily pick rocket 2 as our reference frame. Specifying who is moving, and with what speed, is meaningless without a proper origin or frame of reference.

Has anything really changed physically? No. An analogy of sorts is to think about driving along side other cars on the highway, keeping pace with them. You might report your speed as 60 mi/hr. Relative to what? Clearly, in this case it is implied that the ground beneath you provides a reference frame, and you are talking about your velocity relative to the earth. You wouldn't say you are traveling at 60 mi/hr relative to the other cars (we hope) – your speed relative to the other cars is zero if you are staying along side them. Indeed, if you look out your window, the cars next to you appear to be sitting still. This is only true at constant velocity – we can easily detect accelerated motion, or an accelerated frame of reference due to the force experienced. This is the realm of *general* relativity, Sect. 2.5.

In the end, one of the fundamental principles of special relativity is that *the description of relative constant velocity does not matter, so far as the laws of physics are concerned.* The laws of physics apply the same way to all objects in uniform (non-accelerated) motion, no matter how we measure the velocity. We *cannot* devise an experiment to measure uniform motion absolutely, only relative to a specific chosen frame of reference. More succinctly:

> **Principle of relativity:**
> All laws of nature are the same in all uniformly moving (non-accelerating) frames of reference. No frame is preferred or special.

As another simple example, Fig. 2.6, consider Joe and Moe running at different (constant) speeds in the same direction, initially separated by a distance d_o. Without specifying any particular common frame of reference, we must be able to describe their relative motion, or how the separation between Joe and Moe changes with time, even though we can't speak of their absolute velocities in any sense.

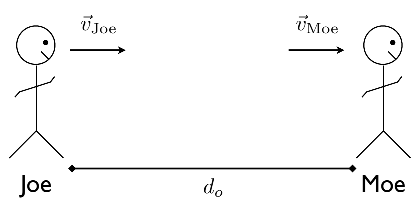


Figure 2.6: *Joe and Moe running at different speeds in the same direction. Both Joe and Moe measure the same relative velocity with respect to each other.*

Let's say we arbitrarily choose Joe's position at $t=0$ as our reference point. It is easy then to write down what Joe and Moe's positions are at any later time interval Δt:

$$x_{\text{Joe}} = v_{\text{Joe}}\Delta t \qquad\qquad x_{\text{Moe}} = d_0 + v_{\text{Moe}}\Delta t \tag{2.2}$$

We can straightforwardly write down the separation between them (their relative displacement) as well:

$$\Delta x_{\text{Moe-Joe}} = x_{\text{Moe}} - x_{\text{Joe}} = d_0 + v_{\text{Moe}}\Delta t - v_{\text{Joe}}\Delta t = d_0 + (v_{\text{Moe}} - v_{\text{Joe}})\,\Delta t \tag{2.3}$$

Sure enough, their relative displacement only depends on their *relative* velocity, $v_{\text{Moe}}-v_{\text{Joe}}$. Further, both Joe and Moe would agree with this, since we could arbitrarily choose *Moe's* position at $t=0$ as our reference point, and *we would end up with the same answer.* Since there is nothing special about either position, we can choose *any* point whatsoever as a reference, and wind up with the same result. We end up with the same physics no matter what reference point we choose, which one we choose is all a matter of convenience in the end.

Choosing a coordinate system:

1. Choose an origin. This may coincide with a special point or object given in the problem - for instance, right at an observer's position, or halfway between two observers. Make it convenient!
2. Choose a set of axes, such as rectangular or polar. The simplest are usually rectangular or *Cartesian* x-y-z, though your choice should fit the symmetry of the problem given - if your problem has circular symmetry, rectangular coordinates may make life difficult.
3. Align the axes. Again, make it convenient - for instance, align your x axis along a line connecting two special points in the problem. Sometimes a thoughtful but less obvious choice may save you a lot of math!
4. Choose which directions are positive and negative. This choice is arbitrary, in the end, so choose the least confusing convention.

This seems simple enough, but if we think about this a bit longer, more problems arise. Who measures the initial separation d_0, Joe or Moe? Who keeps track of the elapsed time Δt? Does it matter at all, can the measurement of distance or time be affected by relative motion? Of course, the answer is an awkward 'yes' or we would not dwell on this point. If we delve deeper on the problem of relative motion, we come to the inescapable conclusion that not only is velocity a relative concept, our notions of distance and time are relative as well, and depend on the relative motion of the observer. In order to properly understand these deeper ramifications, however, we need to perform a few more thought experiments.

2.2.3 Invariance of the Speed of Light

Already, relativity has forced us to accept some rather non-intuitive facts. This is only the beginning! A more fundamental and far-reaching principle of relativity is that *the speed of light is a constant, independent of the observer.* No matter how we measure it, no matter what our motion is relative to the source of the light, we will always measure its velocity to be the same value, c. Light does not obey the principle of relative motion!

Speed of Light in a Vacuum:

$$c = 3 \times 10^8 \text{ m/s}$$

There is a relatively simple way to experimentally demonstrate that this seems to be true, depicted in Fig. 2.7. The earth itself is in constant motion in its orbit around the sun, moving at $\sim 3 \times 10^4$ m/s measured relative to distant stars (this in itself is a measurable quantity). Imagine now that we carefully set up three lasers, each oriented in a different direction relative to earth's orbital velocity – one parallel (A), one antiparallel (B), and one at a right angle (C). We will further set up each laser to emit short pulses of light, and carefully measure the time between pulses. In this way, we can determine the speed of the light coming out of each laser.

Based on simple Newtonian mechanics and velocity addition, we would expect to measure a slightly different velocity for each laser. In case A, we would expect the Earth's velocity to *add* to that of light, $v_{\rm A}=v_{\rm light}+v_{\rm orbit}$, while in case B, it should *subtract*, $v_{\rm B}=v_{\rm light}-v_{\rm orbit}$. In case C, we have to add vectors, $\vec{\mathbf{v}}_{\rm C}=\vec{\mathbf{v}}_{\rm light}+\vec{\mathbf{v}}_{\rm orbit}$, but the idea is the same.

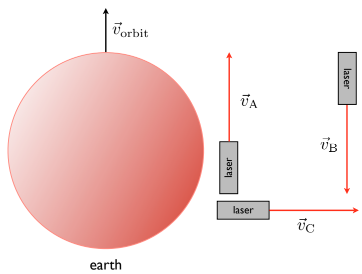


Figure 2.7: *If the velocity of light obeys Newtonian mechanics, then measuring the speed of light from a laser pointed in different directions compared to Earth's orbital velocity should yield different results. In case A, we would expect the Earth's velocity to add to that of light, $v_A=v_{light}+v_{orbit}$, while in case B, it should subtract, $v_B=v_{light}-v_{orbit}$. In case C, we have to add vectors, but the idea is the same. The effect should be small ($\sim$0.01 %), but easily measurable. No effect is observed, the speed of light is always the same value c.*

The effect should be small ($\sim 0.01\,\%$), but easily measurable. No effect is observed, the speed of light is *always* the same value c. This experiment has been performed with increasingly fantastic precision over the last 100 years[4], and no matter what direction we shine the light, we always measure the same speed! (The current best limit[4] on the constancy of the speed of light is about 1 part in 10^{16}.) One straightforward result of this experiment is that the idea of an Æther is clearly not right, as we discussed above. There are much more far-reaching consequences, which we must consider carefully. First, let us re-iterate this idea more formally:

The speed of light is invariant

The speed of light in free space is *independent* of the motion of the source or observer. It is an invariant constant.

This is not just idle speculation or theory, it has been confirmed again and again by careful experiments. These experiments have established, for instance, that the speed of light[ii] does not depend on the wavelength of light, on the motion of the light source, or the motion of the observer. As examples, lack of a wavelength dependence can be strongly ruled out by astronomical observations of gamma ray bursts (to better than 1 part in 10^{15}), while binary pulsars can rule out any dependence on source motion. The lack of a dependence on observer motion was disproved along with the æther (Sec. 2.2.1), which also proved that light requires no medium for propagation.

As an example of this, we turn again to Joe and Moe (Fig. 2.8). Joe is in a rocket (O'), traveling at 90% of the speed of light ($v=0.9c$), while Moe is on the ground (O) with a flashlight. Moe shines the flashlight parallel to Joe's trajectory in the rocket. On first sight, we would think that Moe would measure the speed of the light leaving the flashlight as c, while Joe would measure $v=c-0.9c=0.1c$.

[ii] Throughout this chapter, we refer to the speed of light in a *vacuum*.

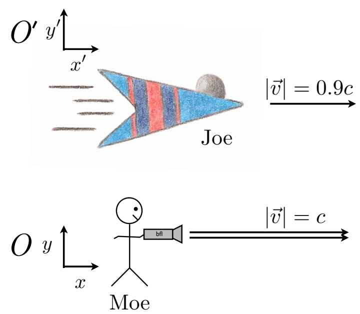


Figure 2.8: *Joe is traveling on a rocket at $|\vec{\mathbf{v}}| = 0.9c$, while Moe on the ground shines a flashlight parallel to Joe's path. Both Joe and Moe observe the light from the flashlight to travel at $|\vec{\mathbf{v}}| = c$ – contrary to our intuition from Newtonian Mechanics.*

Both Joe and Moe measure *the same speed of light c*, despite their relative motion! What if we gave Joe the flashlight inside the rocket? No difference, both Joe and Moe measure the speed of the light to be c. Think back to our example of relative motion in Fig. 2.3. It doesn't seem to make sense that light behaves differently, but that is how it is. As we shall see shortly, our normal intuitions about everyday phenomena at relatively low velocities is no longer valid when velocities approach that of light. The physics is fundamentally different, and our Newtonian instincts are in the end only a low-speed approximation to reality. By the end of this chapter, though, we will be armed with the proper tools to analyze this situation correctly from both viewpoints.

2.2.4 Principles of special relativity

From our discussions so far, relativity when non-accelerating (inertial) reference frames are considered has two basic principles which underpin the entire theory:

Principles of special relativity

1. **Special principle of relativity:** Laws of physics look the same in all inertial (non-accelerating) reference frames. There are no preferred inertial frames of reference.
2. **Invariance of c:** The speed of light in a vacuum is a universal constant, c, independent of the motion of the source or observer.

This theory of relativity restricted to inertial reference frames is known as the *special theory of relativity*, while the more general theory of relativity which also handles accelerated reference frames is simply known as the *general theory of relativity* (which we will touch on in Sect. 2.5).

The second postulate of special relativity - the invariance of the speed of light - can actually be considered as a consequence of the first according to some mathematical formulations of special relativity. That is, the constancy of the speed of light is *required* in order to make the first postulate true. We will continue to hold it up as a second primary postulate of special relativity, however, as some of the more non-intuitive consequences of special relativity are (in our view) more readily apparent when one keeps this fact in mind.

The first principle of relativity essentially states that all physical laws should be exactly the same in any vehicle moving at constant velocity as they are in a vehicle at rest. As a consequence, at constant velocity we are incapable of determining absolute speed or direction of travel, we are only able to describe motion relative to some other object. This idea does not extend to accelerated reference frames, however. When acceleration is present, we feel fictitious forces that betray changes in velocity that would not be present if we were at rest. All experiments to date agree with this first principle: physics is the same in all inertial frames, and no particular inertial frame is special.

The principle of relativity is by itself more general than it appears. The principle of relativity describes a symmetry in the laws of nature, that the laws must look the same to one observer as they do to another. In physics, any symmetry in nature also implies a *conservation law*, such as conservation of energy or conservation of momentum. If the symmetry is in time, such that two observers at different times must observe the same laws of nature, then it is energy that must be conserved. If two observers at different physical locations must observe the same laws of physics (*i.e.,* the laws of physics are independent of spatial translation), it is linear momentum that must be conserved. The relativity principles imply deep conservation laws about space and time that make testable predictions – predictions which must be in accordance with experimental observations in order to be taken seriously. Relativity is not just a principle physicists have proposed, it is a postulate that was in the *required* in order to describe nature as we see it. The consequences of these postulates will be examined presently.

2.3 Consequences of Relativity

We have our principles laid forth, and their rationale clearly provided by our series of thought experiments. All experimental results to date are on the side of these two principles. So, enough already, what are the consequences of these two innocent-looking principles? In some sense you may ask what the big deal is. How often do we deal with objects traveling close to the speed of light? The glib answer is "plenty" – we use light itself pretty much constantly! The more formal answer is that the invariance of the speed of light and the principles of relativity force us to modify our very notions of perception and reality. It is not just fiddling with a few equations to handle special high-velocity cases, we must reevaluate some of our deepest intuitions and physical models. Many books have been written about the implications that relativity has had on philosophy in fact ... however, we will stick to physics.

