UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair Fall 2008

Problem Set 4: Solutions

1. *Jackson 1.6* Two long, cylindrical conductors of radius a_1 and a_2 are parallel and separated by a distance d which is large compared with either radius. Find the capacitance per unit length of the two conductors.

In order to find the capacitance per unit length between the two cylinders, we need to find the potential difference between them assuming that one carries a charge Q and the other $-Q$. Since the cylinders supposed to be "long," we will say instead that each has a charge *per unit length* of λ, with $Q=\lambda l$, where l is the total length of the cylinder. Since we want capacitance per unit length in the end, this will be convenient.

Let us choose a coordinate system which has its origin on the center of the first cylinder of radius a_1, which means the center of the second cylinder of radius a_2 is located at $r=d$. Take the $+\hat{\mathbf{r}}$ direction to be along a line connecting the center of the two conductors toward the conductor of radius r_2.

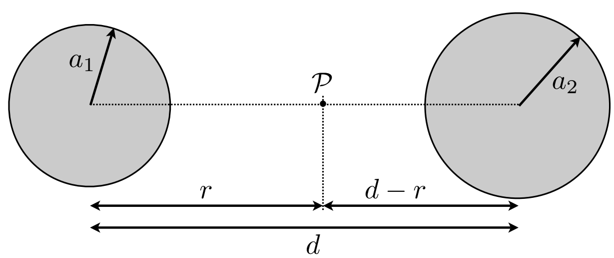

Figure 1: *Geometry for problem 1.*

Because the electric field obeys superposition, the total field at any point $\mathcal{P}$ is just the sum of the fields due to each conductor separately. From Gauss' law, we know that the field of each charged conductor is the same as that of a charged rod of length l and charge per unit length λ. Taking a point $\mathcal{P}$ along the axis connecting the center of the two conductors, we can find the total field readily:

$$
\begin{aligned}
\vec{\mathbf{E}}_{\text{left}} &= \frac{\lambda}{2\pi\epsilon_o r}\,\hat{\mathbf{r}} \\
\vec{\mathbf{E}}_{\text{right}} &= \frac{-\lambda}{2\pi\epsilon_o\,(d-r)}\,(-\hat{\mathbf{r}}) = \frac{\lambda}{2\pi\epsilon_o\,(d-r)}\,\hat{\mathbf{r}} \\
\vec{\mathbf{E}}_{\text{tot}} &= \frac{\lambda}{2\pi\epsilon_o}\left(\frac{1}{r}+\frac{1}{d-r}\right)\hat{\mathbf{r}}
\end{aligned}
$$

The potential difference between the two conductors can be found by integrating $\vec{\mathbf{E}}_{\text{tot}}\cdot d\vec{\mathbf{l}}$ over a path connecting the surface of the two conductors. Since $\vec{\mathbf{E}}$ is conservative, we can take any path we like, and the most natural choice is to take a straight line path along $\hat{\mathbf{r}}$ along a line connecting the centers of the two conductors (horizontal dashed line above), viz., $\hat{\mathbf{r}}\,dr$. Thus,

$$\begin{aligned}
\Delta V &= \int_{a_1}^{d-a_2} \vec{\mathbf{E}}_{\text{tot}} \cdot d\vec{\mathbf{l}} = \frac{\lambda}{2\pi\epsilon_o} \int_{a_1}^{d-a_2} \left(\frac{1}{r} + \frac{1}{d-r}\right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dr = \frac{\lambda}{2\pi\epsilon_o} \int_{a_1}^{d-a_2} \frac{1}{r} + \frac{1}{d-r}\, dr \\
&= \frac{\lambda}{2\pi\epsilon_o} \Big[\ln r - \ln(d-r)\Big]_{a_1}^{d-a_2} = \frac{\lambda}{2\pi\epsilon_o} \left[\ln\left(\frac{d-a_2}{a_1}\right) - \ln\left(\frac{a_2}{d-a_1}\right)\right] \\
\Longrightarrow \quad \Delta V &= \frac{\lambda}{2\pi\epsilon_o} \ln\left[\frac{(d-a_1)(d-a_2)}{a_1 a_2}\right] = \frac{Q}{2\pi\epsilon_o l} \ln\left[\frac{(d-a_1)(d-a_2)}{a_1 a_2}\right]
\end{aligned}$$

