# Exam II: Solutions

□ **1.** The circuit at right is known as a *Wheatstone Bridge*, and it is a useful circuit for measuring small changes in resistance. Perhaps you can figure out why. Three of the four branches on our bridge have identical resistance R, but the fourth has a slightly different resistance, by an amount δR such that its total resistance is $R + \delta R$.

In terms of the source voltage V_s, base resistance R and change in resistance δR, what is the reading on the voltmeter, ΔV? You may assume the voltmeter and voltage source are perfect (drawing no current and having no internal resistance, respectively).

Bonus: Simplify your expression for $\delta R \ll R$, i.e., the change in resistance is small. Show that this results in $\Delta V \propto \delta R$. (Worth +20% credit on this question.)
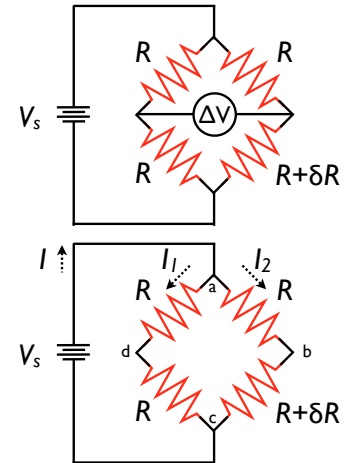
Problem 1: Wheatstone Bridge. Top: figure as given. Bottom: choosing current directions and labeling nodes.

Remember that ideal voltmeters draw no current, so the meter in the center of the bridge doesn't really *do* anything. If we label the nodes on the bridge a-d, as shown in the figure at right, the voltmeter simply tells us the potential difference between points d and b, ΔV_{db}. Knowing that, we will simply leave it out of our diagram to make things a bit more clear.

Looking more carefully at the bridge, we notice that it is nothing more than two sets of series resistors, connected in parallel with each other. This immediately means that the voltage drop across the left side of the bridge, following nodes a→d→c, must be the same as the voltage drop across the right side of the bridge, following nodes a→b→c – both are ΔV_{ac}, and both must be the same as the source voltage: $\Delta V_{ac} = V_s$. If we can find the current in each resistor, then with the known source potential difference we will know the voltage at any point in the circuit we like, and finding ΔV_{db} is no problem.

Let the current from the source V_s be I. This current I leaving the source will at node a split in to separate currents I_1 and I_2; conservation of charge requires $I = I_1 + I_2$. At node c, the currents recombine into I. On the leftmost branch of the bridge, the current I_1 creates a voltage drop $I_1 R$ across each resistor. Similarly, on the rightmost branch of the bridge, each the resistor R has a voltage drop $I_2 R$ and the lower resistor has a voltage drop $I_2 (R + \delta R)$. Equating the total voltage drop on each branch of the bridge:

$$V_s = I_1 R + I_1 R = I_2 R + I_2 (R + \delta R)$$

$$\implies \quad I_1 = \frac{V_s}{2R}$$

$$I_2 = \frac{V_s}{2R + \delta R}$$

Now that we know the currents in terms of known quantities, we can find ΔV_{db} by "walking" from point d to point b and summing the changes in potential difference. Starting at node d, we move toward node a *against* the current I_1, which means we *gain* a potential difference $I_1 R$. Moving from node a to node b, we move *with* the current I_2, which means we *lose* a potential difference $I_2 R$. Thus, the total potential difference between points d and b must be

$$\Delta V_{db} = I_1 R - I_2 R = R(I_1 + I_2) = R\left(\frac{V_s}{2R} - \frac{V_s}{2R + \delta R}\right)$$

$$\Delta V_{db} = V_s\left(\frac{1}{2} - \frac{R}{R + \delta R}\right) = V_s\left(\frac{\delta R}{4R + 2\delta R}\right)$$