2.3.1 Lack of Simultaneity

The speed of light is more than just a constant, it is a sort of 'cosmic speed limit' – no object can travel faster than the speed of light, and no information can be transmitted faster than the speed of light. If either were possible, causality would be violated: in some reference frame, information could be received before it had been sent, so the ordering of cause-effect relationships would be reversed. It is a bit much to go into, but the point is this: the speed of light is really a *speed limit*, because if it were not, either cause and effect would not have their usual meaning, or sending information backward in time would be possible. Neither is an easily-stomached possibility. A more readily grasped consequence of all of this is that we must give up on the notion of two events being simultaneous in any absolute sense – whether events are viewed as simultaneous depends on ones reference frame! It should seem odd that a seemingly simple principle like the speed of light being constant would muck things up so much, but in fact we can demonstrate that this *must* be true with a simple thought experiment.

Imagine that Joe is flying in a spaceship at $v=0.9c$ (we will call his reference frame O'), and Moe is observing him on the ground (in frame O), as shown in Fig. 2.9. Joe, sitting precisely in the middle of the ship, turns on a light at time $t=0$ also in the middle of the spaceship. A small amount of time Δt later, Joe's superhuman eyes observe the rays of light reach the front and the back of the spaceship simultaneously. So far this makes sense – if the light is exactly in the middle of the ship, light rays from the bulb should reach the front and back at the same time.

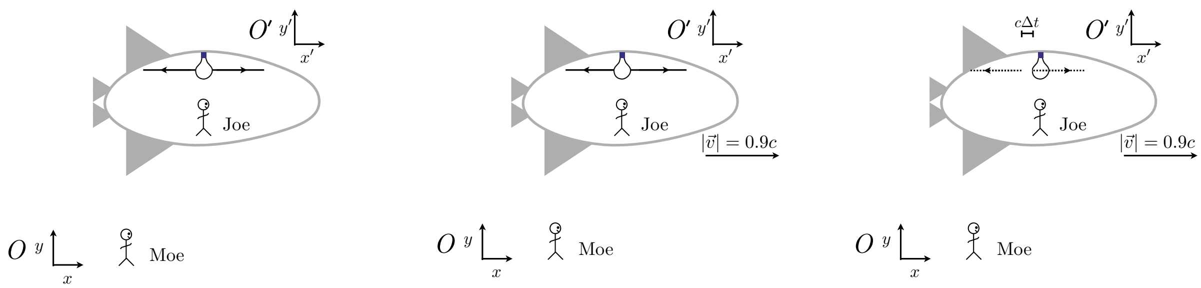


Figure 2.9: left: *Joe is traveling in a (transparent) rocket ship, and turns on a light bulb in the exact center of the rocket.* **middle** *A short time Δt later, in his frame O' Joe sees the light rays hit both sides of the ship at the same time.* **right:** *Moe on the ground observes Joe in his rocket moving at $v=0.9c$. From his frame O, a time Δt after the light leaves the bulb, the ship moves forward by an amount $c\Delta t$ but the light rays do not. Moe sees the light hit the back of the ship first – Moe and Joe cannot agree on the simultaneity of events.*

Now, what will Moe on the ground see?[iii] From his frame O, Moe sees the light emitted from the bulb at $t=0$. The ship and the light bulb are both moving relative to Moe at $v=0.9c$, but we have to be careful. First, Moe observes the same speed of light as Joe, even though the bulb is moving. Once the light bulb is turned on, the first light leaves the bulb at $v=c$ and diverges radially outward from its point of creation. As this first light leaves the bulb, however, *the ship is still moving forward.* The front of the ship moves away from the point of the light's creation, while the back moves *toward* it.

> **Consequence of an invariant speed of light:**
> Events that are simultaneous in one reference frame are **not** simultaneous in another reference frame moving relative to it – and no particular frame is preferred. Simultaneity is not an absolute concept.

In some sense, once the light is created, it isn't really in either reference frame – it is traveling at $v=c$ no matter who observes it. The ship moved forward, but the point at which the light was

[iii] You might think nothing, as we neglected to mention that Joe's ship is transparent.

created did not. We attempt to depict this in Fig. 2.9, where from Moe's point of view, after a time Δt the light rays emitted from the bulb seem to have emanated from a point somewhat behind the rocket – a distance $c\Delta t$ behind it. Thus, after some time, Moe sees the light hit *the back* of the ship first! Joe and Moe seem to observe different events, and they can not agree on whether the light hits the front and the back of the ship simultaneously. Events which are simultaneous in Joe's reference frame are **not** in Moe's reference frame, moving relative to him. Think about how this plays out from Joe and Moe's reference frames carefully. It is strange and non-intuitive, but if we accept the speed of light as invariant, the conclusion is inevitable.

2.3.2 Time Dilation

Now we have already seen that the constancy of the speed of light has some rather unintuitive and bizarre consequences. For better or worse, it gets stranger! Not only is our comfortable notion of simultaneity sacrificed, our concept of the passage of time itself must be "corrected." Just as the notion of two events being simultaneous or not depend on one's frame of reference, the relative passage of time also depends on the frame of reference in which the measurement of time is made. Again, to illustrate this, we will perform a thought experiment.

First, we need a way to measure the passage of time. The constancy of the speed of light fortunately provides us with a straightforward – if not necessarily experimentally simple – manner in which to do this. We will measure the passage of time by bouncing light pulses between two parallel mirrors, carefully placed a distance d apart. Since we know the speed of light is an immutable constant, so long as the space between the mirrors remains fixed at d, the round-trip time Δt for a pulse of light to start at one mirror, bounce off the second, and return to the first will be a constant. The light pulse travels the distance d between the mirrors, and back again, at velocity c, so the time interval for a round trip is just $\Delta t = 2d/c$.

Now, let's imagine Joe is performing this experiment in a boxcar moving at velocity v relative to the ground, as shown in Fig. 2.10a. We will label Joe's own reference frame inside the boxcar as O', such that the boxcar moves in the x' direction. Both Joe, the mirrors, and the light source are stationary relative to one another, and the mirrors and light source have been carefully positioned a distance d apart such that the light pulses propagate vertically in the y' direction. In Joe's reference frame, he can measure the passage of time by measuring the number of round trips that an individual light pulse makes between the two mirrors. For one round trip, Joe would measure a time interval

$$\Delta t' = \frac{2d}{c} \tag{2.4}$$

So far so good. Since Joe is not moving relative to the mirrors, nothing unusual happens – assuming he has superhuman vision, he just sees the light pulses bouncing back and forth between the mirrors, Fig. 2.10a, straight up and down, and counts the number of round trips. Moe monitors this situation from the ground, in his own reference frame O. Thankfully, the boxcar is transparent, and Moe is able to see the light pulses and mirrors as well as the boxcar, moving at a velocity v from his point of view.

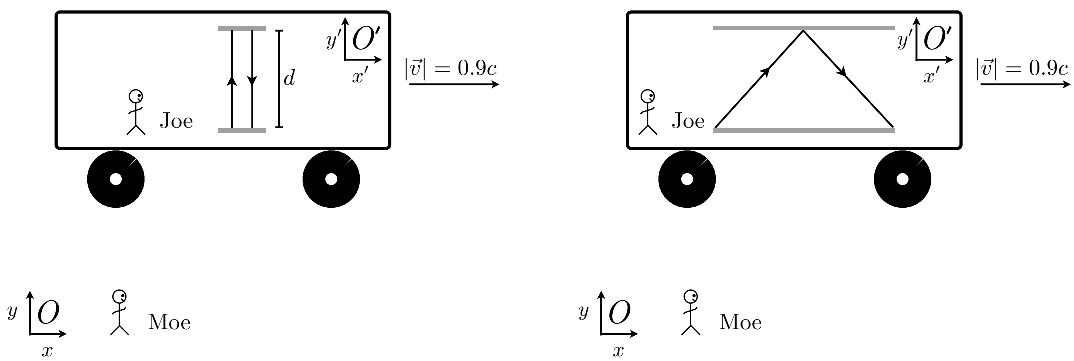


Figure 2.10: left: *Joe is traveling in a (transparent) boxcar, and he bounces laser beams between two mirrors inside the boxcar. Since the distance between the mirrors is known, and the speed of light is constant, he can measure time in this way. Joe measures the round trip time it takes the light to bounce from the bottom mirror, to the top, and back again.* **right:** *Moe observes the mirrors from the ground. From his frame O, the boxcar and mirrors are moving but the light is not. He therefore sees the light bouncing off of the mirrors at an angle. Using geometry and the constant speed of light, Moe also measures a round trip time interval, but since the path he observes for the light is different, he measures a different time interval than Joe.*

What does Moe see inside the boxcar? From his point of view, a light pulse is created at the bottom mirror while the whole assembly moves in the x direction – mirrors, light pulse, and all! Just like in the example of the light being flicked on in a space ship, the boxcar and mirrors have moved, but the point at which the light was created has not – Moe appears to see the light traveling at an angle. A light pulse is created at the bottom mirror, and it travels upward horizontally to reach the top mirror some time later, a bit further along the x axis. Rather than seeing the pulses going straight up and down, from Moe's point of view, they zig-zag sideways along the x axis, as shown in Fig. 2.10b.

So what? We know the speed of light is a constant, so both Joe and Moe must see the light pulses moving at a velocity c, even though they appear to be moving in along a different trajectory. If Moe also uses the light pulses' round trips to measure the passage of time, what time interval does he measure? The speed of light is constant, but the apparent distance covered by the light pulses is larger in Moe's case. Not only has the light traveled in the y direction a distance $2d$, over the course of one round trip it has also moved horizontally due to the motion of the boxcar. If the light has apparently traveled farther from Moe's point of view, and the speed of light is constant, then the apparent passage of time from Moe's point of view must also be greater!

Just how long does Moe observe the pulse round trip to be? Let us examine one half of a round trip, the passage of the light from the bottom mirror to the top. In that interval, from either reference frame, the light travels a vertical distance of d. From Joe's reference frame O', the light does not travel horizontally, so the entire distance covered is just d, and he measures the time interval $\frac{1}{2}\Delta t' = d/c$. From Moe's reference frame, the car has also travelled horizontally. Since he sees the car moving at a velocity v, he would say that in his time interval $\frac{1}{2}\Delta t$ for one half round trip, the car has moved forward by $\frac{1}{2}v\Delta t$. Thus, Moe would see the light cover a horizontal distance of $\frac{1}{2}v\Delta t$ and a vertical distance $\frac{1}{2}c\Delta t$, as shown in Fig. 2.11.

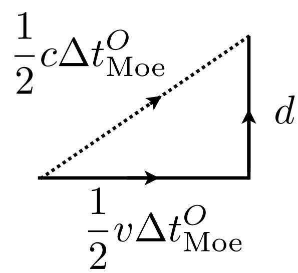


Figure 2.11: *Velocity addition for light pulses leading to time dilation. Within the boxcar (frame O', Joe observes the light pulses traveling purely vertically, covering a distance d. On the ground (frame O), Moe sees the light cover the same vertical distance, but also sees them move horizontally due the motion of the boxcar at velocity v in his reference frame. The total distance the light pulse travels, according to Moe, is then the Pythagorean sum of the horizontal and vertical distances.*

According to Moe, total distance that the light pulse covers in one half of a round trip is the Pythagorean sum of the horizontal and vertical distances:

$$(\text{distance observed by Moe})^2 = d^2 + \left(\frac{1}{2}v\Delta t\right)^2 \tag{2.5}$$

Further, he must also observe the speed of light to be c just as Joe does. If he measures the passage of time by counting the light pulses as Joe does, then he would say that after one half round trip, the light has covered this distance at a speed c, and would equate this with a time interval in his own reference frame Δt. Put another way, he would say that the distance covered by the light in one half round trip is just $\frac{1}{2}c\Delta t$, in which case we can rewrite the equation above:

$$\left(\frac{1}{2}c\Delta t\right)^2 = d^2 + \left(\frac{1}{2}v\Delta t\right)^2 = \left(\frac{1}{2}c\Delta t'\right)^2 + \left(\frac{1}{2}v\Delta t\right)^2 \tag{2.6}$$

Here we made use of the fact that we already know that $d=\frac{1}{2}c\Delta t'$ based on Joe's measeurement. Now we see that if the speed of light is indeed constant, *there is no way that the time intervals measured by Joe and Moe can be the same!* The pulse seems to take longer to make the trip from Moe's perspective, since it also has to travel sideways, not just up and down. Solely due to the constant and invariant speed of light, Joe and Moe must measure different time intervals, and Moe's must be the longer of the two. We can solve the equation above to find out just what time interval Moe measures:

$$\left(\frac{1}{2}c\Delta t\right)^2 = d^2 + \left(\frac{1}{2}v\Delta t\right)^2 \tag{2.7}$$

$$\frac{1}{4}c^2\left(\Delta t\right)^2 = d^2 + \frac{1}{4}v^2\left(\Delta t\right)^2 \tag{2.8}$$

$$\frac{1}{4}\left(\Delta t\right)^2\left(c^2 - v^2\right) = d^2 \tag{2.9}$$

$$\left(\Delta t\right)^2 = \frac{4d^2}{c^2 - v^2} \tag{2.10}$$

$$\Longrightarrow \Delta t = \frac{2d}{\sqrt{c^2 - v^2}} \tag{2.11}$$

The time interval that Joe measures is still just $\Delta t' = 2d/c$. If we factor this out of the expression above, we can relate the time intervals measured by the two observers:

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.12}$$

$$\Delta t_{\text{Moe}} = \Delta t'_{\text{Joe}}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \Delta t'_{\text{Joe}}\gamma \tag{2.13}$$

Here we defined a dimensionless quantity $\gamma = 1/\sqrt{1 - v^2/c^2}$ to simplify things a bit, we'll return to that shortly. So long as $v < c$, the time interval that Moe measures is always *larger* than the one Joe measures, by an amount which increases as the boxcar's velocity increases. This is a general result in fact: the time interval measured by an observer in motion is always longer than that measured by a stationary observer. Typically, we say that the moving observer measures a *dilated* time interval, hence this phenomena is often referred to as *time dilation.* The time dilation phenomena is symmetric – if Moe also had a clock on the ground, Joe would say that Moe's clock runs slow by precisely the same amount. It is only the relative motion that matters.