For the last line, we noted that $\lambda = Q/l$. Now using the definition of capacitance, $Q = C\Delta V$, or $C = Q/\Delta V$, we have for the total capacitance

$$C = \frac{2\pi\epsilon_o l}{\ln\left[\frac{a_1 a_2}{(d-a_1)(d-a_2)}\right]}$$

or, as asked, the capacitance per unit length C/l:

$$\frac{C}{l} = \frac{2\pi\epsilon_o}{\ln\left[\frac{a_1 a_2}{(d-a_1)(d-a_2)}\right]} \approx \frac{2\pi\epsilon_o}{\ln\left[\frac{a_1 a_2}{d^2}\right]} = \frac{\pi\epsilon_0}{\ln\left[\frac{\sqrt{a_1 a_2}}{d}\right]}$$

The last two steps are valid when $d \gg a_1, a_2$, and indicate that the capacitance is governed by the ratio of the *geometric mean* of the two conductors' radii to their separation.

2. *Serway 26.72,75* Find the equivalent capacitance for *both* combinations shown below. Be sure to consider the symmetry involved and the relative electric potential at different points in the circuits.

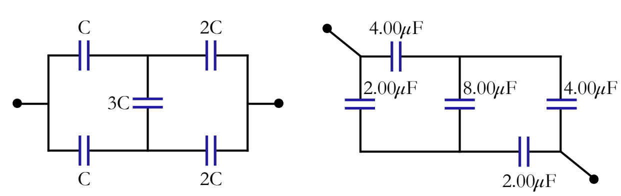

Figure 2: *Capacitor combinations.*

In the first case, by symmetry the $3C$ capacitor has no potential difference across it. Since capacitors in parallel have the same potential difference, both of the C capacitors have the same potential difference, and that means that both ends of the $3C$ capacitor are at the same potential. If that is true, then no charge is stored $Q = C\Delta V = 0$, and we can simply replace the $3C$ capacitor with a plain wire. Once we have done that, we have a pair of C capacitors connected in parallel, in series with a pair of $2C$ capacitors which are also connected in parallel. The two C's give an equivalent capacitance of $2C$, and the two $2C$'s give an equivalent capacitance of $4C$, so the whole circuit is equivalent to $2C$ in *series* with $4C$. This gives an equivalent capacitance of $4C/3$.

In the second case, we can apply the same argument to the $8\,\mu$F capacitor - it cannot have a potential difference across it, and it can therefore be replaced with a plain old wire. That leaves us with two pairs of $4\,\mu$F and $2\,\mu$F capacitors in parallel, each of which can be replaced with a single $6\,\mu$F equivalent capacitor. The whole circuit is then equivalent to two $6\,\mu$F capacitors in series, which is itself equivalent to a single $3\,\mu$F capacitor.

3. *Serway 26.64* A capacitor is constructed from two square plates of sides l and separation d. A material of dielectric constant κ is inserted a distance x into the capacitor, as shown below. **(a)** Find the equivalent capacitance of this device as a function of x. **(b)** Calculate the energy stored in the capacitor, letting ΔV represent the potential difference. **(c)** Find the direction and magnitude of the force exerted on the dielectric, assuming a constant potential difference ΔV. Ignore friction.

