UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair Fall 2008

Problem Set 5: Solutions

1. *Serway 28.62* Two resistors R_1 and R_2 are in parallel with each other. Together they carry total current I. **(a)** Determine the current in each resistor. **(b)** Prove that this division of the total current I between the two resistors results in less power delivered to the combination than any other. It is a general principle that current in a dc circuit distributes itself so that the total power delivered is a minimum.

a: Since the two resistors are in parallel, they will have the same voltage drop, but in general different currents (unless $R_1 = R_2$, in which case the currents are the same). Let I_1 and I_2 be the currents in resistors R_1 and R_2, respectively, with the total current then given by $I = I_1 + I_2$. Given the current through each resistor, we can readily calculate the voltage drop on each, which again must be the same for both resistors:

$$\begin{aligned}
\Delta V_1 &= I_1 R_1 \\
\Delta V_2 &= I_2 R_2 \\
\Delta V_1 &= \Delta V_2 \qquad \Longrightarrow \qquad I_1 R_1 = I_2 R_2
\end{aligned}$$

We can find the current in each resistor from the known total current I by noting that $I = I_1 + I_2$, and thus $I_2 = I - I_1$

$$\begin{aligned}
I_1 R_1 &= I_2 R_2 \\
I_1 R_1 &= (I - I_1) R_2 \\
I_1 R_1 + I_1 R_2 &= I R_2 \\
\Longrightarrow \qquad I_1 &= \frac{R_2}{R_1 + R_2} I = \left[\frac{1}{\frac{R_1}{R_2} + 1}\right] I
\end{aligned}$$

Thus, the fraction of the total current in first resistor depends on the ratio of the two resistors. The larger resistor 2 is, the more current that will flow through the first resistor - not shocking! Given the above expression for I_1, we can easily find I_2 from $I_2 = I - I_1$, which yields

$$I_2 = \frac{R_1}{R_1 + R_2} I = \left[\frac{1}{\frac{R_2}{R_1} + 1}\right] I$$

Our derivation of the currents in each resistor has so far only relied on conservation of energy (components in parallel have the same voltage) and conservation of charge ($I = I_1 + I_2$), we have not invoked any special "laws" about combining parallel resistors. In fact, that is what we have just derived!

b: Now, what about the power then? We want to find the distribution of currents that results in minimum power dissipation in the most general way, specifically *not* using the results of the previous portion of this problem. We will only assume that resistors R_1 and R_2 carry currents I_1 and I_2, respectively, and that these two currents add up to the total current, $I = I_1 + I_2$. In other words, we assume conservation of charge, but do not even restrict ourselves by applying conservation of energy. In this most general case, the total power dissipated is just the sum of the individual power dissipations in the two resistors:

$$\mathscr{P}_{\text{tot}} = \mathscr{P}_1 + \mathscr{P}_2 = I_1^2 R_1 + I_2^2 R_2 = I_1^2 R_1 + (I - I_1)^2 R_2 = I_1^2 (R_1 + R_2) + I^2 R_2 - 2IR_2 I_1$$

For the last part, we did invoke our conservation of charge equation ($I = I_1 + I_2$). What to do next? We have now the total power $\mathscr{P}_{\text{tot}}$ in both resistors as a function of the current in R_1. If we minimize the total power with respect to I_1, we will have found the value of I_1 which leads to the minimum power dissipation. Since I_2 is then fixed by the total current I once we know I_1, $I_2 = I - I_1$, this is sufficient to establish the values of *both* I_1 and I_2 that lead to minimum power dissipation. Of course, to find the minimum of $\mathscr{P}_{\text{tot}}$ for any value of I_1, we need to take a derivative[i] ...

$$\frac{d\mathscr{P}_{\text{tot}}}{dI_1} = 2I_1 (R_1 + R_2) - 2IR_2 = 0$$
$$\Longrightarrow I_1 = \frac{R_2}{R_1 + R_2} I$$

Lo and behold, the minimum power dissipation occurs when the currents are distributed exactly as we expect for parallel resistors. At this point, you can easily find I_2 as well, given $I_2 = I - I_1$. The general rule is that current in a dc circuit distributes itself such that the total power dissipation is minimum, which we will not prove here.

Of course ... hold on a minute. We missed one small point: by finding $\frac{d\mathscr{P}_{\text{tot}}}{dI_1}$ and setting it to zero, we have certainly found an extreme value for $\mathscr{P}_{\text{tot}}$. We did not prove whether it is a maximum or a minimum however! This is important ... so we should apply the *second derivative* test.
Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a} = 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a} < 0$, you have a maximum, if $d^2f/dx^2|_{x=a} > 0$ you have a minimum, and if $d^2f/dx^2|_{x=a} = 0$, the test basically wasted your time. Anyway:

$$\frac{d^2\mathscr{P}_{\text{tot}}}{dI_1^2} = 2(R_1 + R_2) > 0$$

Since resistances are always positive, we have in fact found a minimum of $\mathscr{P}_{\text{tot}}$. Sweet.