If the change in resistance δR is small compared to R ($\delta R \ll R$), the term in the denominator can be approximated $4R + \delta R \approx 4R$, and we have

$$\Delta V_{db} = V_s\frac{\delta R}{4R} \qquad (\delta R \ll R)$$

Thus, for small changes in resistance, the voltage measured across the bridge is directly proportional to the change in resistance, which is the basic utility of this circuit: it allows one to measure small changes on top of a large 'base' resistance. Fundamentally, it is a *difference* measurement, meaning that one directly measures *changes* in the quantity of interest, rather than measuring the whole thing and trying to uncover subtle changes. This behavior is very useful for, e.g., strain gauges, temperature sensors, and many other devices.
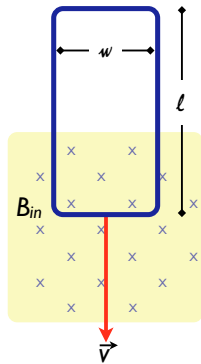
□ **2.** A conducting rectangular loop of mass M, resistance R, and dimensions w by l falls from rest into a magnetic field $\vec{B}$, as shown at left. At some point before the top edge of the loop reaches the magnetic field, the loop attains a constant terminal velocity v_T. Show that the terminal velocity is:

$$v_T = \frac{MgR}{B^2 w^2}$$

Hint: what is true at terminal velocity?

Problem 2: Falling loop in a magnetic field

First, let us analyze the situation qualitatively. As the loop falls into the region of magnetic field, more of its area is exposed to the field, which increases the total flux through the loop. This increase in magnetic flux will cause an induced potential difference around the loop, via Faraday's law, which will create a current that tries to counteract this change in magnetic flux. Since the flux is increasing, the induced current in the loop will try to act *against* the existing field to reduce the change in flux, which means the current will circulate counterclockwise to create a field out of the page.

Once there is a current flowing in the loop, each current-carrying segment will feel a magnetic force. The left and right segments of the loop will have equal and opposite forces, leading to no net effect, but the current flowing (to the right) in the bottom segment will lead to a force $F_B = BIw$ upward. Again, this is consistent with Faraday's (and Lenz's) law - any magnetic force on the loop must act in such a way to reduce the rate at which the flux changes, which in this case clearly means slowing down the loop. The upward force on the loop will serve to counteract the gravitational force, which is ultimately responsible for the flux change in this case anyway. The faster the loop falls, the larger the upward force it experiences, and at some point the magnetic force will balance the gravitational force perfectly, leading to no net acceleration, and hence constant velocity. This is the "terminal velocity." Of course, once the whole loop is inside the magnetic field, the flux is again constant, and the loop just starts to fall normally again.[i]

Quantitatively, we must first find the induced voltage around the loop, which will give us the current. The current will give us the force, which will finally give us the acceleration. As the loop falls into the magnetic field, at some instant t we will say that a length x of the loop has moved into the field, out of the total length l. At this time, the total flux through the loop is then:

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[i]We would still have eddy currents, which would provide some retarding force, but for thin wires eddy current forces are probably going to be negligible. This is basically what we demonstrated with our conducting pendulums swinging through a magnetic field. The pendulums that had only thin segments of conductor (it looked like a fork) experienced very little damping compared to a plain flat plate.

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = B \oint dA = BA = Bwx$$

(Since the area normal of the loop $\hat{n}$ is parallel to $\vec{B}$ at all times, we can deal only in magnitudes and drop the vector notation.) From the flux, we can easily find the induced voltage from Faraday's law.