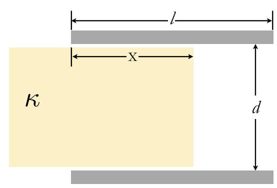

Figure 3: *Capacitor combinations.*

Whether the dielectric is there or not, we still have two plates held at a potential difference of ΔV, and inserting the dielectric will not changes this. Therefore, once we have the dielectric part way inserted, we can think of the situation as two capacitors in parallel - one filled with dielectric of width x and length l, the other without dielectric of width $l-x$ and length l. Both effective capacitors still have a potential difference of ΔV applied. We can calculate the capacitance of each, and the total equivalent, capacitance easily:

$$\begin{aligned}
C_{\text{filled}} &= \frac{\kappa\epsilon_o A_{\text{filled}}}{d} = \frac{\kappa\epsilon_o l x}{d} \\
C_{\text{empty}} &= \frac{\epsilon_o A_{\text{empty}}}{d} = \frac{\epsilon_o\,(l-x)\,l}{d} \\
C_{\text{equiv}} &= C_{\text{filled}} + C_{\text{empty}} = \frac{\epsilon_o l}{d}\left[(\kappa-1)\,x + l\right] = \frac{\epsilon_o l^2}{d}\left[\frac{x}{l}\,(\kappa-1)+1\right]
\end{aligned}$$

The last form is perhaps more pleasing, since it tells us the equivalent capacitance compared to having no dielectric at all ($\epsilon_0 l^2/d$). The total energy stored can now be found easily from the equivalent capacitance and voltage. Remember: an equivalent capacitance is equivalent in every way, so the energy in the equivalent capacitor is the same as that in the individual constituents.

$$U = \frac{1}{2}C_{\text{equiv}}\,(\Delta V)^2 = \frac{1}{2}\left[\frac{\epsilon_o l^2\,(\Delta V)^2}{d}\right]\left[\frac{x}{l}\,(\kappa-1)+1\right]$$

How about the force? We know the energy as a function of position, so this too is easy:

$$|\vec{\mathbf{F}}| = -\frac{dU}{dx} = -\frac{\epsilon_o l}{2d}\left(\kappa - 1\right)\left(\Delta V\right)^2$$

The force in this case acts to the right, pulling the sheet in. This is because the dielectric is polarizable - the top surface of the dielectric near the positive plate would develop a negative charge, and be attracted to the capacitor, pulling the dielectric farther in. Another way to think about it is that the capacitor stores more energy with the dielectric inside, so it will try to pull it in and maximize its stored energy.[i]

4. *Serway 26.65* Using the same figure as the previous question, imagine now that the block being inserted is metal, rather than dielectric. Assume that $d \ll l$, and that the plates carries charges $+Q_o$ and $-Q_o$. **(a)** Calculate the stored energy as a function of x. **(b)** Find the direction and magnitude of the force acting on the metallic block. *Hint: a metal can be considered a perfect dielectric, $\kappa \to \infty$, which allows no electric field to penetrate it.*

Once again, we can consider this to be two capacitors in parallel: one filled with metal, and the other empty. We have to imagine that the metal fills the left half of the capacitor, but doesn't touch the plates - otherwise, we would short out the capacitor and no charge would be stored anywhere. Most likely, shorting out the capacitor like this would cause something to break in a Bad way.

Anyway: the metal-filled half of the capacitor doesn't store any energy at all. A metal can be considered analogous to a dielectric with $\kappa \to \infty$ Thus, *for a fixed amount of charge Q_0 and a fixed voltage ΔV* the stored energy $Q_o^2/2C \to 0$. Only the unfilled portion of the capacitor stores any energy. If each plate in total has a charge Q_o, then the unfilled portion of the plate must store charge proportional to the uncovered area of the plate:

$$Q_{\text{unfilled area}} = \frac{A_{\text{unfilled}}}{A_{\text{total}}} Q_o = \frac{(l-x)\,x}{l^2} = \frac{l-x}{l} Q_o$$

The capacitance of the unfilled region we have already calculated above. The stored energy in the unfilled region, and thus the whole capacitor, is thus

$$U = \frac{Q^2_{\text{unfilled area}}}{2C_{\text{unfilled}}} = \frac{\left(\frac{l-x}{l}Q_o\right)^2}{2\frac{\epsilon_o (l-x) l}{d}} = \frac{d\,(l-x)\,Q_o^2}{2\epsilon_o l^3} = \frac{Q_o^2 d}{2\epsilon_o l^2}\left[\frac{l-x}{l}\right]$$