Don't let that lull you into complacency, however: you need to apply the second derivative test to see what you've really found, and not just take derivatives and set them to zero all willy-nilly.

2. *Purcell 4.x* Show that if a battery of fixed internal voltage ΔV and internal resistance R_i is connected to a variable external resistance R the maximum power is delivered to the external resistor when $R_i = R$.

The circuit we are considering is just a series combination of the (ideal) internal voltage source ΔV, the

[i] Keep in mind that the total current I is fixed, so $dI/dI_1 = 0$. And, yes we should technically be using partial derivatives here (differentiating with respect to I_1 while holding everything else constant), but since only I_1 varies it is not really crucial. Plus, I think that would be using math beyond the course prerequisites.

internal resistance R_i, and the external resistance R. Since all the elements are in series, the current is the same in each, which we will call I. Applying Kirchhoff's "loop" rule (i.e., conservation of energy),

$$\Delta V - IR - IR_i = 0 \qquad \Longrightarrow \qquad I = \frac{\Delta V}{R + R_i}$$

The power delivered *to the external resistor* $\mathscr{P}_R$ is just $I^2 R$:

$$\mathscr{P}_R = I^2 R = \left(\frac{\Delta V}{R + R_i}\right) R = (\Delta V)^2 \frac{R}{(R + R_i)^2}$$

Similar to the last problem, we can maximize the power delivered to the resistor R by differentiating the power with respect to R and setting the result equal to zero:

$$\begin{aligned}
\frac{d\mathscr{P}_R}{dR} &= \frac{d}{dR}\left[(\Delta V)^2 \frac{R}{(R+R_i)^2}\right] = (\Delta V)^2 \left[\frac{1}{(R+R_i)^2} + \frac{-2R}{(R+R_i)^3}\right] = 0 \\
\Longrightarrow \qquad \frac{1}{(R+R_i)^2} &= \frac{2R}{(R+R_i)^3} \\
1 &= \frac{2R}{R+R_i} \\
R + R_i &= 2R \\
\Longrightarrow \qquad R_i &= R
\end{aligned}$$

The power is indeed extremal when the external resistor matches the internal resistance of the battery. Again, we apply the second derivative test to see whether this is a maximum or a minimum. First, let's find the second derivative, and simplify it as much as possible.

$$\begin{aligned}
\frac{d^2\mathscr{P}_R}{dR^2} &= \frac{d}{dR}\left[(\Delta V)^2 \left(\frac{1}{(R+R_i)^2} - \frac{2R}{(R+R_i)^3}\right)\right] \\
&= (\Delta V)^2 \left[\frac{-2}{(R+R_i)^3} - \frac{2}{(R+R_i)^3} + \frac{6R}{(R+R_i)^4}\right] \\
&= (\Delta V)^2 \left[\frac{-4}{(R+R_i)^3} + \frac{6R}{(R+R_i)^4}\right] \\
&= \frac{(\Delta V)^2}{(R+R_i)^3}\left[\frac{6R}{R+R_i} - 4\right]
\end{aligned}$$

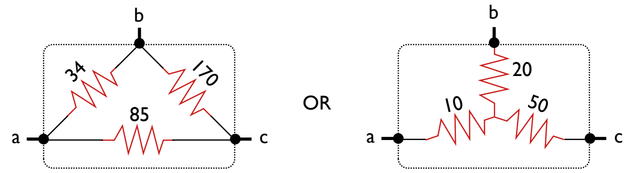
We are concerned with the value of the second derivative at the point $R = R_i$, the extreme point:

$$\left.\frac{d^2\mathscr{P}_R}{dR^2}\right|_{R=R_i} = \frac{(\Delta V)^2}{(R+R)^3}\left[\frac{6R}{R+R} - 4\right] = \frac{(\Delta V)^2}{8R^3}\left[3 - 4\right] = -\frac{(\Delta V)^2}{8R^3} < 0$$

The second derivative is always negative, so we have found a maximum: the power delivered to the external resistor is maximum when $R = R_i$.

3. *Purcell 4.x* A black box with three terminals, a, b, and c, contains nothing but three resistors and

connecting wire. Measuring the resistance between pairs of terminals, you measure $R_{ab} = 30\,\Omega$, $R_{ac} = 60\,\Omega$, and $R_{bc} = 70\,\Omega$. Show that the box could be either of those below.