Time dilation

The time interval Δt between two events *at the same location* measured by an observer moving with respect to a clock is always *larger* than the time interval between the same two events measured by an observer stationary with respect to the clock. The 'proper' time Δt_p is that measured by the stationary observer.

$$\Delta t'_{\text{moving}} = \gamma\Delta t_{\text{stationary}} = \gamma\Delta t_p \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.14}$$

In other words, clocks run slow if you are moving with respect to them.

In the example above, it is Moe who is in a reference frame moving relative to our light 'clock' and Joe is the stationary observer. Therefore, Joe measures the 'proper' time interval, while Moe measures the dilated time interval. Incidentally, for discussions involving relativity, we basically assume that there is always a clock sitting at every possible point in space, constantly measuring time intervals, even though this is clearly absurd. What we really mean is the elapsed time that a clock at a certain position *would* read, if we had one there. For the purpose of illustration, it is just simpler to presume that everyone carries a fantastically accurate clock at all times.

Caveat for time dilation

The analysis above used to derive the time dilation formula relies on both observers measuring the same events taking place at the same physical location at the same time, such as two observers measuring the same light pulses. When timing between spatially separated events or dealing with questions of simultaneity, we must follow the formulas developed in Sect. 2.3.4.

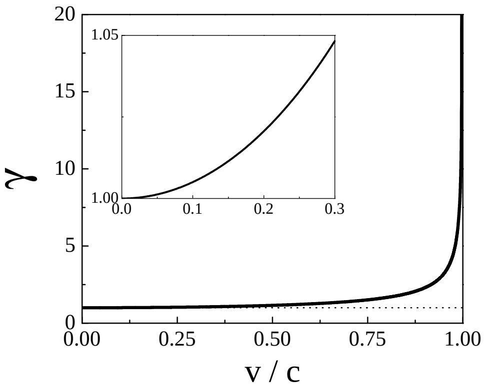
The quantity dimensionless quantity γ is the ratio of the time intervals measured by the observers moving (Moe) and stationary (Joe) relative to the events being timed. This quantity, defined by Eq. 2.15, comes up often in relativity, and it is called the *Lorentz factor*. Since c is the absolute upper limit for the velocity of anything, γ is always greater than 1. So long as the relative velocity of the moving observer is fairly small relative to c, the correction factor is negligible, and we need not worry about relativity (*e.g.,* at a velocity of $0.2c$, the correction is still only about 2%). In some sense, the quantity γ is sort of a gauge for the importance of relativistic effects – if $\gamma \approx 1$, relativity can be neglected, while if γ is much above 1, we must include relativistic effects like time dilation. Figure 2.12 provides a plot and table of γ versus v/c for reference. Note that as v approaches c, γ increases extremely rapidly.

Lorentz factor γ:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1 \tag{2.15}$$

γ is dimensionless, and $\gamma \geq 1$ for $v \leq c$. γ approaches 1 for low velocities, and increases rapidly as v approaches c. If $\gamma \approx 1$, one can safely neglect the effects of relativity.

In the case above, for velocities much less than c, when $\gamma \approx 1$, Eq. 2.14 tells us that both Joe and Moe measure approximately the same time interval, just as our everyday intuition tells us. In fact, for most velocities you might encounter in your everyday life, the correction factor γ is only different from 1 by a miniscule amount, and the effects of time dilation are negligible. They are note, however, *unmeasurable* or *unimportant*, as we will demonstrate in subsequent sections – time dilation has been experimentally verified to an extraordinarily high degree of precision, and does have some everyday consequences.



v [m/s]	$\frac{v}{c}$	γ	$1/\gamma$
0	0	0	∞
3×10^6	0.01	1.00005	0.99995
3×10^7	0.1	1.005	0.995
6×10^7	0.2	1.02	0.980
1.5×10^8	0.5	1.16	0.866
2.25×10^8	0.75	1.51	0.661
2.7×10^8	0.9	2.29	0.436
2.85×10^8	0.95	3.20	0.312
2.97×10^8	0.99	7.09	0.141
2.983×10^8	0.995	10.0	0.0999
2.995×10^8	0.999	22.4	0.0447
2.996×10^8	0.9995	31.6	0.0316
2.998×10^8	0.9999	70.7	0.0141
c	1	∞	0

Figure 2.12: *The Lorentz factor γ and its inverse for various velocities in table and graph form. The inset to the graph shows an expanded view for low velocities.*

2.3.2.1 Example: The Global Positioning System (GPS)

Before we discuss the stranger implications of time dilation, it is worth discussing at least one practical example in which the consequences of time dilation are important: the global positioning system. As you probably know, the Global Positioning System (GPS) is a network of satellites in medium earth orbit that transmit extremely precise microwave signals that can be used by a receiver to determine location, velocity, and timing. Each GPS satellite repeatedly transmits a message containing the current time, as measured by an onboard atomic clock, as well as other parameters necessary to calculate its exact position. Since the microwave signals from the satellites travel at the speed of light (microwaves are just a form of light, Sect. 9.5), knowing time difference between the moment the message was sent and the moment it was received allows an observer to determine their distance from the satellite. A ground-based receiver collects the signals from at least four distinct GPS satellites and uses them to determine its four space and time coordinates - (x, y, z and t).

How does relativity come into play? The 31 GPS satellites currently in orbit are in a medium earth orbit at an altitude of approximately 20,200 km, which give them a velocity relative to the earth's surface of 3870 m/s.[5][iv] This means that the actual atomic clocks responsible for GPS timing on the satellites are moving at nearly 4000 m/s relative to the receivers on the ground calculating position. Therefore, based on our discussion above, we would expect that the satellite-based GPS clocks would measure longer time intervals that those on the earth – the GPS clocks should run

[iv]You may remember from studying gravitation the the orbital speed can be found from Newton's general law of gravitation and centripetal force, $v = \sqrt{GM/r}$, where G is the universal gravitational constant, M is the mass of the earth, and r is the radius of the orbit, as measured from the earth's center.

slow, a problem for a system whose entire principle is based on precise timing.

How big is this effect? We already know enough to calculate the timing difference. Let us assume that (somehow) at $t=0$ we manage to synchronize a GPS clock with a ground-based one. From that moment, we will measure the elapsed time as measured by both clocks until the earth-based clock reads exactly 24 hours. We will call the earth-based clock's reference frame O, and that on the GPS satellite O', and label the time intervals correspondingly. Since are on the ground in the earth's reference frame, obviously we consider the earth-based clock to be the stationary one, measuring the proper time, and the GPS clock is moving relative to us. Applying Eq. 2.14, the elapsed time measured by the GPS clock and an earth-bound clock are related by a factor γ:

$$\Delta t'_{\text{GPS}} = \gamma \Delta t_{\text{Earth}} \tag{2.16}$$

The difference between the two clocks is then straightforward to calculate, given the relative velocity of the satellite of $v=3870\,\text{m/s}\approx 1.3\times 10^{-5}c$:

$$\begin{aligned}
\text{time difference} &= \Delta t_{\text{Earth}} - \Delta t_{\text{GPS}} && (2.17)\\
&= \Delta t_{\text{Earth}} - \gamma\Delta t_{\text{Earth}} && (2.18)\\
&= \Delta t_{\text{Earth}}\left(1-\gamma\right) && (2.19)\\
&= \Delta t_{\text{Earth}}\left[1-\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right] && (2.20)\\
&= \left[24\,\frac{\text{h}}{\text{day}}\cdot 60\,\frac{\text{min}}{\text{h}}\cdot 60\,\frac{\text{s}}{\text{min}}\right]\left[1-\frac{1}{\sqrt{1-\left(1.3\times 10^{-5}\right)^2}}\right] && (2.21)\\
&\approx \left[86400\,\text{s/day}\right]\left[-8.32\times 10^{-11}\right] && (2.22)\\
&\approx -7.2\times 10^{-6}\,\text{s/day} = -7.2\,\mu\text{s/day} && (2.23)
\end{aligned}$$

A grand total of about $7\,\mu$s slow over an entire day (about $0.3\,\mu$s per hour), only about 80 parts per trillion (8×10^{-11}) per day! This may not seem like a lot, until one again considers that the GPS signals are traveling at the speed of light, and even a small error in timing can translate into a relatively large error in position. Remember, it is the travel time of light signals that determines distance in GPS. If time dilation were not accounted for, a receiver using that signal to determine distance would have an error given by the time difference multiplied by the speed of light. If we presume that, conservatively, position measurements are taken only once per hour:

$$\text{position difference in one hour} = \text{time difference in one hour} \times c \tag{2.24}$$
$$= \left[-3.0 \times 10^{-7}\,\text{s/h}\right]\left[3.0 \times 10^{8}\,\text{m/s}\right] \tag{2.25}$$
$$\approx 90\,\text{m/h} \tag{2.26}$$

In the end, GPS must be far more accurate than this, and the effects of special relativity and time dilation must be accounted for, along with those of general relativity[5] (Sect. 2.5). Both effects together amount to a discrepancy of about $+38\,\mu\text{s}$ per day. Since the orbital velocity of the satellites is well-known and essentially constant, the solution is simple: the frequency standards for the atomic clocks on the satellites are precisely adjusted to run slower and make up the difference. Though time dilation seems a rather ridiculous notion at first, it has real-world consequences we are familiar with, if unknowingly so.

Time dilation on a 747

The cruising speed of a 747 is about 250 m/s. After a 5 hour flight at cruising speed, by how much would your clock differ from a ground-based clock? How about after a year?

Using the same analysis as above, your clock would differ by about 6×10^{-9} s (6 ns) after five hours, and still only $10\,\mu\text{s}$ after one year. Definitely not enough to notice, but enough to measure - current atomic clocks are accurate to $\sim 10^{-10}$ s/day (~ 0.1 ns/day). In fact, in 1971 physicists performed precisely this sort of experiment to test the predictions of time dilation in relativity, and found excellent agreement.[6]

2.3.2.2 Example: The Twin 'Paradox'

Now that we have a realistic calculation under our belt, let us consider a more extreme example. We will take identical twins, Joe and Moe, and send Moe on a rocket into deep space while Joe stays home. At the start of Moe's trip, both are 25 years old. Moe boards his rocket, and travels at $v = 0.95c$ to a distant star, and back again at the same speed. According to Joe's clock on earth, this trip takes 40 years, and Joe is 65 years old when Moe returns. Moe, on the other hand, has experienced time dilation, since relative to the earth's reference frame and Joe's clock he has been moving at $0.95c$. Moe's clock, therefore, runs more slowly, registers a smaller delay:

$$\Delta t_{\text{Joe}} = 40\,\text{yr} \tag{2.27}$$
$$\Delta t'_{\text{Moe}} = \gamma \cdot 40\,\text{yr} \tag{2.28}$$
$$= \frac{40\,\text{yr}}{\sqrt{1 - \left(\frac{0.95c}{c}\right)^2}} \tag{2.29}$$
$$\approx 12.5\,\text{yr} \tag{2.30}$$

It would seem, then that while Joe is 65 years old when Moe returns, having aged 40 years, Moe is 37.5 years old, having aged only 12.5 years! On the other hand, one of the principles of relativity is that there is no preferred frame of reference, it should be equally valid to use the clock on Moe's rocket ship as the proper time. From Moe's point of view, the earth is moving away from him at $0.95c$. In his reference frame, Joe's earth-bound clock should run slow, and *Moe* should be older than Joe!

