UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair Fall 2008

Problem Set 6: Solutions

1. *Serway 29.55* Protons having a kinetic energy of 5.00 MeV are moving in the positive x direction and enter a magnetic field $\vec{\mathbf{B}} = 0.0500\,\hat{\mathbf{k}}\,\mathrm{T}$ directed out of the plane of the page and extending from $x=0$ to $x=1.00\,\mathrm{m}$, as shown below. **(a)** Calculate the y component of the protons' momentum as they leave the magnetic field. **(b)** Find the angle α between the initial velocity vector after the beam emerges from the field. Note that $1\,\mathrm{eV} = 1.60\times 10^{-19}\,\mathrm{J}$.

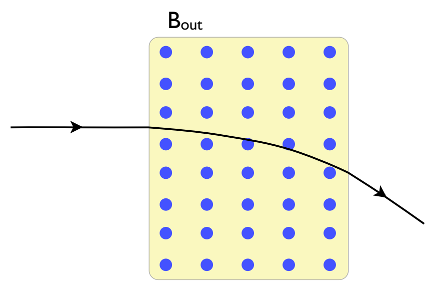

Figure 1: *Problem 1*

First of all, we know that once the proton enters the region of magnetic field it will follow a circular path of radius r, and once it leaves the region it will once again move in a straight line path, tangential to the circular path in the field region. We will need to use some geometry to relate α to the given distance x and the radius of the circular path r. Then we will determine the radius of the circular path in terms of the known kinetic energy, and we will be good to go ... Refer to the figure below:

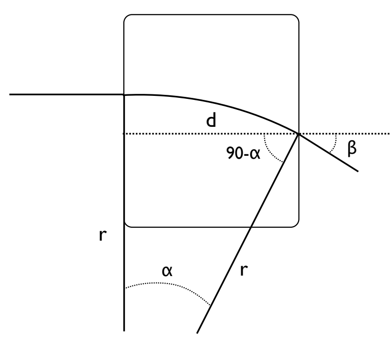

Figure 2: *Problem 1 Solution*

The path of the protons in the region of magnetic field is circular, and described by a radius r. The protons will move through the region of magnetic field across its lateral distance d, and this will define an angle α, carving out an arc of a circle of radius r. We can define a right triangle by the center of the circle defining the path in the magnetic field region, the point at which the protons enter the field, and the point at which they leave. The angle at which the protons leave the field with respect to the horizontal is then $\beta = \alpha$ based on the geometry of the figure above. Once we know r, given d, we can find α by noting $\sin\alpha = d/r$.

The radius of the circular path of the protons can be found by noting that the centripetal force (keeping them in a circular path) must be provided by the magnetic force. Note that a proton has charge e and mass m_p, and let the proton's velocity be v. Also recall that magnetic forces do no work, so the protons' velocity will not change in *magnitude* after passing through the region of magnetic field. Since the motion of the particle is always at a right angle with respect to the field, we can just deal in magnitudes.

$$\begin{aligned} F_{\text{centr}} &= \frac{m_p v^2}{r} = F_B = evB\sin\theta_{Bv} = evB \\ \Longrightarrow \qquad r &= \frac{m_p v}{eB} = \frac{p}{eB} \end{aligned}$$

Here p is the magnitude of the protons' momentum. Now we have the radius of the path in terms of the field, the charge on a proton, and the protons' momentum. We are given the protons' kinetic energy K, which is related to its momentum by $K = p^2/2m_p$. Thus,

$$r = \frac{p}{eB} = \frac{\sqrt{2m_pK}}{eB} \approx 6.46\,\text{m}$$

Remember that $1\,\text{eV} = 1.6\times10^{-19}\,\text{J}$ to make the units come out properly. Given the radius r, we can now find the angle α:

$$\sin\alpha = \frac{d}{r} \qquad \Longrightarrow \qquad \alpha = \sin^{-1}\frac{d}{r} \approx 8.9^\circ$$

Now, since the magnetic force does no work, the protons' momentum does not change in magnitude, so the initial and final momentum are the same. The vertical (y) component of the protons' momentum can thus be easily found: $p_y = p\sin\alpha$

$$p_y = p\sin\alpha = \sqrt{2m_pK}\sin\alpha \approx 8.0\times10^{-21}\,\text{kg}\cdot\text{m/s}$$

2. *Serway 29.67* Consider an electron orbiting a proton and maintained in a fixed circular path of radius $R = 5.29\times10^{-11}$ m by the Coulomb force. Treating the orbiting charge as a current loop, calculate the resulting torque when the system is in a magnetic field of $0.400\,\text{T}$ directed perpendicular to the magnetic moment of the electron.