First, consider the box on the left side. Measuring between points a b (with point c unconnected), we would find a $34\,\Omega$ resistor in parallel with a series combination of $85\,\Omega$ and $170\,\Omega$. The series combination of $85\,\Omega$ and $170\,\Omega$ just gives $255\,\Omega$, and that in parallel with $34\,\Omega$ gives

$$R_{ab} = \frac{(34\,\Omega)\,(255\,\Omega)}{34\,\Omega + 255\,\Omega} = 30\,\Omega$$

Similarly, we can find $R_{bc} = 70\,\Omega$ and $R_{ac} = 60\,\Omega$.

For the box on the right, if we connect only points a and b then the $50\,\Omega$ resistor does nothing - it has one end disconnected. Thus, $R_{ab} = 30\,\Omega$, and similarly $R_{bc} = 70\,\Omega$, $R_{ac} = 60\,\Omega$. Since a measurement of the resistance between any two terminals yields the same result, the two boxes are indistinguishable.

Establishing the equivalence of these two configurations is more generally known as a "Y-Δ" transformation: `http://en.wikipedia.org/wiki/Y-%CE%94_transform`

4. *Purcell 4.8* A copper wire 1 km long is connected across a 6 V battery. The resistivity of the copper is $1.7 \times 10^{-8}\,\Omega\,\mathrm{m}$, and the number of conduction electrons per cubic meter is 8×10^{28}. **(a)** What is the drift velocity of the conduction electrons under these circumstances? **(b)** How long does it take an electron to drift once around the circuit?

Given a conductor of cross-sectional area A and length L, the current will be

$$I = \frac{V}{R} = \frac{VA}{\rho L}$$

and the current density

$$J = \frac{I}{A} = \frac{V}{\rho L}.$$

The current density is also the charge density n times the drift speed v_d, and the charge density is ne where n is the number of electrons per unit volume:

$$v_d = \frac{I}{nqA} = \frac{J}{ne} = \frac{V}{\rho L n e}$$

$$v_d = \frac{6\,V}{(1.7\times10^{-8}\,\Omega\,\mathrm{m})(10^3\,\mathrm{m})(8\times10^{28}\,\mathrm{m}^{-3})(1.6\times10^{-19}\,\mathrm{C})} = 2.8\times10^{-5}\,\mathrm{m/s}$$

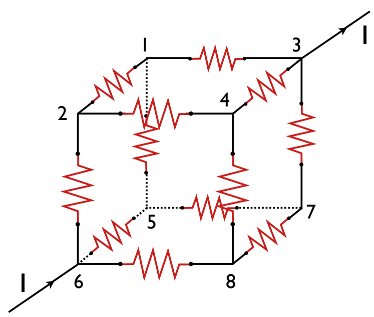
Given the velocity v_d and the length of the conductor L, it is easy to figure out how long the journey will take:

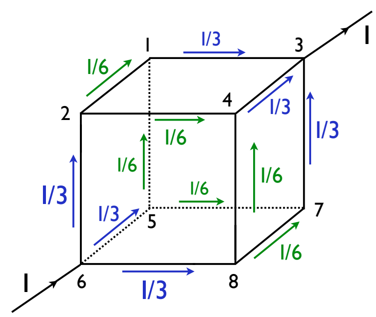
$$t = \frac{L}{v_d} = \frac{10^3\,\mathrm{m}}{2.8\times10^{-5}\,\mathrm{m/s}} = 3.6\times10^7\mathrm{s} \simeq 1\,\mathrm{year}$$

5. Each of the twelve edges of the cube is a resistor R. What is the resistance between two *opposite corners*?

Solutions to this one are all over the internet. For example:
`http://www.physics.ucsb.edu/~lecturedemonstrations/Composer/Pages/64.42.html`

The following figures might help you visualize what is going on:





Take any corner, and let a current I enters at that node. It then has a choice of three directions to go along the cube edges. Each path has an identical resistance, so the current must split up evenly with $I/3$ through each. Likewise, the current exiting the node on the opposite corner comes from three resistors, each with a current $I/3$. This leaves 6 resistors in the middle that must share the whole current I. Because each of those is also identical, they must each have current $I/6$.