This is the so-called Twin 'Paradox' of special relativity. In fact, it is not a paradox, but a misapplication of the notion of time dilation. The principles of special relativity we have been discussing are only valid for *non-accelerating* reference frames. In order for Moe to move from the earth's reference frame to the moving reference frame of the rocket ship at $0.95c$ and back again, he had to have accelerated during the initial and final portions of the trip, plus at the very least to turn around. The reference frame of the earth is for all intents and purposes not accelerating, but the reference frame on the ship *is*, and our calculation of the time dilation factor is not complete.

While the earth-bound clocks to run slow from the ship's point of view *so long as the velocity of the spaceship is constant*, during the accelerated portions of the trip the earth-bound clocks actually run *fast* and gain time compared to the rocket's clocks. An analysis including accelerated motion is beyond the scope of this text, but the gains of the earth-bound clock during the accelerated portion of the trip more than make up for the losses during the constant velocity portion of the trip, and no matter *who* keeps track, Joe will actually be younger than Moe from any reference frame. In short, there is no 'paradox' so long as the notions of relativity are applied carefully within their limits.

Inertial reference frames:
The principles of special relativity we have been discussing are only valid in *inertial* or non-accelerating reference frames. When accelerated motion occurs, a more complex analysis must be used.

2.3.3 Length Contraction

If the passage of time itself is altered by relative motion, what else must also be different? If the elapsed time interval depends on the relative motion of the clock and observer, then at constant velocity one would also begin to suspect that distance measurements must also be affected. After all, so far we have mostly talked about time in terms of objects or pulses of light traversing specific distances at constant velocity. Naturally, in order to explore this idea we need another thought experiment. Once again, it needs to involve a spaceship.

This time, the experiment is simple: a spaceship departs from earth toward a distant star, Fig. 2.13. In accordance with our discussion above, we stipulate that we *only consider the portion of the ship's journey where it is traveling at constant velocity,* and there is no acceleration to worry about. According to observations on the earth, the star is a distance L away, and the spaceship is traveling at a velocity v. From the earth's reference frame O, the amount of time the trip should

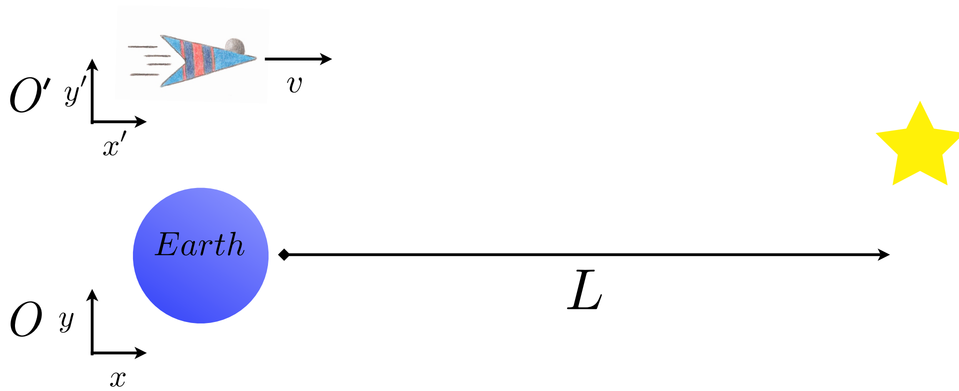


Figure 2.13: *Length contraction and travel to a distant star. A spaceship (frame O') sets out from earth (frame O) at a velocity v toward a distant star. Do the observers in the spaceship and the earth-bound observers agree on the distance to the star?*

take Δt_{E} is easy to calculate:

$$\Delta t_{\text{E}} = \frac{L}{v} \tag{2.31}$$

Fair enough. On the spaceship, however, the passage of time is slowed by a factor γ due to time dilation, and from their point of view, the trip takes less time. Since our spaceship is not accelerating in this example (it doesn't even have to turn around), we can readily apply Eq. 2.14. From the spaceship occupant's point of view, the earth is moving relative to them, so the time interval should be *divided* by γ to reflect their shorter elapsed time interval.

$$\Delta t'_{\text{ship}} = \frac{\Delta t_{\text{E}}}{\gamma} \tag{2.32}$$

Keep in mind, by *clock*, we mean the passage of time itself, this includes biological processes. We already know what Δt_{E} must be from Eq. 2.31, so we can plug that in to Eq. 2.32 above:

$$\Delta t'_{\text{ship}} = \frac{L}{v\gamma} \tag{2.33}$$

Do I divide or multiply by γ?
The Lorentz factor γ is always greater or equal to 1, $\gamma \geq 1$. If you are unsure about whether to divide or multiply by γ, think qualitatively about which quantity should be larger or smaller. In the example above, Eq. 2.32, we know the spaceship's time interval should be larger than that measured on earth, so we know we have to *divide* the earth's time interval by γ.

If the occupants of the ship also measure their velocity relative to the earth (we will pretend they even communicate with earth to make sure all observers agree on the relative velocity, $v'=v$), then they will presume that upon arrival at the distant star, the distance covered must be their velocity times their measured time interval. From the ship occupant's point of view, then, the distance to the star measured in their reference frame, L' is

$$L' = v\Delta t'_{\text{ship}} = \frac{v\Delta t_{\text{E}}}{\gamma} = \frac{L}{\gamma} \neq L \tag{2.34}$$

If you ask the people on the ship, the distance to the star is shorter, because their apparent time interval is! As we might have guessed, the relativity of time measurement also manifests itself in measurements of length, a phenomena known as *length contraction* or *Lorentz contraction*.

Length Contraction The length of an object or the distance to an object as measured by an observer in relative motion is *shorter* than that measured by an observer at rest by a factor $1/\gamma$. The stationary observer measures the proper length, L_p

$$L'_{\text{moving}} = \frac{L_{\text{stationary}}}{\gamma} = \frac{L_p}{\gamma} \tag{2.35}$$

That is, objects and distances appear shorter by $1/\gamma$ if you are moving.

There are a few caveats, however: the length contraction appears *only along the direction of relative motion*. For example, a baseball moving past you at very high velocity would be shortened only along one axis parallel to the direction of motion, and would appear as an ellipsoid, not as a smaller sphere. It would be "squashed" along the direction of the baseball's motion only, as shown in Fig. 2.14.

Just like time dilation, the length contraction effect is negligibly small at everyday velocities. Unlike time dilation, there is as yet no everyday application of time dilation, and no simple and straightforward experimental proof. We have no practical way of measuring the length of an object at extremely high velocities with sufficient precision at the moment. Collisions of elementary particles at very high velocities in particle accelerators provides some strong but indirect evidence for length contraction, and in some sense, since length contraction follows directly from time dilation, the experimental verifications of time dilation all but verify length contraction.

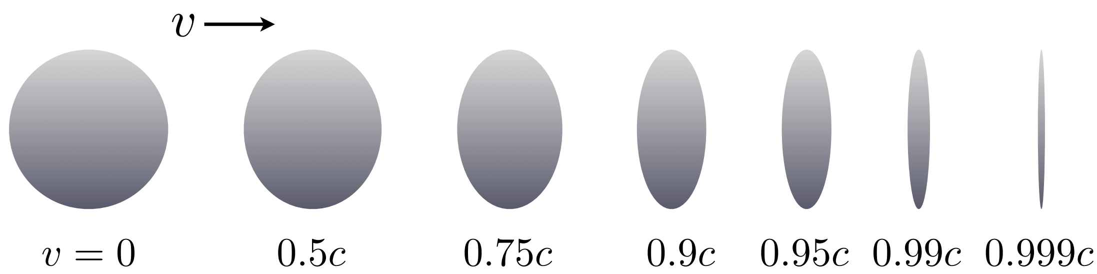


Figure 2.14: *Length contraction of a sphere traveling at various speeds, viewed side-on. The length contraction occurs only along the direction of motion. Hence, to a stationary observer, the moving sphere appears 'flattened' along the direction of motion into an ellipsoid.*

A summary of sorts:

1. objects and distances in relative motion appear shorter by $1/\gamma$
2. the length contraction is only along the direction of motion
3. the objects to not actually get shorter in their own reference frame, it is only apparent to the moving observer

2.3.4 Time and position in different reference frames

Now that we have a good grasp of time dilation and length contraction, we can start to answer the more general question of how we translate between time and position of events seen by observers in different reference frames. For example, consider Fig. 2.15. A girl in frame O is stationary relative to a star at point P, known to be a distance x away, which suddenly undergoes a supernova explosion. At precisely this instant, a boy travels past her in on a skateboard at constant relative velocity v along the x axis (frame O'). For convenience, we will assume that at the moment the explosion occurs, he is exactly the same distance away as the girl. When and where does the supernova occur, according to their own observations? How can we relate the distances and times measured by the girl to that measured by the boy, and *vice versa?*

All we need to do is apply what we know of relativity thusfar, and compare what each observer would measure in their own frame with what the *other* would measure. In the girl's case, the situation is fairly straightforward. She is a distance x from the star, and the first light from the explosion travels that distance at a velocity c. Therefore, according to her observations, the first light from the supernova arrives after:

$$t_{\text{arrival}} = \frac{x}{c} \tag{2.36}$$

What about the boy on the skateboard, in frame O? Since he is moving relative to the star, the distance to the star appears length contracted from his point of view. At the instant of the supernova, he measures a distance shorter by a factor γ compared to that measured by the girl. Furthermore, from his point of view in his own reference frame, he is siting still, and *the supernova*

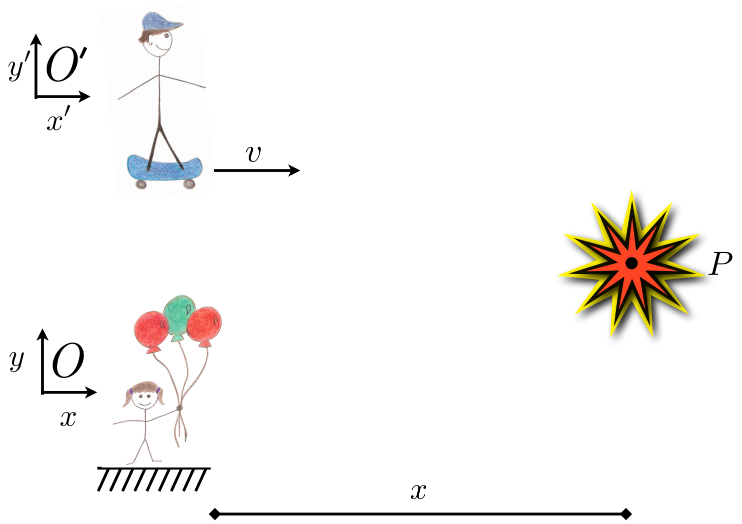


Figure 2.15: *A stationary and moving observer watch a supernova explosion. A girl in frame O is stationary relative to the supernova, a distance x away. A boy on a skateboard in the O' frame is traveling at v relative to frame O. How long does it take before the first light of the supernova reaches each of them?*

is moving toward him at velocity v. Therefore, from his point of view the supernova is getting closer to him. After t' seconds by his clock, the supernova is a distance vt' closer. Putting these two bits together, the distance x' the boy would measure to the supernova is:

$$x' = \frac{x}{\gamma} - vt' \tag{2.37}$$

So the distance to the supernova he claims is the original distance, length contracted due to his motion relative to the supernova, minus the rate at which he gets closer to the supernova.

What would the girl say about all this? The distance between the boy and the supernova, from her point of view, would have to be contracted to x'/γ since the boy is in motion relative to her. Additionally, from her point of view, since the boy is moving away from her at v, the distance between the two is *increasing* by vt after t seconds. We can express her perceived distance to the supernova as the sum of two distances: the distance from her to the boy, and the distance from the boy to the supernova:

$$x = vt + \frac{x'}{\gamma} \tag{2.38}$$

Now we have consistent expressions relating the distance measured by one observer to that measured by the other. If we rearrange Eqs. 2.37 and 2.38 a bit, and put primed quantities on one side and unprimed on the other, we arrive the transformations between positions measured by moving observers in their usual form:

Transformation of distance between reference frames:

$$x' = \gamma\left(x - vt\right) \tag{2.39}$$
$$x = \gamma\left(x' + vt'\right) \tag{2.40}$$

Here (x, t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v, measures the same event to be at position and time (x', t').