First, need to know the current that corresponds to one orbiting electron. From the current I, magnetic field B, and the orbital radius R we can find the torque. An electron in a circular orbit of radius R has a period of $T = 2\pi R/v$, where v it he electron's velocity. If a single electron charge $-e$ orbits once every T seconds, then the current is by definition

$$I = \frac{\Delta q}{\Delta t} = \frac{-e}{T} = \frac{-ev}{2\pi R}$$

We can find the velocity from the condition for circular motion. The only force present (that we know of) is the electric force, which must then provide the centripetal force on the electron. The electric force is just that of two point charges e and $-e$ separated by a distance R.

$$F_{\text{centr}} - F_E = \frac{m_e v^2}{r} - \frac{-k_e e^2}{R^2} = 0 \qquad \Longrightarrow \qquad v = \sqrt{\frac{k_e e^2}{m_e R}}$$

We can now substitute this in our expression for current above:

$$I = \frac{-ev}{2\pi R} = \frac{-e}{2\pi R}\sqrt{\frac{k_e e^2}{m_e R}} = \frac{-e^2}{2\pi}\sqrt{\frac{k_e}{m_e R^3}}$$

Finally, since the magnetic field is perpendicular to the electron's magnetic moment, the magnitude of the torque is given by $\tau = IAB$ where A is the area of the "current loop" formed by the orbiting electron, $A = \pi R^2$. Thus,

$$\tau = IAB = \frac{-e^2}{2\pi}\sqrt{\frac{k_e}{m_e R^3}}\pi R^2 B = -\frac{1}{2}e^2 B\sqrt{\frac{k_e R}{m_e}} \approx 3.7 \times 10^{-24}\,\mathrm{N \cdot m}$$

The negative sign reminds us that current is the direction that *positive* charge flows, and thus the direction of the torque is given by the right hand rule consistent with the current, which is *opposite* the direction that the electron orbits.

3. *Ohanian 29.5* A wire lying along the x axis carries a current of $30\,\mathrm{A}$ in the $+x$ direction. A proton at $\vec{\mathbf{r}} = 2.5\,\hat{\mathbf{y}}$ has instantaneous velocity $\vec{\mathbf{v}} = 2.0\,\hat{\mathbf{x}} - 3.0\,\hat{\mathbf{y}} + 4.0\,\hat{\mathbf{z}}$, where $\vec{\mathbf{r}}$ is in meters and $\vec{\mathbf{v}}$ in meters per second. What is the instantaneous magnetic force on this proton?

If the current flows along the $\hat{\mathbf{x}}$ direction, and the proton is directly above the wire in the $\hat{\mathbf{y}}$ direction, then the magnetic field must be pointing along the $\hat{\mathbf{z}}$ direction at the proton's position. Thus, the magnetic field at the proton's position $\vec{\mathbf{r}}$ is given by

$$\vec{\mathbf{B}} = \frac{\mu_o I}{2\pi r}\,\hat{\mathbf{z}} \equiv B_z\,\hat{\mathbf{z}}$$

Finding the magnetic force is now just a matter of calculating the cross product $\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ and multiplying by the proton's charge e. First, the cross product:

$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = \det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2\,\mathrm{m/s} & -3\,\mathrm{m/s} & 4\,\mathrm{m/s} \\ 0 & 0 & B_z \end{vmatrix} = -3B_z\,\hat{\mathbf{x}} - 2B_z\,\hat{\mathbf{y}}\,\mathrm{m/s}$$