The last form makes it clear that the energy stored is that of a completely unfilled capacitor, times the fraction $(l-x)/l$ that the metal fills the capacitor. Once again, we find the force from the gradient of the potential energy:

$$|\vec{\mathbf{F}}| = -\frac{dU}{dx} = \frac{Q_o^2 d}{2\epsilon_o l^2}$$

[i] In fact, the presence of a force at all is entirely due to the fringing fields at the edges of the plates. After all, if not for that region, the field would be perpendicular to the required direction of the force, and no work could be done! For a good discussion of what is really going on, see *The Feynman Lectures on Physics*, vol. II, ch. 10, pp. 8-9, or an excellent article in the *American Journal of Physics*, vol. 52, pp. 515-518, 1984, online here: `http://link.aip.org/link/?AJPIAS/52/515/1`. The link will only work on campus.

In this case, the force is actually to the left - the metal plate is pushed out of the capacitor, because the capacitor stores more energy without it. Thus, the capacitor will expel the plate to maximize its stored energy.

5. *Purcell 3.5.* A charge Q is located h meters above a conducting plane. How much work is required to bring this charge out to an infinite distance above the plane? *Hint: Consider the method of images.*

There are two ways to approach this one. First, the presence of the conducting plane a distance h from the positive charge means that this problem is equivalent to a dipole of spacing $2h$, as we found in exercise 3 (see figure below).[ii] Thus, we need to find the work required to move a charge q from a distance $2h$ from a second charge $-q$ out to infinity. Let the origin be halfway between the real charge q and its image charge $-q$. The work required to move the positive charge away is:

$$W = -\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}} = -\int_h^\infty q\vec{\mathbf{E}} \cdot \hat{\mathbf{r}}\, dr = -\int_h^\infty \frac{kq^2}{(2r)^2}\, dr = \frac{kq^2}{4}\left[\frac{1}{z}\right]_h^\infty$$

$$\implies \quad W = \frac{-kq^2}{4h}$$

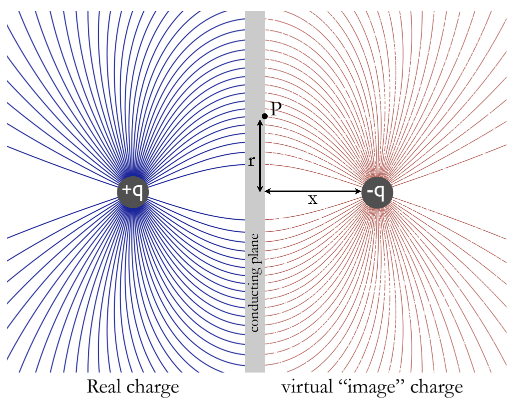

Figure 4: *The field of a charge near a conducting plane, found by the method of images.*

A much sneakier way is to realize that the energy in the electric field must just be half of that due to a real dipole. We learned that the energy of a charge configuration can be found by integrating the electric field over all space, $U \sim \int E^2\, dV$. The electric field due to our point charge above the conducting plane is identical to that of a dipole, but *only for the region of space above the plane.* Below the plane, half of all volume in space, the field is zero. We can immediately conclude that the point charge and infinite plane have half as much energy, since there is no field below the conducting plane. The energy of a dipole we found already when we considered point charges. If the dipole spacing is $2h$,

$$U_{\text{dip}} = \frac{kq^2}{2h} \qquad \implies \qquad U = \frac{1}{2}U_{\text{dip}} = \frac{kq^2}{4h}$$

