Problem Set 2: More Electrostatics

Instructions:

1. Answer all questions below. Show your work for full credit.
2. Due before 5pm, 5 September 2008
3. Email: pleclair@ua.edu; hard copies: Gallalee 206 or Bevill 228
4. You may collaborate, but everyone must turn in their own work

1. *Purcell 1.16* The sphere of radius a was filled with positive charge at uniform density ρ. Then a smaller sphere of radius $a/2$ was carved out, as shown in the figure, and left empty. What are the direction and magnitude of the electric field at A? At B?
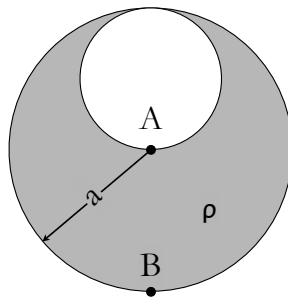

Figure 1: *Problem 1*

By the principle of superposition, a carved out sphere of positive charge is equivalent to a *filled* sphere of positive charge plus a smaller sphere of radius $a/2$ having a charge density $-\rho$ (see figure below).
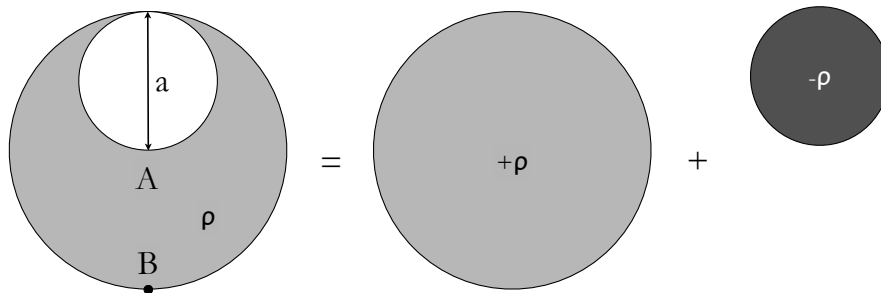

Figure 2: *Problem 1: using superposition*

Thus, the original, nasty-seeming problem is just equivalent to finding the electric field due to two spheres. Further, we earlier derived from Gauss' law that any spherically symmetric charge distribution is equivalent to a point charge, so long as you are considering points outside the distribution. Taking that into account, we actually only have to worry about the field due to point charges - even easier!

At point A, we are at the exact center of the larger sphere, and by symmetry, its net contribution to the electric field should be zero. We are also on the edge of the smaller sphere, and since we are outside its volume, we may consider it as a point charge a distance $a/2$ away. The magnitude of the equivalent point charge is the same as the total charge on the smaller sphere, which we can find by multiplying its volume by the charge density:

$$Q_{\text{small}} = \rho V = \rho \left(\frac{4}{3} \pi r^3 \right) = \frac{4}{3} \pi \rho \left(\frac{a}{2} \right)^3 = \frac{4\pi \rho a^3}{24} = \frac{\pi \rho a^3}{6}$$

Now we can find the electric field at A easily, noting that the field from the smaller sphere (and equivalent point charge) must be pointing in the vertical direction, which we will call the $\hat{\mathbf{y}}$ direction. Formally, we should say that the appropriate unit vector points in the $-\hat{\mathbf{y}}$ direction, from the charge's center to point A, but this is counteracted by the negative charge density:

$$\vec{\mathbf{E}}_{\text{tot}} = \vec{\mathbf{E}}_{\text{big}} + \vec{\mathbf{E}}_{\text{small}} = 0 + \frac{k_e Q_{\text{small}}}{(a/2)^2} \left(-\hat{\mathbf{y}} \right) = \frac{k_e \pi \rho a^3}{6 \, (a/2)^2} \, \hat{\mathbf{y}} = \frac{2 k_e \pi \rho a}{3} \, \hat{\mathbf{y}}$$

Point B? Now we are outside of both spheres, and both contribute to the electric field. The big sphere is equivalent to a point charge Q_{large} a distance a away. We can find Q_{big} just like we found Q_{small}:

$$Q_{\text{big}} = \rho V = \rho \left(\frac{4}{3} \pi r^3 \right) = \frac{4}{3} \pi \rho \, (a)^3 = \frac{4\pi \rho a^3}{3}$$