Thus, the magnetic force is

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = -3eB_z\,\hat{\mathbf{x}} - 2eB_z\,\hat{\mathbf{y}}\,\mathrm{m/s} \approx \left(-3.84 \times 10^{-25}\right)(3\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}})\,\mathrm{N}$$

If you note that $1\,\mathrm{T} = 1\,\mathrm{kg/s^2} \cdot \mathrm{A}$, you should be able to make the units come out properly. For completion, the magnitude of the force is then

$$|\vec{\mathbf{F}}_B| = \left(-3.84\times10^{-25}\right)\sqrt{3^2+2^2} \approx 1.38\times10^{-24}\,\mathrm{N}$$

4. *Ohanian 29.19* The electric field of a long, straight line of charge with λ coulombs per meter is

$$E = \frac{2k_e\lambda}{r}$$

where r is the distance from the wire. Suppose we move this line of charge parallel to itself at speed v. **(a)** The moving line of charge constitutes an electric current. What is the magnitude of this current? **(b)** What is the magnitude of the magnetic field produced by this current? **(c)** Show that the magnitude of the magnetic field is proportional to the magnitude of the electric field, and find the constant of proportionality.

The current can be found by thinking about how much charge passes through a given region of space per unit time. If we were standing next to the wire, in a time Δt, the length of wire that passes by us would be $v\Delta t$. The corresponding charge is then $\Delta q = \lambda v\Delta t$, and thus the current is

$$I = \frac{\Delta q}{\Delta t} = \frac{\lambda v\Delta t}{\Delta t} = \lambda v$$

From the current, we can easily find the magnetic field a distance r from the wire.

$$B = \frac{\mu_o I}{2\pi r} = \frac{\mu_o\lambda v}{2\pi r}$$

If the wire were sitting still (or we were traveling parallel to it at the same velocity v), it would produce the electric field given above. Rearranging the given expression, we can relate λ and E, $\lambda = Er/2k_e$. Substituting this in our expression for the magnetic field,

$$B = \frac{\mu_o\lambda v}{2\pi r} = \frac{\mu_o E r v}{4\pi k_e r} = \mu_o\epsilon_o v E$$

For the last step, we noted that $\epsilon_o = 1/4\pi k_e$.

5. *Purcell 7.14* A metal crossbar of mass m slides without friction on two long parallel rails a distance b apart. A resistor R is connected across the rails at one end; compared with R, the resistance of the bar and rails is negligible. There is a uniform field $\vec{\mathbf{B}}$ perpendicular to the plane of the figure. At time $t=0$, the crossbar is given a velocity v_o toward the right. What happens then? **(a)** Does the rod ever stop moving? If so, when? **(b)** How far does it go? **(c)** How about conservation of energy? *Hint: first find the acceleration, and make use of an instantaneous balance of power.*

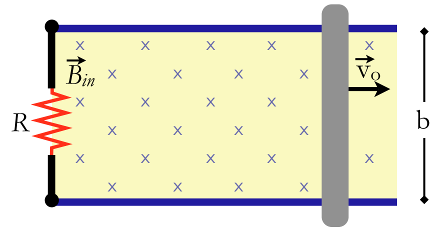

Figure 3: *Problem 5*

The moving rod forms a closed loop with the rails, and once the rod starts moving, the area of this loop increases with time. With a constant magnetic field, this means that the magnetic flux is increasing with time, and therefore there must be an induced voltage. Let the position of the rod be x, with the $\hat{\mathbf{x}}$ direction being to the right, and the $\hat{\mathbf{y}}$ direction upward. This means the magnetic field points in the $-\hat{\mathbf{z}}$ direction, giving in a magnitude $-B$. At time $t=0$, we will say the rod has velocity v_o and position x_o. For any time t, we will just call the velocity v and position x, since we don't know what they are yet.