These equations include the effects of length contraction and time dilation we have already developed, as well as including the relative motion between the observers. If we use Eqs. 2.37 and 2.38 together, we can also arrive at a more direct expression to transform the measurement times as well. To start, we'll take Eq. 2.39 as written, and substitute it into Eq. 2.40:

$$\begin{aligned} x &= \gamma\left(x' + vt'\right) && (2.41)\\ &= \gamma\left(\gamma\left(x - vt\right) + vt'\right) && (2.42)\\ &= \gamma^2 x - \gamma^2 vt + \gamma vt' && (2.43)\end{aligned}$$

So far its a bit messy, but it will get better. Now let's solve that for t'. A handy relationship we will make use of is $\left(1-\gamma^2\right)/\gamma^2 = -v^2/c^2$, which you should verify for yourself.

$$\begin{aligned} \gamma vt' &= \left(1-\gamma^2\right)x + \gamma^2 vt && (2.44)\\ \Longrightarrow \qquad t' &= \gamma t + \frac{\left(1-\gamma^2\right)x}{\gamma v} && (2.45)\\ &= \gamma\left[t + \frac{1-\gamma^2}{\gamma^2}\left(\frac{x}{v}\right)\right] && (2.46)\\ &= \gamma\left[t - \frac{vx}{c^2}\right] && (2.47)\end{aligned}$$

And there we have it, the transformation between time measured in different reference frames. A similar procedure gives us the reverse transformation for t in terms of x' and t'.

Time measurements in different non-accelerating reference frames:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.48}$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{2.49}$$

Here (x, t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v, measures the same event to be at position and time (x', t').

The first term in this equation is just the time it takes light to travel across the distance x from point P, corrected for the effects of time dilation we now expect. The second term is new, and it represents an additional *offset* between the clock on the ground and the one in the car, not just one running slower than the other. What it means is that events seen by the girl in frame O do *not* happen at the same time as viewed by the boy in O'!

This is perhaps more clear to see if we make two different measurements, and try to find the elapsed time between two events. If our girl in frame O sees one even take place at position x_1 and time t_1, labeled as (x_1, t_1), and a second event at x_2 and t_2, labeled as (x_2, t_2), then she would say that the two events were spatially separated by $\Delta x = x_1 - x_2$, and the time interval between them was $\Delta t = t_1 - t_2$. If we follow the transformation to find the corresponding times that the boy observes, t'_1 and t'_2, we can also calculate the boy's perceived time interval between the events, $\Delta t'$:

Elapsed times between events in non-accelerating reference frames:

$$\Delta t' = t'_1 - t'_2 = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \tag{2.50}$$

If observer in O stationary relative to the events (x_1, t_1) and (x_2, t_2) measures a time difference between them of $\Delta t = t_1 - t_2$ and a spatial separation $\Delta x = x_1 - x_2$, an observer in O' measures a time interval for the same events $\Delta t'$. Events simultaneous in one frame $(\Delta t = 0)$ are only simultaneous in the other $(\Delta t' = 0)$ when there is no spatial separation between the two events $(\Delta x = 0)$.

For two events to be simultaneous, there has to be no time delay between them. For the girl to say the events are simultaneous requires that she measure $\Delta t = 0$, while for the boy to say the same requires $\Delta t' = 0$. We cannot satisfy both of these conditions based on Eq. 2.50 unless there is no relative velocity between observers $(v = 0)$, or the events being measured are not spatially separated $(\Delta x = 0)$. This means *two observers in relative will only find the same events simultaneous if the events are not spatially separated!* **Put simply, events are only simultaneous in both reference frames if they happen at the same spot.** At a given velocity, the larger the separation between the two events, the greater the degree of non-simultaneity. Similarly, for a given separation, the larger the velocity, the greater the discrepancy between the two frames. This is sometimes called **failure if simultaneity at a distance.**

In the end, this is our *general* formula for time dilation, including events which are spatially separated. If we plough still deeper into the consequences of special relativity and simultaneity, we will find that our principles of relativity have indeed preserved causality - cause always precedes effect - it is just that what one means by "precede" depends on which observer you ask. What relativity says is that cause must precede its effect according to all observers in inertial frames, which equivalently prevents both faster than light travel or communication and influencing the past.

2.3.4.1 Summary of sorts: the Lorentz Transformations

We are now ready to make a summary of the relativistic transformations of time and space. Let us consider two reference frames, O and O', moving at a **constant** velocity v relative to one another. For simplicity, we will consider the motion to be along the x and x' axes in each reference frame, so the problem is still one-dimensional. The observer in frame O measures an event to occur at time t and position (x, y, z). The event is *at rest with respect to the O frame.* Meanwhile, the observer in frame O' measures the *same event* to take place at time t' and position (x', y', z'). Based on what we have learned so far, we can write down the general relations between space and time coordinates in each frame, known as the *Lorentz transformations:*

Lorentz transformations between coordinate systems:

$$x' = \gamma\,(x - vt) \quad \text{or} \quad x = \gamma\left(x' + vt'\right) \tag{2.51}$$

$$y' = y \tag{2.52}$$

$$z' = z \tag{2.53}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{or} \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{2.54}$$

Here (x, y, z, t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v along the x axis, measures the same event to be at position and time (x', y', z', t').

Here we have provided both the 'forward' and 'reverse' forms of the transformations for convenience. Again, the distance is only contracted along the direction of motion, the x and x' directions – the y and z coordinates are thus unaffected. When the velocity is small compared to c ($v \ll c$), the first equation gives us our normal Newtonian result, the position in one frame relative to the other is just offset by their relative velocity times the time interval, and the time is the same. These compact equations encompass all we know of relativity so far - length contraction, time dilation, and lack of simultaneity.

Why are the transformations the way they are?
They take this form because they are the ones that leave the velocity of light constant at c in *every* reference frame.

Relativity for observers in relative motion at constant velocity:

1. Moving observers see lengths contracted along the direction of motion.
2. Moving observers' clocks 'run slow', less time passes for them.
3. Events simultaneous in one frame are not simultaneous in another unless they occur at the same position
4. All observers measure the same speed of light c

2.3.5 Addition of Velocities in Relativity

The invariance of the speed of light has another interesting consequence, namely, that one can no longer simply add velocities together to compute relative velocities in different reference frames in the way we did at the beginning of this chapter. Think about one of our original questions regarding relative motion, Fig. 2.3, in which a bully threw a dart off of a moving skateboard at a little girl's balloon. In that case, we said that the girl observed the dart to move at a velocity which was the *sum* of the velocities of the skateboard relative to the girl and the dart relative to the skateboard. When velocities are an appreciable fraction of the speed of light, this simple velocity addition breaks down.

In the end, it *has to*, or the speed of light could not be an absolute cosmic speed limit. Think about this: if you are driving in your car at 60 mi/hr down the freeway and turn on your headlights, do the light beams travel at c, or c plus 60 mi/hr? We know already that the answer must be c, but that is not at all consistent with our usual ideas of relative motion. If we can't just add the velocities together, what do we do? Is there a way to combine relative velocities such that the speed of light remains a constant and an upper limit? There is a relatively simple mathematical way to accomplish this. Once again, we will derive the result in the context of yet another thought experiment and try to show you how to use it.

The present thought experiment is just a variation the dart thrown from the skateboard, and is shown in Fig. 2.16. An observer on the ground (frame O) sees a person on a cart (frame O') moving at velocity v_a, as measured in the ground-based reference frame O. The person on the cart throws a ball at a velocity v'_b relative to the cart, which is measured as v_b in the ground-based frame. The ground-based observer measures v_a and v_b, while the observer on the cart measures the cart's velocity as v'_a and the ball's velocity as v'_b. How do we relate the velocities measured in the different frames O and O', without violating the principles of relativity we have investigated so far?

We can't simply add and subtract the velocities like we want to, our thought experiment of Sect. 2.2.3 involving a flashlight and a rocket ruled this out already, since this does not keep the speed of light invariant. So how *do* we properly add the velocities? Velocity is just displacement

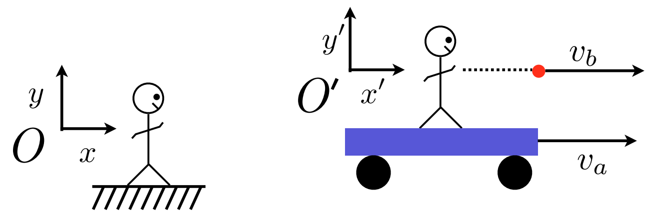


Figure 2.16: *Relativistic addition of velocities. An observer on the ground (frame O) sees a person on a cart (frame O') traveling at velocity v_a throw a ball off of the car at a velocity v_b relative to the ground. How do we relate the velocities as measured in the different reference frames?*

per unit time. If we calculate the displacement and time in one reference frame, then transform *both* to the other reference frame, we can divide them to *correctly* find velocity.

Let's start with the velocity of the ball as measured by the observer on the cart, v'_b. The displacement of the ball relative to the cart at some time t' after it was thrown, also measured in the cart's frame O', is just $x'_b = v'_b t'$. This is just how far ahead of the car the ball is after some time t'. We can substitute this into Eq. 2.51 to find out what displacement the observer on the ground in O should measure, remembering that v_a is the relative velocity of the observers:

$$x_b = \gamma \left(x'_b + v_a t' \right) = \gamma \left(v'_b t' + v_a t' \right) \tag{2.55}$$

But now we have x, the displacement of the ball seen from O, in terms of t', the time measured in O'. If we want to find the velocity of the ball as measured by an observer in O, *we have to divide the distance measured in O by the time measured in O!* We can't divide one person's position by another person's time, we have to transform *both*. So we should use Eq. 2.54 to find out what t is from t' too:

$$t = \gamma \left(t' + \frac{v_a x'}{c^2} \right) = \gamma \left(t' + \frac{v_a v'_b t'}{c^2} \right) \tag{2.56}$$

Now we have the displacement of the ball x and the time t as measured by the observer on the ground in O. The velocity in O is just the ratio of x to t:

$$v_b = \frac{x}{t} \tag{2.57}$$

$$= \frac{\gamma\left(v'_b t' + v_a t'\right)}{\gamma\left(t' + \frac{v_a v'_b t'}{c^2}\right)} \tag{2.58}$$

$$= \frac{v'_b + v'_a}{1 + \frac{v_a v_b}{c^2}} \tag{2.59}$$

For the last step, we divided out $\gamma t'$ from everything, by the way. So, this is the proper way to compute relative velocity of the ball observed from the ground, consistent with our framework of relativity.

$$\text{velocity of ball observed from the ground} = v_b = \frac{v_a + v'_b}{1 + \frac{v_a v'_b}{c^2}} \tag{2.60}$$

In the limiting case that the velocities are very small compared to c, then it is easy to see that the expression above reduces to $v_b = v_a + v'_b$ – the velocity of the ball measured from the ground is the velocity of the car relative to the ground plus the velocity of the ball relative to the car. But, this is *only* true when the velocities are small compared to c.[v] Similarly, we could solve this equation for v'_b instead and relate the velocity of the ball as measured from the car to the velocities measured from the ground:

$$\text{velocity of ball observed from the cart} = v'_b = \frac{v_b - v_a}{1 - \frac{v_a v_b}{c^2}} \tag{2.61}$$

The equation above allows us to calculate the velocity of the ball as observed from the car if we only had ground-based measurements. Again, for low velocities, we recover the expected result $v'_b = v_b - v_a$. What about the velocity of the cart? We don't need to transform it, since it is already the *relative* velocity between the frames O and O', and hence between the ground-based observer and the car. **We only need the velocity addition formula when a third party is involved.** Out of the three relevant velocities, we only ever need to know two of them.

So this is it. This simple formula is all that is needed to properly add velocities and obey the principles of relativity we have put forward. Below, we put this in a slightly more general formula.

[v] Or, more precisely, when the *product* of the velocities is small compared to c^2.

Relativistic velocity addition:
We have an observer in a frame O, and a second observer in another frame O' who are moving relative to each other at a velocity v. Both observers measure the velocity of another object in their own frames (v_{obj} and v'_{obj}). We can relate the velocities measured in the different frames as follows:

$$v_{\text{obj}} = \frac{v + v'_{\text{obj}}}{1 + \frac{vv'_{\text{obj}}}{c^2}} \qquad v'_{\text{obj}} = \frac{v_{\text{obj}} - v}{1 - \frac{vv_{\text{obj}}}{c^2}} \tag{2.62}$$

Again, v_{obj} is the object's velocity as measured from the O reference frame, and v_{obj} is its velocity as measured from the O' reference frame.

Velocities greater than c?
The velocity addition formula shows that one cannot accelerate something past the speed of light. No matter what subluminal velocities you add together, the result is *always* less than c. Try it! Our relativistic equations for momentum and energy will further support this.

Remember, c isn't just the speed of light, it is a limiting speed for *everything*!

2.3.5.1 Example: throwing a ball out of a car

Just to be clear, let us make our previous example more concrete. Let's say we have Joe in reference frame O, sitting on the ground, while Moe is in a car (frame O') moving at $v_{\text{car}} = \frac{3}{4}c$. Moe throws a ball *very hard* out of the car window, such that he measures its velocity to be $v'_{\text{ball}} = \frac{1}{2}c$ in his reference frame. What would Joe say that the velocity of the ball is, relative to his reference frame on the ground?