$$\Delta V = -\frac{d\Phi_B}{dt} = -Bw\frac{dx}{dt} = -Bwv$$

Here we made use of the fact that the rate at which the length of the loop exposed to the magnetic field changes is simply the instantaneous velocity, $dx/dt = v$. Once we have the induced voltage, given the resistance of the loop R, we know the current via Ohm's law:

$$I = \frac{\Delta V}{R} = -\frac{Bwv}{R}$$

From Lenz's law we know the current circulates counterclockwise. In the right-most segment of the loop, the current is flowing up, and the magnetic field into the page. The right-hand rule then dictates that the force on this current-carrying segment must be to the left. The left-most segment of the loop has a force equal in magnitude, since the current I, the length of wire, and the magnetic field are the same, but the force is in the opposite direction. Thus, taken together, the left and right segments of the loop contribute no net force. The bottom segment, however, experiences an upward force, since the current is to the right. For a constant magnetic field and constant current (true at least instantaneously), the force is easily found:

$$F_B = \int_{\substack{\text{bottom} \\ \text{segment}}} I d\vec{l} \times \vec{B} = \int_0^w IB\,dl = BIw$$
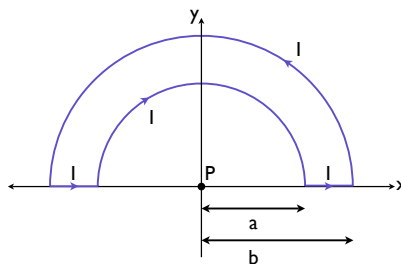
We can substitute our expression for I above:

$$F_B = BIw = -\frac{B^2 w^2 v}{R}$$

At the terminal velocity v_T, this upward force will exactly balance the downward gravitational force:

$$\sum F = mg - \frac{B^2 w^2 v_T}{R} = 0$$
$$\implies \quad v_T = \frac{mgR}{B^2 w^2}$$

□ **3.** Find the magnetic field at point P due to the current distribution shown below. *Hint: Break the loop into segments, and use superposition.*

Problem 3: A current loop

The easiest way to do solve this is by superposition. Since the magnetic field obeys superposition, we can consider our odd current loop above to be the same as two semicircles plus two small straight segments. We know that the magnetic field at the center of a circular loop of radius r carrying a current I is

$$B = \frac{\mu_o I}{2r} \qquad \text{(loop radius r)}$$

If you did not remember this, it is easily derived from the Biot-Savart law ... or you could notice that problem 5 actually gives you this expression if you set $z = 0$. As a quick reminder, for a circle of radius r, our infinitesimal length element dl is just $r d\theta$. For a current circulating around the ring in the $\hat{\theta}$ direction, a vector length element pointing along the current direction is then $d\vec{\mathbf{l}} = r d\theta \, \hat{\theta}$. We can now apply the Biot-Savart law:

$$d\vec{\mathbf{B}} = \frac{\mu_o I}{4\pi} \frac{d\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_o I}{4\pi} \frac{r d\theta \, \hat{\theta} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_o I}{4\pi r} d\theta \, \hat{\mathbf{z}}$$

$$\vec{\mathbf{B}} = \int_{\text{circle}} d\vec{\mathbf{B}} = \int_0^{2\pi} \frac{\mu_o I}{4\pi r} d\theta \, \hat{\mathbf{z}} = \frac{\mu_o I}{4\pi r} d\theta \, \hat{\mathbf{z}} \, (2\pi) = \frac{\mu_o I}{2r} \, \hat{\mathbf{z}}$$

This is the field at the center of a *full* circle. Since the magnetic field obeys superposition, we could just as well say that our full circle is built out of two equivalent half circles, like the one above! The field from each half circle, by symmetry, must be half of the total field, so the field at the center of a semicircle must simply be

$$B = \frac{\mu_o I}{4r} \qquad \text{(semicircle, radius r)}$$

A more formal derivation goes just like the one above: simply replace the upper integration limit with π instead of 2π. Fundamentally, integrating the little dB's using the Biot-Savart law is just saying the field from any current distribution can be built out of the fields of infinitesimal line segments by superposition. That is what the integral is really "doing," it is building a circle out of tiny bits.

Anyway: for the problem at hand, we have two semicircular current segments contributing to the magnetic field at P: one of radius b, and one of radius a. The currents are in the opposite directions for the two loops, so their fields are in opposing directions. Based on the axes given, it is the outer loop of radius b that has its field pointing out of the page in the $\hat{\mathbf{z}}$ direction, and the inner loop of radius a in the $-\hat{\mathbf{z}}$ direction.