We still have the smaller sphere as well, now acting like a point charge a distance $3a/2$ away. The total electric field is just the superposition of that from the two effective point charges, we just need to keep in mind that the smaller points in the $\hat{\mathbf{y}}$ direction (up), while the larger points in the $-\hat{\mathbf{y}}$ direction (down), since their charges are opposite:

$$\begin{aligned}
\vec{\mathbf{E}}_{\text{tot}} = \vec{\mathbf{E}}_{\text{big}} + \vec{\mathbf{E}}_{\text{small}} &= \frac{k_e Q_{\text{big}}}{a^2} \left(-\hat{\mathbf{y}} \right) + \frac{k_e Q_{\text{small}}}{(3a/2)^2} \left(-\hat{\mathbf{y}} \right) \\
&= \frac{-4 k_e \pi \rho a^3}{3a^2} \, \hat{\mathbf{y}} + \frac{k_e \pi \rho a^3}{6 \, (3a/2)^2} \, \hat{\mathbf{y}} = \left(\frac{-4 k_e \pi \rho a}{3} + \frac{2 k_e \pi \rho a}{27} \right) \hat{\mathbf{y}} \\
&= \frac{-34 k_e \pi \rho a}{27} \, \hat{\mathbf{y}}
\end{aligned}$$

Any point inside the cavity? As a bonus, it turns out we can find the field at *any* point inside the cavity, it is not much harder at all. Take the origin to be the center of the large sphere. From this coordinate system, the center of the small sphere can be represented by a position vector $\frac{a}{2} \hat{\mathbf{y}}$. The electric fields generated by each sphere is most easily represented with their respective radial vectors $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$ as shown in the figure below - $\vec{\mathbf{r}}$ gives the field from the larger sphere, and $\vec{\mathbf{r}}'$ from the smaller. However, we can see from the figure that $\vec{\mathbf{r}} = \vec{\mathbf{r}}' + \frac{a}{2} \hat{\mathbf{y}}$. For any point within the hollow cavity, at an arbitrary point represented by $\vec{\mathbf{r}}$, we can easily find the amount of charge enclosed from the larger and smaller spheres:

$$Q_{\text{big}} = \frac{4}{3}\pi r^2 \rho$$

$$Q_{\text{small}} = \frac{4}{3}\pi r'^2 \rho$$

From the enclosed charges, we can now readily find the total field:

$$\vec{E}_{\text{tot}} = \vec{E}_{\text{big}} + \vec{E}_{\text{small}} = \frac{k_e Q_{\text{big}}}{r^2}\,(-\hat{y}) + \frac{k_e Q_{\text{small}}}{r'^2}\,(-\hat{y})$$

$$= \frac{4k_e \pi \rho}{3}\vec{r} + \frac{4k_e \pi \rho}{3}\left(\vec{r} - \frac{a}{2}\,\hat{y}\right) = \frac{2k_e \pi \rho a}{3}\,\hat{y}$$
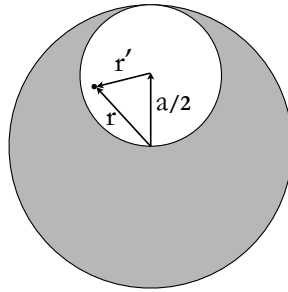

Figure 3: *Problem 1: using superposition*

2. *Purcell 1.33* Imagine a sphere of radius a filled with negative charge $-2e$ of uniform density. Imbed in this jelly of negative charge two protons and assume that in spite of their presence the negative charge remains uniform. Where must the protons be located so that the force on each of them is zero?

The protons should be placed at a distance $a/2$ from the center of the sphere of negative charge, symmetric about the sphere's midpoint.

The forces on the protons from each other will be equal and opposite. Therefore, the forces on them from the negative charge distribution must be equal and opposite also. This requires that they lie on a line through the center and are equidistant from the center. The force on each proton at radius r from the negative charge will be proportional to the amount of negative charge lying inside a sphere of radius r. For purposes of finding the electric field, we may treat all of this charge as if it were a point charge sitting in the center. We ignore all negative charge outside the radius of the proton positions. The negative charge inside the radius r is:

$$q(r) = -2e \left(\frac{\text{volume enclosed by sphere of radius } r}{\text{total volume}} \right)$$

$$= -2e \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a_0^3} \right)$$

$$= -\frac{2er^3}{a_0^3}$$

This charge $q(r)$ will give an electric field at the position of each proton. Since the charge $q(r)$ is spherically symmetric, it will be the same as the field from a point charge $q(r)$ at a distance r:

$$E(r) = \frac{k_e q(r)}{r^2} = -\frac{2ek_e r^3}{a_0^3 r^2} = -\frac{2k_e er}{a_0^3}$$

