UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair Fall 2008

Problem Set 1: random hints

Problem 2: The integration to get the electric force between the two rods should look something like this:

$$\vec{\mathbf{F}}_{12,\text{tot}} = \int_0^{L_1} dx_1 \int_{L_1+L}^{L_1+L_2+L} \frac{k_e\lambda_1\lambda_2}{(x_2-x_1)^2}\, dx_2\, \hat{\mathbf{x}} \tag{1}$$

We set that up in class, and we will review how to get this far again on Friday. If you perform the integrals correctly, you should end up with something like this:

$$\vec{\mathbf{F}}_{12,\text{tot}} = -k_e\lambda_1\lambda_2 \log\left[\frac{(L+L_2)(L+L_1)}{L(L_1+L_2+L)}\right] = -k_e\lambda_1\lambda_2\left[\ln\left(\frac{L_1+L}{L}\right)+\ln\left(\frac{L_2+L}{L_2+L_1+L}\right)\right]\hat{\mathbf{x}}$$

Now …what happens when $L_2 \gg L_1$? This is equivalent to saying $\frac{L_1}{L_2} \ll 1$. That means that if we can rewrite the expression above to contain the small fraction $\frac{L_1}{L_2}$, we should be able to use some sort of approximation. For example, the Taylor expansion for $\ln(1+x)$:

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \tag{2}$$

To first order then, $\ln(1+x) \approx x$. Thus, if we can break the force expression into bits that look like $\ln\left(1+\frac{L_1}{L_2}\right)$, we can approximate those terms by $\ln\left(1+\frac{L_1}{L_2}\right) \approx \frac{L_1}{L_2}$. Remembering how to manipulate logarithms, we first need to 'massage' our previous result a bit:[i]

$$\begin{aligned}
\vec{\mathbf{F}}_{12,\text{tot}} &= -k_e\lambda_1\lambda_2\left[\ln\left(\frac{L_2+L}{L_2+L_1+L}\right)+\ln\left(\frac{L_1+L}{L}\right)\right]\hat{\mathbf{x}} && (3)\\
&= -k_e\lambda_1\lambda_2\left[-\ln\left(\frac{L_1+L_2+L}{L_2+L}\right)+\ln\left(\frac{L_1+L}{L}\right)\right]\hat{\mathbf{x}} && (4)\\
&= -k_e\lambda_1\lambda_2\left[-\ln\left(1+\frac{L_1}{L_2+L}\right)+\ln\left(\frac{L_1}{L}+1\right)\right]\hat{\mathbf{x}} && (5)\\
& && (6)
\end{aligned}$$

[i]For example, recalling that $\ln(ab) = \ln a + \ln b$ and $\ln a = -\ln(1/a)$.

Now we are close. The second term in square brackets is fine as it is - it does not involve $\frac{L_1}{L_2}$ at all, so it is not necessary to approximation (yet). The first term will be nearly zero so long as $L_2 \gg L_1$. More explicitly, we can use our Taylor expansion from above:

$$-\ln\left(1+\frac{L_1}{L_2+L}\right) \approx -\frac{L_1}{L+L_2} = -\frac{1}{\frac{L_2}{L_1}+\frac{L}{L_1}} \approx 0 \tag{7}$$

So long as $L_2 \gg L_1$, the denominator of the right-most fraction above will be very large, which makes the whole term very small ...negligible in fact. Thus, to first order, we can approximate this term as zero. Putting it all together,

$$\begin{aligned} \vec{\mathbf{F}}_{12,\text{tot}} &= -k_e\lambda_1\lambda_2\left[-\ln\left(1+\frac{L_1}{L_2+L}\right)+\ln\left(\frac{L_1}{L}+1\right)\right]\hat{\mathbf{x}} && (8) \\ &\approx -k_e\lambda_1\lambda_2\left[0+\ln\left(\frac{L_1}{L}+1\right)\right]\hat{\mathbf{x}} && (9) \\ \Longrightarrow \quad \vec{\mathbf{F}}_{12,\text{tot}} &= -k_e\lambda_1\lambda_2\ln\left(\frac{L_1}{L}+1\right)\hat{\mathbf{x}} && (10) \end{aligned}$$

This is the desired result for the second part of the question.

Finally, we are asked to additionally consider $L \gg L_1$ and $L \gg L_2$. Under these conditions, $\frac{L_1}{L} \ll 1$, and $\frac{L_2}{L} \ll 1$. Starting from our last non-approximated expression,

$$\begin{aligned} \vec{\mathbf{F}}_{12,\text{tot}} &= -k_e\lambda_1\lambda_2\left[\ln\left(\frac{L_1+L}{L_2+L_1+L}\right)+\ln\left(\frac{L_2+L}{L}\right)\right]\hat{\mathbf{x}} && (11) \\ &= -k_e\lambda_1\lambda_2\left[-\ln\left(1+\frac{L_1}{L_2+L}\right)+\ln\left(\frac{L_1}{L}+1\right)\right]\hat{\mathbf{x}} && (12) \\ & && (13) \end{aligned}$$

We first notice that the second term is now readily approximated:

$$\ln\left(\frac{L_1}{L}+1\right) \approx \frac{L_1}{L} \tag{14}$$

We don't need to further simplify this yet; we will first plug it back into our original expression and simplify in the end. The first term we have already approximated before:

$$-\ln\left(\frac{L_1+L_2+L}{L_2+L}\right) \approx -\frac{L_1}{L+L_2} \tag{15}$$

Now our full expression becomes:

$$\vec{\mathbf{F}}_{12,\text{tot}} = -k_e\lambda_1\lambda_2 \left[-\ln\left(1 + \frac{L_1}{L_2 + L}\right) + \ln\left(\frac{L_1}{L} + 1\right)\right] \hat{\mathbf{x}} \tag{16}$$

$$\approx -k_e\lambda_1\lambda_2 \left[-\frac{L_1}{L + L_2} + \frac{L_1}{L}\right] \hat{\mathbf{x}} \tag{17}$$

$$= -k_e\lambda_1\lambda_2 \left[\frac{L_1 L_2}{L\left(L + L_2\right)}\right] \hat{\mathbf{x}} \tag{18}$$

$$= -k_e\lambda_1\lambda_2 \left[\frac{L_1 L_2}{L^2} \frac{1}{1 + \frac{L_2}{L}}\right] \hat{\mathbf{x}} \tag{19}$$

$$\approx -k_e\lambda_1\lambda_2 \frac{L_1 L_2}{L^2} \tag{20}$$

In the second to last line, we found one more negligibly small factor of L_2/L to get rid of. Finally, we note that the total charge on rod 1 is just $Q_1 = \lambda_1 L_1$, and for rod 2 the total charge is $Q_2 = \lambda_2 L_2$. Thus,

$$\vec{\mathbf{F}}_{12,\text{tot}} = -\frac{k_e Q_1 Q_2}{L^2} \hat{\mathbf{x}} \qquad (L_1, L_2 \ll L) \tag{21}$$