What about the straight bits of wire? The Biot-Savart law tells us that the magnetic field from a segment of the straight wire is proportional to $d\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. For the straight segments, $d\vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$ are parallel, and their cross product is zero. There is no field contribution at P from the straight segments! Thus, the total field is just that due to the semicircular bits,

$$\vec{\mathbf{B}} = \frac{\mu_o I}{4b} \, \hat{\mathbf{z}} - \frac{\mu_o I}{4a} \, \hat{\mathbf{z}} = \frac{\mu_o I}{4} \left(\frac{1}{b} - \frac{1}{a} \right) \hat{\mathbf{z}}$$

□ **4.** If the voltage at the terminals of an automobile battery drops from ΔV_1 to ΔV_2 $(\Delta V_1 > \Delta V_2)$ when a resistor R is connected across the battery, what is the internal resistance r of the battery?

Our model of a real battery is an ideal voltage source in series with an internal resistance r. What we measure at the terminals of the battery at any given time is the ideal voltage minus the voltage drop across the internal resistance. Before we connect the resistor R across our battery, we have an open circuit – there is no current flow. If there is no current flow, there is no voltage drop across the internal resistance, and *what we measure on the battery under open circuit conditions, ΔV_1 is the ideal voltage.* Once we connect our resistance R, it is in series with the ideal voltage source ΔV_1 and the internal resistance r. This completes the circuit, and there will be a current flowing through the internal and external resistors.

With the external resistor R connected, with a current I flowing, the voltage measured at the battery terminals ΔV_2 is the ideal voltage minus the voltage drop on the internal resistor:

$$\Delta V_2 = \Delta V_1 - Ir$$

This is not quite enough, since we don't know I. We can still apply conservation of energy (or Kirchhoff's "loop" rule): the total of all voltage drops in the circuit must be zero. First we add the battery voltage, and then subtract the potential drops on the internal resistance and the external resistance:

$$\Delta V_1 - Ir - IR = 0$$

Combining the two equations above, we immediately have

$$I = \frac{\Delta V_2}{R}$$
$$\Delta V_1 = I\,(R + r)$$

It is not surprising that $\Delta V_2 = IR$ – the voltage at the terminals in the second case is just the voltage applied to the resistor R. The latter equation can be rearranged to find r, and then combined with the former equation:

$$r = \frac{\Delta V_1 - IR}{I} = \frac{\Delta V_1 - \Delta V_2}{I} = \frac{\Delta V_1 - \Delta V_2}{\Delta V_2/R} = R\left[\frac{\Delta V_1 - \Delta V_2}{\Delta V_2}\right]$$

☐ **5.** A Helmholtz coil, useful for producing a region of uniform field, consists of two identical circular coils separated by a distance equal to their radius R, as shown at right. Show that dB_z/dz vanishes at the midpoint P, indicating a homogeneous field.

Hint: The field from a single current loop of radius R a distance z from the center of the loop along the coil axis is

$$B = \frac{\mu_o I}{2}\frac{R^2}{\left(z^2 + R^2\right)^{3/2}} \quad \text{(single loop)}$$

Bonus: show that $d^2 B_z/dz^2$ is also zero at P. (Worth $+20\%$ credit on this question.)
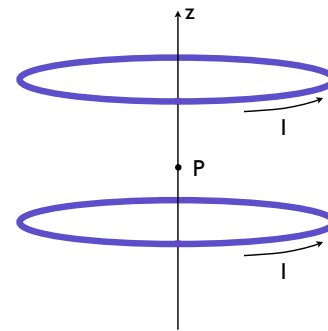
Problem 5: A Helmholtz coil

In order to establish uniformity, we first want to find the field at an *arbitrary* point z somewhere between the two coils, then find dB/dz and evaluate it at the midpoint. We don't just want the field at the midpoint; we need its spatial variation to find the derivative(s) and only *then* do we evaluate our quantities at the midpoint between the two coils.