The force on each proton must be zero, the sum of the attractive force due to the charge $q(r)$ and the repulsive force from the other proton. Since a proton has a charge e, the attractive force is $qE(r)$. The repulsive force between the protons is easily calculated noting their charge e and separation $2r$:

$$\sum F = eE(r) + \frac{k_e e^2}{(2r)^2} = -\frac{2k_e e^2 r}{a_0^3} + \frac{k_e e^2}{4r^2} = 0$$

$$\implies \quad \frac{2k_e e^2 r}{a_0^3} = \frac{k_e e^2}{4r^2}$$

$$8k_e e^2 r^3 = e^2 k_e a_0^3$$

$$r = \frac{a_0}{2}$$
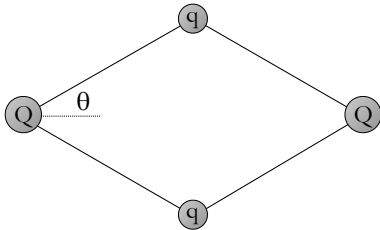
3. *Purcell 1.34* Four positively charged bodies, two with charge Q and two with charge q, are connected by four unstretchable strings of equal length. In the absence of external forces they assume the equilibrium configuration shown in the diagram. Show that $\tan^3 \theta = q^2/Q^2$. This can be done in two ways. You could show that this relation must hold if the total force on each body, the vector sum of string tension and electrical repulsion, is zero. Or you could write out the expression for the energy U of the assembly and minimize it.

Energy-based approach. Let the length of the strings connecting adjacent Q and q charges be d. Call the distance between the two Q charges horizontally l, and the vertical distance between the two q charges h. Using trigonometry, then:

$$\cos \theta = \frac{l/2}{d} = \frac{l}{2d}$$

$$\sin \theta = \frac{h/2}{d} = \frac{h}{2d}$$

The total potential energy of this system can be found by adding the potential energies of all unique pairs of charges, recalling that for a pair of point charges q_1 and q_2 separated by a distance r_{12} the potential energy is $k_e q_1 q_2 / r_{12}$. We also note that there are four equivalent pairings of the q and Q charges, all separated by a distance d.

$$
\begin{aligned}
U &= \frac{k_e Q^2}{l} + \frac{k_e q^2}{h} + 4 \frac{k_e Q q}{d} \\
&= \frac{k_e Q^2}{2d \cos \theta} + \frac{k_e q^2}{2d \sin \theta} + 4 \frac{k_e Q q}{d}
\end{aligned}
$$

Now we have the potential energy of the entire system as a function of the angle θ. Notice that the last term - the potential energy due to the charges fixed by the string - does not depend on θ, since the distance between adjacent q and Q charges is fixed by the length of the string.

At equilibrium, this potential energy should be at a *minimum* with respect to any angular variation. If $U(\theta)$ should be at a minimum, we must have $dU/d\theta = 0$:

$$
\begin{aligned}
\frac{dU}{d\theta} &= \frac{d}{d\theta} \left[\frac{k_e Q^2}{2d \cos \theta} + \frac{k_e q^2}{2d \sin \theta} + 4 \frac{k_e Q q}{d} \right] = 0 \\
\frac{dU}{d\theta} &= \frac{k_e Q^2}{d} \frac{\sin \theta}{2 \cos^2 \theta} - \frac{k_e q^2}{d} \frac{\cos \theta}{2 \sin^2 \theta} = 0
\end{aligned}
$$

Solving the equation above,

$$
\begin{aligned}
\frac{k_e Q^2}{d} \frac{\sin \theta}{2 \cos^2 \theta} &= \frac{k_e q^2}{d} \frac{\cos \theta}{2 \sin^2 \theta} \\
Q^2 \frac{\sin \theta}{2 \cos^2 \theta} &= q^2 \frac{\cos \theta}{2 \sin^2 \theta} \\
\frac{q^2}{Q^2} &= \frac{\sin^3 \theta}{\cos^3 \theta} = \tan^3 \theta
\end{aligned}
$$

Now, we have forgotten to be careful about one thing: is this a maximum, a minimum, or an inflection point? Setting $dU/d\theta = 0$ only ensures we have found one of the three; recall from Calculus I *which one it is* depends on the sign of $d^2 U/d\theta^2$. One can argue on physical grounds that it must be a minimum, but mathematically one must show that $d^2 U/d\theta^2 > 0$ to be certain.