Basically, Joe wants to know the velocity of the ball relative to the ground, not relative to the car. What we need to do is relativistically combine the velocity of the car relative to the ground and the velocity of the ball relative to the car. Classically, we would just add them together:

$$v_{\text{ball}} = v_{\text{car}} + v'_{\text{ball}} = \frac{3}{4}c + \frac{1}{2}c = \frac{5}{4}c = 1.25c \qquad \text{WRONG!}$$

Clearly this is an absurdity - the ball cannot be traveling faster than the speed of light in *anyone's* reference frame. We need to use the proper relativistic velocity addition formula, Eq. 2.62. We know the velocity of the ball relative to the car in frame O', v'_{ball} and the velocity of the car relative to the ground in the O frame, v_{car}, so we just substitute and simplify:

$$v_{\text{ball}} = \frac{v_{\text{car}} + v'_{\text{ball}}}{1 + \frac{v_{\text{car}} v'_{\text{ball}}}{c^2}} \tag{2.63}$$

$$= \frac{\frac{3}{4}c + \frac{1}{2}c}{1 + \frac{\left(\frac{3}{4}c\right)\left(\frac{1}{2}c\right)}{c^2}} \tag{2.64}$$

$$= \frac{\frac{5}{4}c}{1 + \frac{3}{8}c} = \frac{10}{11}c \approx 0.91c \tag{2.65}$$

So, in relativity, three quarters plus one half is only about 0.9! But this is the result we are looking for - no matter what velocities $v < c$ we add together, we always get an answer less than c. Put another way, no matter what reference frame we consider, the velocity of an object will *always* observed to be less than c. So our relativistic velocity addition works so far. But what about applying it to light, which is actually traveling right at c. Does everything still come out ok?

2.3.5.2 Example: shining a flashlight out of a rocket

What if, instead of throwing a ball out of the window, Moe uses a flashlight to send out a light pulse? In that case, we have to find that the velocity of light is c no matter which frame we use. Remember our problem in Sect. 2.2.3? We had Joe traveling on a rocket at $0.99c$, while Moe on the ground shines a flashlight parallel to his path, shown again in Fig: 2.17. Our claim at the time was that both Moe and Joe should measure the same speed of light. Does our new velocity addition formula work for this case?

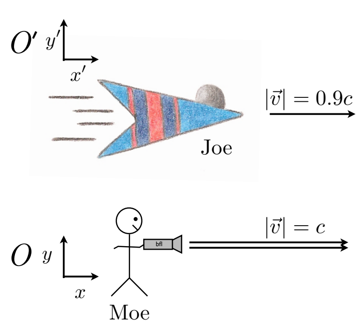


Figure 2.17: *Joe is traveling on a rocket at $|\vec{\mathbf{v}}| = 0.99c$, while Moe on the ground shines a flashlight parallel to Joe's path. Both Joe and Moe observe the light from the flashlight to travel at $|\vec{\mathbf{v}}| = c$, consistent with our relativistic velocity addition formula!*

In this case, Joe is on a rocket (frame O') moving at $v_{\text{rocket}} = 0.99c$ relative to Moe on the ground. Moe knows that in his frame O, the light from the flashlight travels away from him at velocity $v_{\text{light}} = c$. What is the velocity of light observed by Joe in the rocket, $v_{\text{light}'}$, if we use the velocity addition formula? All we have to do is subtract the speed of light as measured by Moe from Joe's speed on the rocket ship, according to the second equation in 2.62:

$$v'_{\text{light}} = \frac{v_{\text{light}} - v_{\text{rocket}}}{1 - \frac{v_{\text{rocket}} v_{\text{light}}}{c^2}} \tag{2.66}$$

$$= \frac{c - 0.99c}{1 - \frac{(0.99c)(c)}{c^2}} \tag{2.67}$$

$$= \frac{0.01c}{1 - 0.99} = c \tag{2.68}$$

Lo and behold, the thing works! Our velocity addition formula correctly calculates that both Joe and Moe have to measure the same speed of light, since the speed of light is the same when observed from any reference frame. We shouldn't be too surprised, however: the velocity addition formula was *constructed* to behave in exactly this way. How about if Joe holds the flashlight while in the rocket, what is the speed of light as measured by Moe on the ground? Now we have to add the velocities of the light coming out of the rocket and the velocity of the rocket itself, according the first equation in 2.62. Still no problem:

$$v_{\text{light}} = \frac{v_{\text{rocket}} + v'_{\text{light}}}{1 + \frac{v_{\text{rocket}} v'_{\text{light}}}{c^2}} \tag{2.69}$$

$$= \frac{0.99c + c}{1 + \frac{(0.99c)(c)}{c^2}} \tag{2.70}$$

$$= \frac{1.99c}{1 + 0.99} = c \tag{2.71}$$

In the end, we have succeeded in constructing a framework of mechanics that keeps the speed of light invariant in all reference frames, and answers (nearly) all the questions raised at the beginning of the chapter.

Is everything relative then?

Not quite!

- All observers will agree on an objects *rest* length
- All observers will agree on the proper time
- All observers will agree on an objects rest mass
- The speed of light is an upper limit to physically attainable speeds

2.4 Mass, Momentum, and Energy

So far, the simple principles of relativity have had enormous consequences. Our basic notions of time, position, and even simultaneity all needed to be modified. If position and time must be altered, then it stands to reason that *velocity* - the change of position with time - must also be altered. Sure enough, the velocity addition formula was also a required change. What next? If our notions of relative velocity need to be altered, then the next thing must surely be momentum and kinetic energy. As it turns out, even our concept of *mass* needs to be tweaked a bit.

2.4.1 Relativistic Momentum

First, let's consider momentum. Classically, we define momentum in terms of mass and velocity, $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. A basic principle of classical mechanics you have learned is that momentum must be conserved, no matter what. What about in relativity? In relativity, exactly what $\vec{\mathbf{v}}$ is depends on the reference frame in which it is measured. That means that our usual definition of momentum above depends on the reference frame as well. It gets worse. Using our simple $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, not only would the total amount of momentum depend on the choice of reference frame, conservation of momentum in one frame would not necessarily be true in another. How can a fundamental conservation law depend on the frame of reference?

It cannot - this is one of our basic principles of relativity, *viz.*, the laws of physics are the same for all non-accelerating frames of reference. We *must* have conservation of momentum, independent of what frame in which the momentum us measured. How do we construct a new equation for momentum, one for which conservation of momentum is always valid, but at low velocities reduces to our familiar $\vec{\mathbf{p}}=m\vec{\mathbf{v}}$? The result is not surprising: we only need to transform velocity the same way we transformed position:

Relativistic momentum:

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}} \tag{2.72}$$

Here $\vec{\mathbf{p}}$ is the momentum vector for an object of mass m moving with velocity $\vec{\mathbf{v}}$.

The derivation is a bit beyond the scope of our discussion, but defining momentum in this way makes it independent of the choice of reference frame, and restores conservation of momentum as a fundamental physical law. For low velocities ($v \ll c$), $\gamma \approx 1$, and this reduces to the familiar result. For velocities approaching c, the momentum grows much more quickly than we would expect. In fact, an object traveling at c would require *infinite* momentum (and therefore infinite kinetic energy), clearly an absurdity. This is one good reason why nothing with finite mass can ever travel at the speed of light! Only light itself, with no mass, can travel at the speed of light.

2.4.2 Relativistic Energy

The relativistic correction to momentum is straightforward. Given that kinetic energy depends on the momentum of an object (one can write $KE = p^2/2m$), one would expect a necessary revision for kinetic energy as well. This one is not so straightforward, however. First, we need to think about what we mean by energy in the first place.

In classical mechanics, for a single point mass in linear motion (*i.e.,* not rotating), the kinetic energy simply goes to zero when the body stops, $KE = \frac{1}{2}mv^2 = p^2/2m$. For an arbitrary body, however, the result is not so simple. If a composite object contains multiple, independently moving bodies (such the individual atoms making up matter, for instance), the individual entities may interact among themselves and move about, and the object possesses *internal energy* E_i as well as the kinetic energy due to the motion of the whole mass. Overall, classically the kinetic energy of such a body is the sum of these two energies – the energy due to the motion of the object as a whole, and the energy due to the motion of the constituents of the object, $KE = \frac{1}{2}mv^2 + E_i$. Any moving body more complex than a single point mass has a contribution due to its internal energy.

In relativity, the kinetic energy does still depend on the motion of a body as a whole as well as its internal energy content. As with momentum, conservation of energy requires that *the energy of a body is independent of the choice of reference frame*, the *total energy* of a body cannot depend on the frame in which it is measured. The total energy – kinetic plus internal – must be the same in all reference frames. A derivation requires somewhat more math than we would like, but the result is simple:

Relativistic energy of a moving body:

$$E = \gamma mc^2 \tag{2.73}$$

This equation already tells us that the energy content of a body grows rapidly as v approaches c, and reaching the speed of light would require a body to have infinite energy. What is more interesting, however, is when the velocity of the body is *zero*, *i.e.,* $\gamma = 1$. In this case, $E = mc^2$ - the body has finite energy even when not in motion! This is Einstein's most famous equation, and it represents the fundamental equivalence of mass and energy. Any object has an *intrinsic, internal energy* associated with it by virtue of having mass. This constant energy is called the *rest energy*:

Rest Energy:

$$E_R = mc^2 \tag{2.74}$$

As Einstein himself put it, "Mass and energy are therefore essentially alike; they are only different expressions for the same thing."[7] Matter is basically an extremely dense form of energy – is convertible into energy, and *vice versa*. In fact, the rest energy content of matter is enormous, owing to the enormity of c^2 - one gram of normal matter corresponds to about 9×10^{13} J, the same energy content as 21 ktons of TNT! It is the conversion of matter to energy that is responsible for the enormous energy output of nuclear reactions, such as those that power the sun, a subject we will return to.

The equivalence of matter and energy, or, if you like, the presence of an internal energy due solely to a body's matter content, is an unexpected consequence of relativity. But we still have not determined the actual kinetic energy of a relativistic object! Again, the derivation is somewhat laborious, but the result is easy enough to understand. If we take the total energy of an object, Eq. 2.73, and subtract off the velocity-independent rest energy, Eq. 2.74, what we are left with is the part of a body's energy that depends solely on velocity. This is the kinetic energy we are looking for, and it means the *total energy of a body is the sum of its rest and kinetic energies:*

Relativistic kinetic energy:

$$KE = (\gamma - 1)\, mc^2 \tag{2.75}$$

Total energy:

$$E_{\text{total}} = KE + E_R \tag{2.76}$$

Since $\gamma = 1$ when $v = 0$, the kinetic energy of a stationary body is zero, as we expect. At low velocities ($v \ll c$), one can show that this expression correctly reduces to $\frac{1}{2}mv^2$. As with the total energy, for a body to actually acquire a velocity of c it would need an infinite kinetic energy, again, a primary reason why no object with mass can travel at the speed of light.

For completion, we should note that it is still possible to relate relativistic energy and momentum, just like it was possible to relate classical kinetic energy and momentum, though we will not derive the expressions here:

Relativistic energy-momentum equations:

$$E^2 - (pc)^2 = \left(mc^2\right)^2 \tag{2.77}$$

$$Ev = pc^2 \tag{2.78}$$

here p is the momentum of a body, m its mass, v its velocity, E its energy, and c is the speed of light. We can use this to write the relativistic kinetic energy and momentum equations in a different form:

$$KE = \sqrt{p^2c^2 + m^2c^4} - mc^2 \qquad \text{and} \qquad p = \sqrt{\frac{E^2}{c^2} - m^2c^4} \tag{2.79}$$

The energy content of a body still scales with its momentum, and for a body at rest ($p = 0$), the energy content is purely the rest energy mc^2. Once again we have an unexpected result, however: *objects with no mass must also have momentum, so long as they have energy.* For massless particles – such as the photons that make up a beam of light – we have the result $E = pc$, or $p = E/c$. This is truly another odd result of relativity, completely unexpected from classical physics! How can objects with no mass still have momentum? Since matter and energy are equivalent according to relativity, having energy is just as good as having mass, and still leads to a net momentum. This will become an important consideration when we begin to study optics and modern physics.

Momentum of massless objects:

$$p = \frac{E}{c} \tag{2.80}$$

If you combine Eqs. 2.78 and 2.80, you come to an even wilder conclusion. If the particle has zero mass, but *some* energy greater than zero, then we can write

$$v = \frac{pc^2}{E} = \frac{\frac{E}{c}c^2}{E} = c \tag{2.81}$$

A particle with zero mass always moves at the speed of light, and can never stop moving! It doesn't matter what the energy of the particle is, anything with finite energy but zero mass has to travel at the speed of light. The converse is true as well – anything moving at the speed of light must be massless. Just to drive the point home one last time: *the speed of light is an upper limit to physically attainable speeds for material bodies.*

2.4.3 Relativistic Mass

About the only thing left we have not modified with relativity is mass. Most modern interpretations of relativity consider mass to be an *invariant* quantity, properly measured when the body is at rest (or measured within its own reference frame). This rest mass of an object in its own reference frame is called the *invariant mass* or *rest mass*, and is an observer-independent quantity synonymous with our usual definition of "mass."