Let $z = 0$ at the intersection of the plane of the bottom coil and the z axis. The field from the bottom coil at an arbitrary point a distance z along the axis due to the bottom coil is just the quantity given above. At a position z, since the separation of the coils is R, we are a distance $R - z$ from the upper coil. We need only replace z with $R - z$ in the expression above to find the field from the upper coil at a distance $z < R$ from the bottom coil. Since the currents are in the same directions for both coils, the magnetic fields are in the same direction, and we may just add them together:

$$B_{\text{tot}} = B_{\text{lower}} + B_{\text{upper}} = \frac{\mu_o I}{2}\frac{R^2}{\left(z^2 + R^2\right)^{3/2}} + \frac{\mu_o I}{2}\frac{R^2}{\left[(R-z)^2 + R^2\right]^{3/2}}$$

Now we only need calculate dB/dz:

$$\frac{dB_{\text{tot}}}{dt} = \frac{\mu_o I R^2}{2}\left[\frac{d}{dz}\frac{1}{\left(z^2+R^2\right)^{3/2}} + \frac{d}{dz}\frac{1}{\left(\left(R-z\right)^2+R^2\right)^{3/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{-\frac{3}{2}\left(2z\right)}{\left(z^2+R^2\right)^{5/2}} + \frac{-\frac{3}{2}\left(2z-2R\right)}{\left(\left(R-z\right)^2+R^2\right)^{5/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{-3z}{\left(z^2+R^2\right)^{5/2}} + \frac{3R-3z}{\left(\left(R-z\right)^2+R^2\right)^{5/2}}\right]$$

Evaluating this at point P, where $z = R/2$,

$$\left.\frac{dB_{\text{tot}}}{dt}\right|_{z=\frac{R}{2}} = \frac{\mu_o I R^2}{2}\left[\frac{-3\frac{R}{2}}{\left(\frac{R^2}{4}+R^2\right)^{5/2}} + \frac{3R-3\frac{R}{2}}{\left(\left(R-\frac{R}{2}\right)^2+R^2\right)^{5/2}}\right]$$

$$= \frac{\mu_o I R^2}{2}\left[\frac{-3R}{2\left(\frac{5R^2}{4}\right)^{5/2}} + \frac{3R}{2\left(\frac{5R^2}{4}\right)^{5/2}}\right] = 0$$

Indeed, the derivative of the field with respect to position is zero, implying high uniformity near the center of the Helmholtz arrangement.

One can also show that $d^2 B/dz^2$ and $d^3 B/dz^3$ are zero as well (but it is messy), and in fact the field is extremely uniform over a reasonably large volume between the two coils.

Incidentally, the field midway between the two coils, with $z = R/2$ is $B = 8\mu_o I/5^{3/2}R$, about 1.4 times the field one would get directly at the center of a single coil. Not only is the field more uniform, it is larger. How convenient!
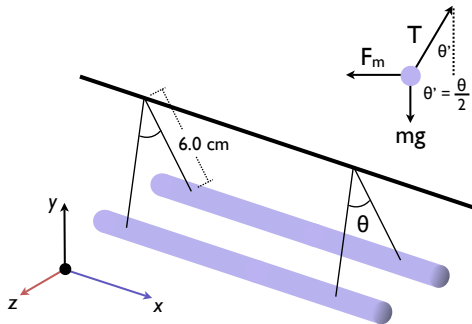
Problem 6: Parallel current-carrying wires

□ **6.** Two long parallel wires, each with a mass per unit length of λ, are supported in a horizontal plane by strings of length d, as shown at left. Each wire carries the same current I, causing the wires to repel one another, which causes the supporting strings to make an angle θ with one another.

Find the magnitude of the current.

Hint: consider the free-body diagram in the upper right.