Finding the second derivative of $U(\theta)$ is rather messy; you should find something like this once you grind through it:

$$
\frac{d^2 U}{d\theta^2} = \frac{d}{d\theta} \frac{dU}{d\theta} = \frac{k_e}{2d} \left(\frac{Q^2}{\cos \theta} + \frac{q^2}{\sin \theta} \right)
$$

For the present problem, the angle θ can only be between 0 and $90°$ without breaking the strings. The equation above is positive over that entire range of angles (though singular at the endpoints 0 and $90°$), which means that $d^2 U/d\theta^2 > 0$ for any physically possible choice of θ, and we have indeed found a *minimum* of potential energy, rather than a maximum or inflection point. Thus, our condition represents not only a minimum, but in fact a *stable* minimum energy situation.

Force-based approach. First, refer to the figure below, where we have drawn a simple free-body diagram about one of the q charges, and one of the Q charges.
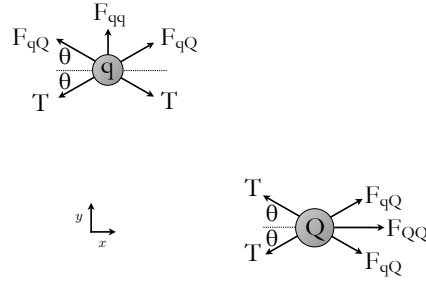
Figure 4: *Problem 3: free-body diagram*

We will call the force between adjacent Q and q charges F_{qQ}, the force between two q charges F_{qq}, the force between two Q charges F_{QQ}, and finally, the tension in the strings is T. All four strings must have the same tension, based on the symmetry of the system and Newton's third law. Since we know the distances between the charges (see above), we already know the electrostatic forces involved:

$$F_{QQ} = \frac{k_e Q^2}{4d^2 \cos^2 \theta}$$

$$F_{qq} = \frac{k_e q^2}{4d^2 \sin^2 \theta}$$

$$F_{qQ} = \frac{k_e Q q}{d^2}$$

Next, focus on one of the q charges. We will pick the uppermost one just to be concrete. As indicated in the free body diagram above, there will be two repulsive F_{qQ} forces from the two adjacent Q charges, and these forces will be directed at an angle θ above the indicated x axis. The string tensions will act opposite these two repulsive forces. At equilibrium, all forces must sum to zero. Summing the forces along the x and y axes, we have:

$$\text{on } q \text{ charge:} \quad \sum F_x = F_{qQ} \cos \theta - F_{qQ} \cos \theta + T \cos \theta - T \cos \theta = 0$$

$$\sum F_y = 2F_{qQ} \sin \theta - 2T \sin \theta + F_{qq} = 0$$

The forces in the x direction give us nothing useful, but those in the y direction do. Plugging in our expressions for the forces:

$$\frac{2k_e Q q}{d^2} \sin \theta - 2T \sin \theta + \frac{k_e q^2}{4d^2 \sin^2 \theta} = 0 \tag{1}$$

This looks useful, but it is not enough. We must eliminate the tension T, and the only way to get enough equations to do so is to also perform a force balance around one of the Q charges. Pick the rightmost one:

on Q charge: $\sum F_x = F_{QQ} + 2F_{qQ}\cos\theta - 2T\cos\theta$

$$\sum F_y = F_{qQ}\sin\theta - F_{qQ}\sin\theta + T\sin\theta - T\sin\theta = 0$$

This time, the y force balance is useless, but the x force balance gives us another interesting equation. Again, plugging in our expressions for the forces:

$$\frac{k_e Q^2}{4d^2\cos^2\theta} + \frac{2k_e Qq}{d^2}\cos\theta - 2T\cos\theta = 0 \tag{2}$$

Now: compare equations (1) and (2). We can solve both equations for $2T$, and eliminate the tensions entirely:

$$2T = \frac{2k_e Qq}{d^2} + \frac{k_e q^2}{4d^2\sin^3\theta} \qquad \text{from (1)}$$

$$2T = \frac{2k_e Qq}{d^2} + \frac{k_e Q^2}{4d^2\cos^3\theta} \qquad \text{from (2)}$$

$$\implies \quad \frac{k_e q^2}{4d^2\sin^3\theta} = \frac{k_e Q^2}{4d^2\cos^3\theta}$$

$$\frac{q^2}{Q^2} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$$

Thus, as it must, the force-based approach yields the same answer as the energy-based approach.