The voltage drop between the two corner nodes can be found by following (any) path from one to the other and adding up the individual voltage drops. The most direct path has us going first through a resistor R carrying $I/3$, then through a resistor R carrying $I/6$, and finally exiting through a resistor R carrying $I/3$. Thus:

$$\Delta V = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}RI$$

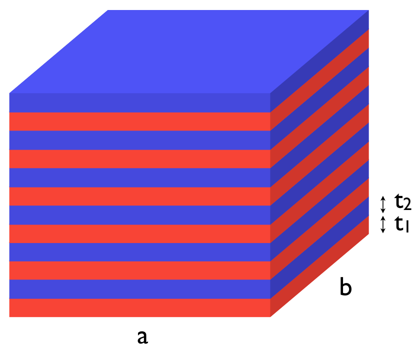
$$R_{eq} = \frac{\Delta V}{I} = \frac{5}{6}R$$

You can find the solutions for all Platonic solids and all node spacings here:

`http://cs.nyu.edu/%7Egottlieb/tr/2008-may-jun-2.pdf`

6. *Purcell 4.7* A laminated conductor was made by depositing, alternately, layers of silver 10 nm thick and layers of tin 20 nm thick. The composite material, consdiered on a larger scale, may be considered a homogeneous but anisotropic material with electrical conductivity $\sigma_\perp$ for currents perpendicular to the planes of the layers, and a different conductivity $\sigma_{||}$ for currents parallel to that plane. Given that the conductivity of silver is 7.2 times that of tin, find the ratio $\sigma_\perp/\sigma_{||}$.

First, let us sketch out the situation given:



We are not told how many layers of each type we have, and it will not matter in the end. For now, however, assume we have n_1 layers of tin of conductivity σ_1 and n_2 layers of silver of conductivity σ_2. Instead of conductivity, we can equivalently use resistivity ϱ when it is more convenient, with $\varrho = 1/\sigma$. We will also say the tin layers have thickness t_1, and the silver layers thickness t_2. The total thickness of our entire multi-layer stack is then $t_{\text{tot}} = n_1 t_1 + n_2 t_2$.

First, consider the perpendicular conductivity, the case where we pass current upward through the stack, perpendicular to the planes of the layers. When a current is flowing, electrons pass through each layer in sequence, and we can consider the stack of layers to be resistors in series. If the layers have an area of $A = ab$ (see the Figure above) and a thickness t_1 or t_2, we can readily calculate the resistance presented by a single tin or silver layer with current perpendicular to the layers:

$$\begin{aligned} R_{1,\perp} &= \frac{\varrho_1 t_1}{A} = \frac{t_1}{\sigma_1 A} \\ R_{2,\perp} &= \frac{\varrho_2 t_2}{A} = \frac{t_2}{\sigma_2 A} \end{aligned}$$

For reasons that should become apparent below, it will be convenient in this case to work with the resistivity rather than the conductivity, and invert the result later. The total resistance of the stack is then just a series combination of n_1 resistors of value R_1 and n_2 resistors of value R_2:

$$R_{\text{tot},\perp} = n_1 R_{1,\perp} + n_2 R_{2,\perp} = \frac{1}{A}\left(\rho_1 t_1 n_1 + \rho_2 t_2 n_2\right)$$

If we measure the whole stack and find this resistance, we can define an effective resistivity or conductivity for the whole stack in terms of the total resistance and total thickness of the multilayer. If the resistivity of the whole stack for perpendicular currents is $\varrho_\perp = 1/\sigma_\perp$, then:

$$R_{\text{tot},\perp} = \frac{\varrho_\perp t_{\text{tot}}}{A} \qquad \Longrightarrow \qquad \varrho_\perp = \frac{AR_{\text{tot},\perp}}{t_{\text{tot}}}$$

Now we just need to plug in what we know and simplify ...

$$\varrho_\perp = \frac{AR_{\text{tot},\perp}}{t_{\text{tot}}} = \frac{A}{t_{\text{tot}}}\left[\frac{1}{A}\left(\rho_1 t_1 n_1 + \varrho_2 t_2 n_2\right)\right]$$
$$\varrho_\perp = \frac{\varrho_1 t_1 n_1 + \varrho_2 t_2 n_2}{t_{\text{tot}}} = \frac{\varrho_1 t_1 n_1 + \rho_2 t_2 n_2}{n_1 t_1 + n_2 t_2}$$

We can simplify this somewhat if we realize that we have the *same* number of silver and tin layers - we are told that the layers are deposited alternatingly. If we let $n_1 = n_2 \equiv n_{bi}$, meaning we count the number of bilayers instead, then $t_{\text{tot}} = n_{bi}\left(t_1 + t_2\right)$, and

$$\varrho_\perp = \frac{n_{bi}\varrho_1 t_1 + n_{bi}\varrho_2 t_2}{n_{bi} t_1 + n_{bi} t_2} = \frac{\varrho_1 t_1 + \varrho_2 t_2}{t_1 + t_2}$$