The induced voltage can be found from the magnetic flux through the loop, which is itself is easily found, since the magnetic field is constant and everywhere perpendicular to the plane of the loop. We only need the area of the loop. If the width of the loop is b, and the position of the rod is x, the area is just bx, and that is enough to find the flux:

$$\Phi_B = \oint\limits_{\text{loop}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = B \oint\limits_{\text{loop}} d\vec{\mathbf{A}} = -BA = -Bxb$$

The induced voltage is found from the time variation of the flux via Faraday's law. The induced voltage - which is now applied to the resistor - will lead to a counterclockwise current in the loop, since it wants to stop the increase in flux by creating a magnetic field opposing the external magnetic field.

$$\Delta V = -\frac{d\Phi_B}{dt} = Bb\frac{dx}{dt} = Bbv = IR$$

The presence of a current in the conducting rod will lead to a magnetic force. Since the field is into the page ($-\hat{\mathbf{z}}$ direction), and the current is flowing up through through the rod ($\hat{\mathbf{y}}$ direction), the force must be in the $\hat{\mathbf{y}}$ direction.

$$\vec{\mathbf{F}}_B = I\vec{\mathbf{L}} \times \vec{\mathbf{B}} = IbB\,\hat{\mathbf{y}} \times (-\hat{\mathbf{z}}) = -IbB\,\hat{\mathbf{x}} = -\frac{B^2b^2v}{R} = ma$$

Recall that the direction of $\vec{\mathbf{L}}$ is the same as the direction of the current. Since the magnetic force is the only force acting on the rod (in the absence of friction), it must also give the acceleration of the rod, as indicted in the last step. Incidentally, we could have gotten here much more quickly with a little intuition. If we recognize that there must be a current flowing in the resistor due to the induced voltage caused by the motion of the rod, then we know there is power dissipated in the resistor. This power must be the same as that supplied to the rod. The mechanical power is $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$, and the electrical power is I^2R. Conservation of energy requires that these two powers be equal, which along with the motional voltage leads directly to the equation above.

Anyway: now we have a small equation relating v and its rate of change, $dv/dt = a$. We can solve it by separation of variables, which is totally cool since none of our quantities are zero. Dividing by zero is not cool.

$$
\begin{aligned}
-\frac{B^2 b^2 v}{R} &= m\frac{dv}{dt} \\
\frac{mR}{B^2 b^2}\frac{dv}{v} &= -dt
\end{aligned}
$$

Now we've got something we can integrate. Our starting condition is velocity v_o at time $t = 0$, going until some later time t where the velocity is v.

$$
\begin{aligned}
\int\limits_{v_o}^{v} \frac{mR}{B^2 b^2}\frac{dv}{v} &= \int\limits_{0}^{t} -dt \\
\frac{mR}{B^2 b^2} \ln v\Big|_{v_o}^{v} &= -t\Big|_{0}^{t} \\
\frac{mR}{B^2 b^2} \ln \frac{v}{v_o} &= -t \\
\implies \quad v = v_o e^{-t/\tau} & \qquad \text{with} \qquad \tau = \frac{mR}{B^2 b^2}
\end{aligned}
$$

The velocity is an exponentially decreasing function of time, which means it never stops moving - the velocity approaches, but does not reach, zero. The rod will also approach a final target displacement in spite of this fact, which we can find readily by integrating the velocity.

$$
\Delta x = \int\limits_{o}^{\infty} v\,dt = \int\limits_{o}^{\infty} v_o e^{-t/\tau} = v_o \left(-\tau e^{-t/\tau}\right)\Big|_{o}^{\infty} = v_o \tau = \frac{mRv_o}{B^2 b^2}
$$

Once again, if you note that $1\,\mathrm{T} = 1\,\mathrm{kg/s^2} \cdot \mathrm{A}$ and $1\,\mathrm{V} \cdot 1\,\mathrm{A} = 1\,\mathrm{W}$, you should be able to make the units come out correctly.