Basically, we have two current-carrying wires, both carrying the same current I, and we know the magnitude of the force per unit length between them must be:

$$\frac{F_m}{l} = \frac{\mu_0 I^2}{2\pi d} \tag{1}$$

Here d is the *lateral* separation of the wires. If the wires are hanging as shown, then the magnetic force between the wires must be balancing the weight of each wire and the tension in the strings holding them up. Therefore **the magnetic force must be repulsive,**

and the currents in the opposite direction.

Notice the free-body diagram in the upper right corner. In equilibrium, the sum of all forces is zero. Take $+x$ to the right, and $+y$ upward. Then we can write down the net forces in the x and y directions. **For convenience, we will say from now on $\theta' \equiv \theta/2$, since we only need the half angle to do the problem.**

$$\begin{aligned}
\Sigma F_x &= T\sin\theta' - F_m = 0 & \Rightarrow & & F_m &= T\sin\theta' \\
\Sigma F_y &= T\cos\theta' - mg = 0 & \Rightarrow & & T &= \frac{mg}{\cos\theta'}
\end{aligned}$$

Plug the expression for T into the one for F_m, and divide by the length of the wires, which we will call l, so we can equate this result to Eq. 1:

$$\frac{F_m}{l} = T\sin\theta' = mg\tan\theta' = \frac{m}{l}g\tan\theta' = g\lambda\tan\theta' = \frac{\mu_0 I^2}{2\pi d}$$

Note the substitution $\lambda = m/l$ above. Now the question is, what is d, the separation of the wires? Simple plane geometry relates the separation d to the length of the support wires (the 6 cm, in the figure; we'll call this h), and the angle θ': $d = 2h\sin\theta'$. Put that into the equation above and solve for I ...

$$\begin{aligned}
\frac{|\vec{F}_m|}{l} &= \lambda g\tan\theta' = \frac{\mu_0 I^2}{2\pi d} = \frac{\mu_0 I^2}{4\pi h\sin\theta'} \\
I^2 &= \frac{4\pi}{\mu_0}g\lambda h\sin\theta'\tan\theta' = \frac{4\pi}{\mu_0}g\lambda h\sin\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)
\end{aligned}$$

□ **7.** A conducting bar of length l rotates with a constant angular speed ω about a pivot at one end. A uniform magnetic field $\vec{B}$ is directed perpendicular to the plane of rotation, as shown in the figure below. Find the motional voltage induced between the ends of the bar.

Hint: recall $v = r\omega$ at a distance r from the pivot point.
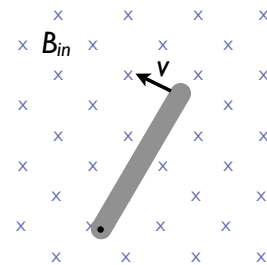
Problem 7: A conducting rod rotates about one end in a magnetic field.

BONUS (worth 20% of a normal question): During an in-class demonstration, we dropped a magnet and a non-magnet of equal weight and size through a copper tube. The non-magnet fell through the tube at the expected rate, but the magnet took many times longer to fall out, due to eddy current braking.

Is it possible to have a magnet strong enough (or a tube conductive enough, *etc*) that it would actually *stop* inside the tube? Explain. No. The eddy current braking comes from induced currents in the copper tube due to the falling magnet. The falling magnet represents a time-varying B field, which creates a time-varying flux through the copper tube. If the magnet actually stopped, there would be no eddy currents at all, and nothing to hold the magnet against gravity.

Once the magnet stops, the very force slowing it down ceases to exist. The flux in the tube is changing only because the magnet has some non-zero velocity. No emf, and therefore no eddy currents result from a stationary magnet giving a constant flux through

the tube.

Putting it another way: the force is due to the relative velocity of the magnet and the charges in the copper. The magnetic force is $F = qvB$, where v is the *relative* velocity o f the tube and magnet. If $v = 0$, there is no force - so if the magnet could actually be stopped, the force holding it up would go to zero, and it would fall again! Clearly, the answer is no.