[ii]See `http://faculty.mint.ua.edu/~pleclair/ph106/Exercises/EX3_SOLN.pdf`

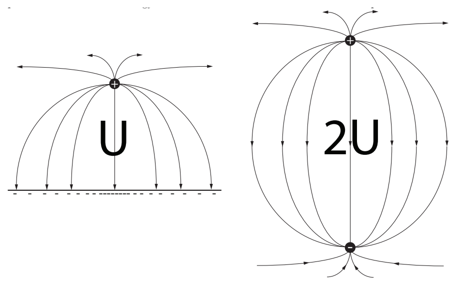

Figure 5: *The field energy of our single charge with a conducting plate is half that of a dipole.*

6. Two capacitors, one charged and the other uncharged, are connected in parallel. **(a)** Prove that when equilibrium is reached, each carries a fraction of the initial charge equal to the ratio of its capacitance to the sum of the two capacitances. **(b)** Show that the final energy is less than the initial energy, and derive a formula for the difference in terms of the initial charge and the two capacitances.

This problem is easiest to start if you approach it from a conservation of energy & charge point of view. We have two capacitors. Initially, one capacitor stores a charge Q_{1i}, while the other is empty, $Q_{2i}=0$. After connecting them together in parallel, some charge leaves the first capacitor and goes to the second, leaving the two with charges Q_{1f} and Q_{2f}, respectively. Now, since there were no sources hooked up, and we just have the two capacitors, the total amount of charge must be the same before and after we hook them together:

$$\begin{aligned} Q_i &= Q_f \\ Q_{1i} + Q_{2i} &= Q_{1f} + Q_{2f} \\ Q_{1i} &= Q_{1f} + Q_{2f} \end{aligned}$$

We also know that if two capacitors are connected in parallel, they will have the same voltage ΔV across them:

$$\Delta V_f = \frac{Q_{1f}}{C_1} = \frac{Q_{2f}}{C_2}$$

The fraction of the total charge left on the first capacitor can be found readily combining what we have:

$$\frac{Q_{1f}}{Q_i} = \frac{Q_{1f}}{Q_{1i}} = \frac{Q_{1f}}{Q_{1f}+Q_{2f}} = \frac{Q_{1f}}{Q_{1f}+\frac{C_2}{C_1}Q_{1f}} = \frac{C_1 Q_{1f}}{C_1 Q_{1f} + C_2 Q_{1f}} = \frac{C_1}{C_1+C_2}$$

The second capacitor must have the rest of the charge:

$$\frac{Q_{2f}}{Q_i} = 1 - \frac{C_1}{C_1+C_2} = \frac{C_2}{C_1+C_2}$$

That was charge conservation. We can also apply energy conservation, noting that the energy of a charged capacitor is $Q^2/2C$:

$$
\begin{aligned}
E_i &= E_f \\
\frac{Q_{1i}^2}{2C_1} &= \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2}
\end{aligned}
$$

The final energy can be simplified using the result of the first part of the problem - we note that $Q_{1f} = Q_i C_1 / (C_1 + C_2)$ and $Q_{2f} = Q_i C_2 / (C_1 + C_2)$

$$
\begin{aligned}
E_f &= \frac{Q_{1f}^2}{2C_1} + \frac{Q_{2f}^2}{2C_2} \\
&= \left(\frac{Q_i C_1}{C_1 + C_2}\right)^2 \frac{1}{2C_1} + \left(\frac{Q_i C_2}{C_1 + C_2}\right)^2 \frac{1}{2C_2} \\
&= \frac{Q_i^2 C_1}{2\,(C_1 + C_2)^2} + \frac{Q_i^2 C_2}{2\,(C_1 + C_2)^2} \\
&= \frac{Q_i^2\,(C_1 + C_2)}{2\,(C_1 + C_2)^2} = \frac{Q_i^2}{2\,(C_1 + C_2)} \\
&= \frac{Q_i^2}{2C_1}\left(\frac{C_1}{C_1 + C_2}\right) = E_i\left(\frac{C_1}{C_1 + C_2}\right)
\end{aligned}
$$

Thus, the final energy will be less than the initial energy, by a factor $C_1 / (C_1 + C_2) < 1$.