Finally, we can calculate the total electrical energy expended. The electrical power dissipated in the resistor is $\mathscr{P} = dU/dt = I^2 R$, so the tiny bit of potential energy dU expended in a time dt is $dU = I^2 R\,dt$. We can integrate over all times to find the total potential energy.

$$
\begin{aligned}
U &= \int\limits_{0}^{\infty} I^2 R\,dt = \int\limits_{0}^{\infty} \left(\frac{Bbv}{R}\right)^2 R\,dt = \frac{B^2 b^2}{R} \int\limits_{0}^{\infty} v^2\,dt \\
&= \frac{B^2 b^2 v_o^2}{R} \int\limits_{0}^{\infty} e^{-2t/\tau}\,dt = \frac{B^2 b^2 v_o^2}{R} \left(-\frac{\tau}{2} e^{-2t/\tau}\right)\Big|_{0}^{\infty} = \frac{B^2 b^2 v_o^2}{R} \left(\frac{mR}{2B^2 b^2}\right) \\
&= \frac{1}{2} m v_0^2
\end{aligned}
$$

As we would expect from conservation of energy, all of the initial kinetic energy of the conducting bar ends up dissipated in the resistor.

6. *Serway 29.66* A uniform magnetic field of magnitude $0.150\,\mathrm{T}$ is directed along the positive x axis. A positron (a positively-charged electron) moving at 5.00×10^6 m/s enters the field along a direction that makes an angle of 85° with the x axis. The motion of the particle is expected to be a helix in this case. Calculate the pitch p and radius r of the trajectory.

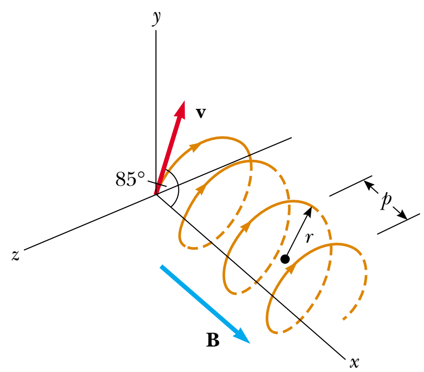

Figure 4: *Problem 6*

The first thing to realize is that a helix is basically a curve described by circular motion in one plane, in this case the $y - z$ plane, and linear motion along the perpendicular direction, in this case the x axis. A helix of circular radius a and pitch p can be described parametrically by

$$
\begin{aligned}
x(t) &= \frac{pt}{2\pi} \\
y(t) &= a\cos t \\
z(t) &= a\sin t
\end{aligned}
$$

As we can see, the motion in the $y - z$ plane obeys $y^2 + z^2 = a^2$, describing a circle of radius a, and along the x axis we just have constant velocity motion. Since the x, y, and z motions are uncoupled (e.g., the equation for $x(t)$ has no y's or z's in it), things are in fact pretty simple.

The circular motion comes from the component of the velocity perpendicular to the magnetic field, the component of velocity lying in the $y - z$ plane, which we will call $v_\perp$. The pitch is just how far forward along the x axis the particle moves in one period of circular motion T. Thus, if the velocity along the x axis is v_x,

$$p = v_x T = (v\cos 85°)\,T$$

We have already discovered that the period and radius of circular motion for a particle in a magnetic field does not depend on the particle's velocity, it only matters that there is always a velocity component perpendicular to the magnetic field

$$T = \frac{2\pi m}{qB} \qquad \text{and} \qquad r = \frac{mv_\perp}{qB}$$

Putting everything together,

$$
\begin{aligned}
p &= \frac{2\pi m v}{Bq}\cos 85^\circ \approx 1.04 \times 10^{-4}\,\mathrm{m} \\
r &= \frac{mv}{qB}\sin 85^\circ \approx 1.89 \times 10^{-4}\,\mathrm{m}
\end{aligned}
$$

By the way, here is an interesting tidbit from MathWorld:[i]

> A helix, sometimes also called a coil, is a curve for which the tangent makes a constant angle with a fixed line. The shortest path between two points on a cylinder (one not directly above the other) is a fractional turn of a helix, as can be seen by cutting the cylinder along one of its sides, flattening it out, and noting that a straight line connecting the points becomes helical upon re-wrapping. It is for this reason that squirrels chasing one another up and around tree trunks follow helical paths.

[i] `http://mathworld.wolfram.com/Helix.html`