These days, we say that while the *momentum* of a body must be the same in all reference frames, and hence must be transformed, the *mass* of a body is just a constant, and is measured in the body's own reference frame. Rest mass is in some sense just counting the number of atoms in an object, something we really only do in the object's reference frame anyway. If we are measuring an object from another reference frame, we will typically be measuring its *momentum*, or kinetic energy, not counting how many atoms it contains. Thus it is momentum and kinetic energy we transform to be invariant in all reference frames and mass we simply say is a property of an object measured in its own reference frame.

2.5 General Relativity

2.6 Quick Questions

1. An astronaut traveling at $v = 0.80c$ taps her foot 3.0 times per second. What is the frequency of taps determined by an observer on earth? (*Hint: be careful about the difference between time and frequency!*)

- ◯ 5.0 taps/sec
- ◯ 6.7 taps/sec
- ◯ 1.8 taps/sec
- ◯ 3.0 taps/sec

2. A spaceship moves away from earth at high speed. How do experimenters on earth measure a clock in the spaceship to be running? How do those in the spaceship measure a clock on earth to be running?

- ◯ slow; fast
- ◯ slow; slow
- ◯ fast; slow
- ◯ fast; fast

3. If you are moving in a spaceship at high speed relative to the earth, would you notice a difference in your pulse rate? In the pulse rate of the people back on earth?

- ◯ no; yes
- ◯ no; no
- ◯ yes; no
- ◯ yes; yes

4. The period of a pendulum is measured to be 3.00 in its own reference frame. What is the period as measured by an observer moving at a speed of $0.950c$ with respect to the pendulum?

- ◯ 6.00 sec
- ◯ 13.4 sec
- ◯ 0.938 sec
- ◯ 9.61 sec

5. The Stanford Linear Accelerator (SLAC) can accelerate electrons to velocities very close to the speed of light (up to about $0.99999999995c$ or so). If an electron travels the 3 km length of the accelerator at $v = 0.999c$, how long is the accelerator from the *electron's* reference frame?

- ◯ 134 m
- ◯ 67.1 km
- ◯ 94.9 m
- ◯ 300 m

6. A spacecraft with the shape of a sphere of diameter D moves past an observer on Earth with a speed $0.5\,c$. What shape does the observer measure for the spacecraft as it moves past?

- ◯ streak
- ◯ ellipsoid
- ◯ sphere
- ◯ cube

7. Suppose you're an astronaut being paid according to the time you spend traveling in space. You take a long voyage traveling at a speed near that of light. Upon your return to earth, you're asked how you would like to be paid: according to the time elapsed by a clock on earth, or according to the ship's clock. Which do you choose to maximize your paycheck?

- ◯ The earth clock.
- ◯ The ship's clock.
- ◯ It doesn't matter.

2.7 Problems

1. In the 1996 movie *Eraser*,[8] a corrupt business Cyrez is manufacturing a handheld rail gun which fires aluminum bullets at nearly the speed of light. Let us be optimistic and assume the actual velocity is $0.75\,c$. We will also assume that the bullets are tiny, about the mass of a paper clip, or $m = 5\times 10^{-4}\,\text{kg}$.

(a) What is the relativistic kinetic energy of such a bullet?
(b) Let us further assume that Cyrez has managed to power the rail guns by matter-energy conversion. What amount of mass would have to be converted to energy to fire a single bullet? (For comparison, note that 1 kg of TNT has an equivalent energy content of about 4×10^{9} J.)

2. Show that the kinetic energy of a (non-relativistic) particle can be written as $KE = p^2/2m$, where p is the momentum of a particle of mass m.

3. A pion at rest ($m_{\pi} = 273\,m_{e^-}$) decays to a muon ($m_{\mu} = 207\,m_{e^-}$) and an antineutrino ($m_{\overline{\nu}} \approx 0$). This reaction is written as $\pi^- \rightarrow \mu^- + \overline{\nu}$. Find the kinetic energy of the muon and the energy of the antineutrino in electron volts. *Hint: relativistic momentum is conserved.*

4. An alarm clock is set to sound in 15 h. At $t = 0$ the clock is placed in a spaceship moving with a speed of $0.77c$ (relative to Earth). What distance, as determined by an Earth observer, does the spaceship travel before the alarm clock sounds?

5. The average lifetime of a pi (π) meson in its own frame of reference (*i.e.*, the proper lifetime) is 2.6×10^{-8} s

(a) If the meson moves at $v = 0.98c$, what is its mean lifetime as measured by an observer on earth?

(b) What is the average distance it travels before decay, measured by an observer on Earth?
(c) What distance would it travel if time dilation did not occur?

6. You are packing for a trip to another star, and on your journey you will travel at $0.99c$. Can you sleep in a smaller cabin than usual, because you will be shorter when you lie down? Explain your answer.

7. A deep-space probe moves away from Earth with a speed of $0.88c$. An antenna on the probe requires 4.0 s, in probe time, to rotate through 1.0 rev. How much time is required for 1.0 rev according to an observer on Earth?

8. A friend in a spaceship travels past you at a high speed. He tells you that his ship is 24 m long and that the identical ship you are sitting in is 18 m long.

(a) According to your observations, how long is your ship?
(b) According to your observations, how long is his ship?
(c) According to your observations, what is the speed of your friend's ship?

9. A Klingon space ship moves away from Earth at a speed of $0.700c$. The starship Enterprise pursues at a speed of $0.900c$ relative to Earth. Observers on Earth see the Enterprise overtaking the Klingon ship at a relative speed of $0.200c$. With what speed is the Enterprise overtaking the Klingon ship as seen by the crew of the Enterprise?

10. An observer sees two particles traveling in opposite directions, each with a speed of $0.99000c$. What is the speed of one particle with respect to the other?

2.8 Solutions to Quick Questions

1. 1.8 **taps/sec.** The 'proper time' Δt_p is that measured by the astronaut herself, which is 1/3 of a second between taps (so that there are 3 taps per second). The time interval *between taps* measured on earth is dilated (longer), so there are *less* taps per second. For the astronaut:

$$\Delta t_p = \frac{1\,\text{s}}{3\,\text{taps}}$$

On earth, we measure the dilated time:

$$\Delta t' = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{0.8^2c^2}{c^2}}} \cdot \left(\frac{1\,\text{s}}{3\,\text{taps}}\right) = \frac{1}{\sqrt{1-0.8^2}} \cdot \left(\frac{1\,\text{s}}{3\,\text{taps}}\right) \approx \frac{0.56\,\text{s}}{\text{tap}} = \frac{1\,\text{s}}{1.8\,\text{taps}}$$

2. slow; slow. The time-dilation effect is symmetric, so observers in each frame measure a clock in the other to be running slow. Put another way, the *relative* velocity of the earth and the ship is the same no matter who you ask – each says the other is moving with some speed v, and they are sitting still. Therefore, the dilation effect is the same in both cases.

3. no; yes. There is no relative speed between you and your own pulse, since you are in the same reference frame, so there is no difference in your pulse rate (possible space-travel-related anxieties aside). There is a relative velocity between you and the people back on earth, however, so you would find their pulse rate *slower* than normal. Similarly, they would find *your* pulse rate slower than normal, since you are moving relative to them. Relativistic effects are always attributed to the other party – you are always at rest in your own reference frame.

4. 9.61 **sec.** The proper time is that measured by in the reference frame of the pendulum itself, $\Delta t_p = 3.00\,\text{sec}$. The moving observer has to observe a *longer* period for the pendulum, since from the observer's point of view, the pendulum is moving relative to it. Observers always perceive clocks moving relative to them as running slow. The factor between the two times is just γ:

$$\Delta t' = \gamma \Delta t_p = \frac{3.0\,\text{sec}}{\sqrt{1 - \frac{0.95^2 c^2}{c^2}}} = \frac{3.0\,\text{sec}}{\sqrt{1 - 0.95^2}} \approx 9.61\,\text{sec}$$

5. 134 **m.** The electron in its own reference frame sees the *accelerator* moving toward it at $0.999c$, and sees a contracted length:

$$L = \frac{L_p}{\gamma} = 3\,\text{km} \cdot \sqrt{1 - \frac{0.999^2 c^2}{c^2}} = 3\,\text{km} \cdot \sqrt{1 - 0.999^2} = 0.134\,\text{km} = 134\,\text{m}$$

6. ellipsoid. The sphere is length contracted only along its direction of motion, *i.e.,* only along one axis. Squishing a sphere along one axis makes an ellipsoid.

7. The earth's clock. Less time will have passed in your reference frame, since you are moving relative to the earth. The earth's clock will have registered more time elapsed than yours.

2.9 Solutions to Problems

1. 2.3×10^{13} **J,** 2.56×10^{-4} **kg**. First part: relativistic kinetic energy is given by:

$$\text{KE} = (\gamma - 1)\, mc^2$$

First, we'll calculate γ based on the given velocity:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.75c)^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.75^2}} = 1.51$$

Next, we'll calculate the mc^2 bit:

$$mc^2 = \left(5 \times 10^{-4}\,\text{kg}\right)\left(3 \times 10^{8}\,\text{m/s}\right)^2 = 4.5 \times 10^{13}\,\text{kg} \cdot \text{m}^2/2^2 = 4.5 \times 10^{13}\,\text{J}$$

Putting it all together:

$$\text{KE} = (\gamma - 1)\, mc^2 = (1.51 - 1)\left(4.5 \times 10^{13}\,\text{J}\right) = 2.30 \times 10^{13}\,\text{J} = 23.0\,\text{TJ}$$

Second part: what rest mass is equivalent to this amount of kinetic energy? We just need to use the mass-energy equivalence formula:

$$\begin{aligned} E_R &= mc^2 = \text{KE} \\ \Longrightarrow m &= \frac{\text{KE}}{c^2} = \frac{(\gamma-1)\, mc^2}{c^2} \\ &= (\gamma - 1)\, m = 0.51m \\ &= 2.56 \times 10^{-4}\,\text{kg} \end{aligned}$$

In other words, it takes fully half the mass of the bullet itself, completely converted to pure energy, to fire one round. Using more conventional propellants, that would mean 5760 kg ($\sim$ 6 tons) of TNT per round.

2. We'll run it both forwards and backwards:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{mv \cdot v}{2} = \frac{mv \cdot v}{2}\frac{m}{m} = \frac{mv \cdot mv}{2m} = \frac{p \cdot p}{2m} = \frac{p^2}{2m}$$

Or, since you know the answer you want ...

$$\frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{m^2v^2}{2m} = \frac{mv^2}{2} = \frac{1}{2}mv^2$$

3. 4.08 **MeV for the muon,** 29.6 **MeV for the antineutrino.** This one is a bit lengthier than most of the others! Before the collision, we have only the pion, and since it is at rest, it has zero momentum and zero kinetic energy. After it decays, we have a muon and an antineutrino created and speed off in opposite directions (to conserve momentum). Both total energy - including rest energy - and momentum must be conserved before and after the collision.