This is a nice, simple result: for current perpendicular to the planes, the effective *resistivity* is just a thickness-weighted average of the resistivities of the individual layers. Given the resistivity in the perpendicular case, we can now find the conductivity $\sigma_\perp$

$$\sigma_\perp = \frac{1}{\varrho_\perp} = \frac{t_1 + t_2}{\varrho_1 t_1 + \varrho_2 t_2} = \frac{t_1 + t_2}{\frac{t_1}{\sigma_1} + \frac{t_2}{\sigma_2}} = \frac{\sigma_1 \sigma_2 \left(t_1 + t_2\right)}{\sigma_2 t_1 + \sigma_1 t_2}$$

As a consistency check, we can take a couple of limiting cases. First, let $\sigma_1 = \sigma_2 \equiv \sigma$. This corresponds to a homogeneous lump of a single material, and we find $\sigma_\perp = \sigma$, as expected. Next, we can check for $\sigma_1 = 0$. In this case, one layer is not conducting at all, and since the layers are in series, this means no current flows through the stack at all, and $\sigma_\perp = 0$ as expected. Finally, we notice that the number of bilayers is irrelevant. Since the layers do not affect each other in our simple model of conduction, there is no reason to expect otherwise. So far so good. What other limiting cases can you check?

Next, let us consider current flowing parallel to the plane of the layers, from (for example) left to right in the figure above. Now the stack looks like many parallel resistors. A single tin layer of thickness t_1 and in-plane dimensions a and b now presents a resistance

$$R_{1,||} = \frac{\varrho_1 a}{t_1 b} = \frac{a}{t_1 b \sigma_1}$$

Similarly, each silver layer presents a resistance

$$R_{2,||} = \frac{\varrho_2 a}{t_2 b} = \frac{a}{t_2 b \sigma_2}$$

One bilayer of silver and tin means a *parallel* combination of these two resistances:

$$\frac{1}{R_{bi,||}} = \frac{1}{R_{1,||}} + \frac{1}{R_{2,||}} = \frac{b}{a}\left(t_1\sigma_1 + t_2\sigma_2\right)$$

If we have n_{bi} bilayers, then the total equivalent resistance is easily found:

$$\frac{1}{R_{\text{tot},||}} = n_{bi}\frac{1}{R_{bi,||}} = n_{bi}\frac{b}{a}\left(t_1\sigma_1 + t_2\sigma_2\right)$$

Given the total resistance, we can now calculate the conductivity directly (in this case, first finding the resistivity does not save us any algebra), noting that the length of the whole stack along the direction of the current is just a, and the cross-sectional area is $bt_{\text{tot}} = bn_{bi}\left(t_1 + t_2\right)$:

$$\sigma_{||} = \frac{1}{\varrho_{||}} = \frac{a}{n_{bi}\left(t_1 + t_2\right) bR_{\text{tot},||}} = \frac{a}{n_{bi}\left(t_1 + t_2\right) b}\left(\frac{b}{a}\right) n_{bi}\left(t_1\sigma_1 + t_2\sigma_2\right) = \frac{t_1\sigma_1 + t_2\sigma_2}{t_1 + t_2}$$

Again, a sensible result: the effective *conductivity* for current parallel to the planes is just a thickness-weighted average of the conductivities of the individual layers. Again, you can convince yourself with a couple of limiting cases that this result makes some sense.

Now that we have both parallel and perpendicular conductivities, we can easily find the anisotropy $\sigma_{\perp}/\sigma_{||}$.

$$\sigma_{\perp}/\sigma_{||} = \frac{\varrho_{||}}{\varrho_{\perp}} = \frac{\sigma_1\sigma_2\left(t_1 + t_2\right)}{\sigma_2 t_1 + \sigma_1 t_2}\frac{t_1 + t_2}{\sigma_1 t_1 + \sigma_2 t_2} = \frac{\sigma_1\sigma_2\left(t_1 + t_2\right)^2}{\left(t_1\sigma_1 + t_2\sigma_2\right)\left(t_1\sigma_2 + t_2\sigma_1\right)}$$

Finally, we are given that the conductivity of silver is 7.2 times that of tin, and the tin layers' thickness is twice that of the silver. Thus, $t_1 = 2t_2$ and $\sigma_2 = 7.2\sigma_1$. The actual values and units do not matter, as this is a dimensionless ratio (you should verify this fact . . .), and you should find $\sigma_{\perp}/\sigma_{||} \approx 0.457$.

And, once again, you can check that for $\sigma_1 = \sigma_2$, we have $\sigma_{\perp}/\sigma_{||} = 1$, as it must if both materials are the same.