First, conservation of momentum. Before the decay, since the pion is at rest, we have zero momentum. Therefore, afterward, the muon and antineutrino must have equal and opposite momenta. This means we can essentially treat this as a one-dimensional problem, and not bother with vectors. A consolation prize of sorts.

$$\begin{aligned}
\text{initial momentum} &= \text{final momentum} \\
p_\pi &= p_\mu + p_\nu \\
0 &= p_\mu + p_\nu \\
\Longrightarrow p_\nu &= -p_\mu = -\gamma_\mu m_\mu v_\mu
\end{aligned}$$

For the last step, we made use of the fact that relativistic momentum is $p = \gamma m v$. Now we can also write down conservation of energy. Before the decay, we have only the rest energy of the pion. Afterward, we have the energy of both the muon and antineutrino. The muon has both kinetic energy and rest energy, and we can write its total kinetic energy in terms of γ and its rest mass, $E = \gamma m c^2$. The antineutrino has negligible mass, and therefore no kinetic energy, but we can still assign it a total energy based on its momentum, $E = pc$.

$$\begin{aligned}
\text{initial energy} &= \text{final energy} \\
E_\pi &= E_\mu + E_\nu \\
m_\pi c^2 &= \gamma_\mu m_\mu c^2 + p_\nu c \\
m_\pi &= \gamma_\mu m_\mu + \frac{p_\nu}{c}
\end{aligned}$$

Now we can combine these two conservation results and try to solve for the velocity of the muon:

$$\begin{aligned}
m_\pi &= \gamma_\mu m_\mu + \frac{p_\nu}{c} \\
m_\pi &= \gamma_\mu m_\mu - \gamma_\mu m_\mu \frac{v_\mu}{c} \\
\frac{m_\pi}{m_\mu} &= \gamma_\mu - \gamma_\mu \frac{v_\mu}{c} \\
\frac{m_\pi}{m_\mu} &= \gamma \left[1 - \frac{v_\mu}{c}\right]
\end{aligned}$$

We will need to massage this quite a bit more to solve for v_μ ...

$$\begin{aligned}
\frac{m_\pi}{m_\mu} &= \gamma\left[1-\frac{v_\mu}{c}\right] = \frac{1-\frac{v_\mu}{c}}{\sqrt{1-\frac{v_\mu^2}{c^2}}} \\
\left(\frac{m_\pi}{m_\mu}\right)^2 &= \frac{\left(1-\frac{v_\mu}{c}\right)^2}{1-\frac{v_\mu^2}{c^2}} \\
&= \frac{\left(1-\frac{v_\mu}{c}\right)^2}{\left(1-\frac{v_\mu}{c}\right)\left(1+\frac{v_\mu}{c}\right)} \\
&= \frac{\left(1-\frac{v_\mu}{c}\right)^{\not 2}}{\cancel{\left(1-\frac{v_\mu}{c}\right)}\left(1+\frac{v_\mu}{c}\right)} \\
&= \frac{1-\frac{v_\mu}{c}}{1+\frac{v_\mu}{c}}
\end{aligned}$$

Now we're getting somewhere. Take what we have left, and solve it for v_μ ... we will leave that as an exercise to the reader, and quote only the result, using the given masses of the pion and muon:

$$\frac{v_\mu}{c} = \frac{1-\left(\frac{m_\pi}{m_\mu}\right)^2}{1+\left(\frac{m_\pi}{m_\mu}\right)^2} \approx -0.270$$

From here, we are home free. We can calculate γ_μ and the muon's kinetic energy first. It is convenient to remember that the electron mass is $511\,\mathrm{keV/c^2}$.

$$\begin{aligned}
\gamma_\mu &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.27c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.27^2}} \approx 1.0386 \\
\mathrm{KE}_\mu &= (\gamma_\mu - 1)\, m_\mu c^2 = (1.0386-1)\,(207 m_{e^-})\, c^2 \\
&= 0.0386\left(207\cdot 511\,\mathrm{keV}/c^2\right)c^2 \approx 4.08\times 10^6\,\mathrm{eV} = 4.08\,\mathrm{MeV}
\end{aligned}$$

Finally, we can calculate the energy of the antineutrino as well:

$$\begin{aligned}
E_\nu &= p_\nu c = -p_\mu c \\
&= -\gamma_\mu m_\mu v_\mu \\
&= -1.0386\cdot\left(207\cdot 5.11\,\mathrm{keV}/c^2\right)\cdot(-0.270c) \\
&\approx 2.96\times 10^7\,\mathrm{eV} = 29.6\,\mathrm{eV}
\end{aligned}$$

4. 1.96×10^{13} **m.** The 15 h set on the alarm clock in the spaceship is the proper time interval, Δt_p. Since the space ship is moving away from the earth at $v=0.77c$, an earthbound observer observes a longer dilated time interval, $\Delta t'$. Based on this longer time interval, the earthbound observer will measure that the space ship has covered a distance of $v\Delta t'$. So, first: we need to calculate γ, then the dilated time interval, then finally the distance measured by the earthbound observer.

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.77c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.77^2}} = 1.57 \\
\Delta t' &= \gamma \Delta t_p \\
&= 1.57 \cdot 15\,\text{h} = 1.57 \cdot 5.4 \times 10^4\,\text{s} \approx 8.48 \times 10^4\,\text{s} \\
d' &= v \Delta t' \\
&= 0.77c \cdot 8.48 \times 10^4\,\text{s} = 0.77 \cdot 3 \times 10^8\,\text{m/s} \cdot 8.48 \times 10^4\,\text{s} \\
d' &\approx 1.96 \times 10^{13}\,\text{m}
\end{aligned}
$$

5. 1.31×10^{-7} **s, 38.4 m, 7.64 m** The π meson's lifetime in its own frame is the proper time interval, $\Delta t_p = 2.6 \times 10^{-8}$ s. An earthbound observer measures a longer dilated time interval $\Delta t'$. To calculate it, we need only calculate γ for the velocity given, $v_\pi = 0.98c$.

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.98c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.98^2}} = 5.03 \\
\Delta t' &= \gamma \Delta t_p \\
&= 5.03 \left(2.6 \times 10^{-8}\,\text{s}\right) \\
&\approx 1.31 \times 10^{-7}\,\text{s}
\end{aligned}
$$

The distance the π meson travels in the earthbound observer's reference frame, d' is the π meson's velocity multiplied by the time interval measured by the earthbound observer. We don't need to worry about whether the velocity is measured in the π meson's or the observer's frame - since it is a relative velocity, it is the same either way.

$$
d' = \gamma v_\pi \Delta t_p = v_\pi \Delta t' = (0.98c) \cdot \left(1.31 \times 10^{-7}\,\text{s}\right) = \left(0.98 \cdot 3 \times 10^8\,\text{m/s}\right) \cdot \left(1.31 \times 10^{-7}\,\text{s}\right) \approx 38.4\,\text{m}
$$

Without time dilation, the distance traveled would just be the proper lifetime multiplied by the meson's velocity:

$$
d = v_\pi \Delta t_p = (0.98c) \cdot \left(2.6 \times 10^{-8}\,\text{s}\right) = \left(0.98 \cdot 3 \times 10^8\,\text{m/s}\right) \cdot \left(2.6 \times 10^{-8}\,\text{s}\right) \approx 7.64\,\text{m}
$$

6. No. There is no relative speed between you and your cabin, since you are in the same reference frame. You and your bed will remain at the same lengths relative to each other.

7. 8.42 s. The time interval in the probe's reference frame is the proper one Δt_p ... which makes sense, since the antenna is part of the probe itself! The probe and antenna are moving relative to the earth, and therefore the earthbound observer measures a longer, dilated time interval $\Delta t'$:

$$
\begin{aligned}
\text{probe} &= \Delta t_p \\
\text{earth} &= \Delta t' \\
\Delta t' &= \gamma \Delta t_p
\end{aligned}
$$

As usual, we first need to calculate γ. No problem, given the probe's velocity of $0.88c$ relative to earth:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.88c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.88^2}} = 2.11$$

The proper time interval for one revolution Δt_p in the probe's reference frame is 4.0 s, so we can readily calculate the time interval observed by the earthbound observer:

$$\Delta t' = \gamma \Delta t_p = 2.11 \cdot (4.0\,\text{s}) = 8.42\,\text{s}$$

8. 24 m; 18 m; $0.661c$**.** Once again: if you are observing something in your own reference frame, there is no length contraction or time dilation. You always observe your own ship to be the same length. If your friend's ship is 24 m long, and yours is identical, you will measure it to be 24 m.

On the other hand, you are moving relative to his ship, so you would observe his ship to be length contracted, and measure a shorter length. Your friend, on the other hand, will observe *exactly the same thing* - he will see *your* ship contracted, by precisely the same amount. Your observation of his ship has to be the same as his observation of his ship - since you are only the two observers, and you both have the same *relative* velocity, you must observe the same length contraction. If he sees your ship as 18 m long, then you would also see his (identical) ship as 18 m long.

Given the relationship between the contracted and proper length, we can find the relative velocity easily. Your measurement of your own ship is the proper length L_p, while your measurement of your friend's ship is the contracted length L':

$$\begin{aligned}
L_p &= \gamma L' \\
\Longrightarrow \gamma &= \frac{L_p}{L'} = \frac{24}{18} = \frac{4}{3} \\
\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} &= \frac{4}{3} \\
1-\frac{v^2}{c^2} &= \frac{3^2}{4^2} = \frac{9}{16} \\
\frac{v^2}{c^2} &= 1-\frac{9}{16} = \frac{7}{16} \\
v &= \sqrt{\frac{7}{16}}c = \frac{\sqrt{7}}{4}c \approx 0.661c
\end{aligned}$$

9. $0.541c$**.** This is just a problem of relativistically adding velocities, if we can keep them all straight. Let the unprimed system denote velocities measured relative to the earth, and the primed system those measured relative to the enterprise. We have, then:

$$\begin{aligned} v_e = 0.900c &= \text{Enterprise relative to earth} \\ v_k = 0.700c &= \text{Klingon ship relative to earth} \\ v_k' = ? &= \text{Klingon ship, relative to Enterprise} \end{aligned}$$

Since the Enterprise is moving faster relative to the earth than the Klingon ship, that means that from the Enterprise's point of view, the Klingons are actually moving backwards toward them. If we plug what we know into the velocity addition formula ...

$$v_k = \frac{v_e + v_k'}{1 + \frac{v_e v_k'}{c^2}}$$

It takes a bit of algebra, but we can readily solve this for v_k':

$$v_k' = \frac{v_e - v_k}{1 - \frac{v_e v_k}{c^2}}$$

Not so surprisingly, what we have just done is to re-write the 'velocity addition formula' as a 'velocity subtraction formula.' It is just rearranging same formula (you can verify that both equations above are equivalent ...), but the second form is far more convenient for our present purposes.

We can find the velocity of the Klingon ship relative to the enterprise in terms of both ships' velocities relative to the earth. In the limit that both velocities are much smaller than c, we see that $v_k' \approx v_e - v_k = 0.200c$, just as we would expect from normal Newtonian physics. Since in this case, neither velocity is negligible compared to c, the actual v_k' will be significantly larger. At this point, we can just plug in the numbers we have and see:

$$\begin{aligned} v_k' &= \frac{v_e - v_k}{1 - \frac{v_e v_k}{c^2}} \\ &= \frac{0.900c - 0.700c}{1 - \frac{(0.900c)(0.700c)}{c^2}} \\ &= \frac{0.200c}{1 - (0.900)(0.700)} = \frac{0.200c}{0.37} \\ v_k' &\approx 0.541c \end{aligned}$$

So, as far as the crew of the Enterprise is concerned, they are overtaking the Klingon ship at a rate of $0.541c$.

10. $0.99995c$. Let the observer be in frame O'. In the reference frame of one of the particles, labeled O, the observer is traveling at $v=0.99c$, and the second particle is traveling at $v_2'=0.99c$ relative to the observer. We can then find the velocity of the second particle relative to the first, v_2, through velocity addition:

$$v_2 = \frac{v + v_1'}{1 + \frac{vv_1'}{c^2}} \tag{2.82}$$

$$= \frac{0.99c + 0.99c}{1 + \frac{(0.99c)(0.99c)}{c^2}} \tag{2.83}$$

$$= \frac{1.98c}{1 + 0.9801} \approx 0.99995c \tag{2.84}$$

This is an example of a problem where you need to make sure to use enough significant digits!

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[32] See **http://en.wikipedia.org/List_of_indices_of_refraction**. Image from http://en.wikipedia.org/wiki/Image:Dispersion-curve.png. The creator of this image has given permission to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts.

[33] The project software, hardware schematics, installation instructions, sample laboratory procedures utilizing the system, and many other details can be found at the project home-page, listed above. Binary downloads of the software package are also available.

[34] The GNU public license is available at http://www.gnu.org/copyleft/gpl.html.

[35] See http://www.labjack.com for information including pricing, software and documentation. The LabJack U3 is currently listed at $99, or $90 after educational discount.

[36] We used the NTE976 which we had in stock, $8.93 from http://www.mouser.com. Many cheaper substitutes are available.

[37] Details regarding this course can be found at http://ph102.blogspot.com.

[38] A PASCO Xplorer GLX configured with voltage/current sensor is another low-cost equivalent. It provides a portion of the functionality of the current system at ~$400 per seat. The proprietary software and hardware represents, in our view, a lack of flexibility however. The National Instruments USB-6008 device compares quite favorably to the LabJack U3, and could be readily substituted. It does have a slightly higher cost (~$60 more).

[39] Radio Shack, 8x6x3in ABS project enclosure, $6.99.

[40] Adhesive-backed velcro is usually available at any local fabric store, *e.g.,* http://www.hancockfabrics.com/.

[41] See, for example, McMaster-Carr p/n 7124K42, $12.14 per package of 10.

[42] The smaller size prevents students from plugging sensitive test loads directly into 18 V from the batteries.

[43] See http://www.ni.com/lwcvi/ for details. Academic site-licenses available. An effort is underway to make the software build-able with a free cross-platform development environment.

[44] A separate article on this experiment is in preparation. A sample procedure is available at `http://code.google.com/p/bamalab/`.

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