

UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 106-4 / LeClair

Fall 2008

Problem Set 7: Solutions

1. *Serway 31.23.* Very large magnetic fields can be produced using a procedure called *flux compression*. A metallic cylindrical tube of radius R is placed coaxially in a long solenoid of somewhat larger radius. The space between the tube and the solenoid is filled with a highly explosive material. When the explosive is set off, it collapses the tube to a cylinder of radius $r < R$. If the collapse happens very rapidly, induced current in the tube maintains the magnetic flux nearly constant inside the tube, even though the area shrinks. If the initial magnetic field in the solenoid is 2.50 T, and $R/r = 12.0$, what is the maximum field that can be reached?

The basic idea here is that the *flux* through the tube is the same before and after the explosion. Since after the explosion the cross-sectional area is severely reduced, the field must be much larger in order to make the flux the same. Here's a bit from the Wikipedia about flux compression, explaining things in more detail:

Magneto-explosive generators use a technique called "magnetic flux compression", which will be described in detail later. The technique is made possible when the time scales over which the device operates are sufficiently brief that resistive current loss is negligible, and the magnetic flux on any surface surrounded by a conductor (copper wire, for example) remains constant, even though the size and shape of the surface may change.

This flux conservation can be demonstrated from Maxwell's equations. The most intuitive explanation of this conservation of enclosed flux follows from the principle that any change in an electromagnetic system provokes an effect in order to oppose the change. For this reason, reducing the area of the surface enclosed by a conductor, which would reduce the magnetic flux, results in the induction of current in the electrical conductor, which tends to return the enclosed flux to its original value. In magneto-explosive generators, this phenomenon is obtained by various techniques which depend on powerful explosives. The compression process allows the chemical energy of the explosives to be (partially) transformed into the energy of an intense magnetic field surrounded by a correspondingly large electric current.

- Wikipedia, "Flux Compression"

So: all we need to do is calculate the flux before and after the explosion, set them equal to each other, and solve for the field after the explosion. Quantities with 'i' subscripts refer to before the explosion, those with 'f' after the explosion, and all symbols have their usual meanings.

$$\begin{aligned}
\Phi_{B,i} &= B_i A_i = B_i \cdot \pi R^2 \\
\Phi_{B,f} &= B_f A_f = B_f \cdot \pi r^2 \\
\Phi_{B,i} &= \Phi_{B,f} \\
\implies B_i \pi R^2 &= B_f \pi r^2 \\
B_f &= \left(\frac{R}{r}\right)^2 B_i = (12.0)^2 \cdot 2.50 \text{ T} = 360 \text{ T}
\end{aligned}$$

2. Coaxial cables are used to shield conductors carrying small signals from stray electric fields by creating a *Faraday cage* around the central conductor. In order to shield extremely sensitive signals from stray *magnetic* fields, a Faraday cage will not work. Instead, so-called “twisted pair” wiring is used. Explain how this works. How would you shield a signal from *both* electric and magnetic interference?

See http://en.wikipedia.org/wiki/Twisted_pair for a description of how twisted-pair wiring works. Basically: Rather than having one enormous loop making up your circuit, you wind the wires into a helix to make many, many tiny loops. Each tiny loop of the helix has alternating ‘handedness,’ so each adjacent loop has an equal and opposite induced voltage. All the tiny voltages end up (mostly) canceling out, virtually eliminating unwanted inductive signals.

3. One common way to make a resistor is simply to wind a coil of high resistivity wire of the appropriate length - for a wire radius r and resistivity ρ , the resistance of a coil of wire of total length l is $\rho l / \pi r^2$. This is known as a “wire wound resistor,” not surprisingly. Another common method for constructing resistors is to use thin, patterned metal films instead of wires, reducing the cross-sectional area and allowing useful values of resistance. Why might one have a preference for which type of resistor is used when designing circuits, for example, audio amplifiers?

In fact, there are many reasons to prefer one over the other. Wire-wound resistors - being big coils of wires - can have significant inductance, which can be detrimental in circuits handling high-frequency signals. Wire-wound resistors also tend to be bulky. On the other hand, they have excellent precision, and can easily be made to tolerate high power. Metal film resistors have lower power ratings, but are dirt cheap and have negligible inductance.

We should note that one *can* make a wire-wound resistor with no inductance at all. It is simple - you wind a solenoid, but alternate the handedness of each layer of wire. Wrap one coil around the axis in a left-handed manner, but wrap the second layer coil in a right-handed manner. Just like in the last problem, the induced voltages in each layer will (mostly) cancel each other, leading to a small net inductance. One can buy this type of resistor ... for a price. This can be good for high-power amps where one needs a high power rating and precision, but inductance cannot be tolerated.

See <http://en.wikipedia.org/wiki/Resistor#Wirewound> for more information.

4. *Purcell 7.1.* What is the maximum voltage induced across a coil of 4000 turns, average radius 12 cm, rotating at 30 revolutions per second in the earth's magnetic field, where the field is approximately 5×10^{-5} T?

The coil is rotating at constant angular velocity, while the magnetic field is static. As the loop rotates, the magnetic flux oscillates in time, since the area of the facing the magnetic field is oscillating in time. At some point in time, the area normal of the loop will make an angle θ with the magnetic field. Since the loop rotates with constant angular velocity, we know $\theta = \omega t$. Thus, the flux through the loop must be

$$\Phi_B = \vec{B} \cdot \vec{A} = BA \cos \theta = BA \cos \omega t$$

The induced voltage is given by Faraday's law, noting that there are $N = 4000$ turns in the loop, and that we know the area of the loop in terms of the given radius r .

$$\Delta V = -N \frac{d\Phi_B}{dt} = -NBA (-\sin \omega t) \omega = NB\omega\pi r^2 \sin \omega t$$

We are given the rotation rate in revolutions per second, which is the *frequency* f , not the angular frequency $\omega = 2\pi f$. Making the substitution, and plugging in the values given,

$$\Delta V = 2\pi^2 NBfr^2 \sin(2\pi ft) \approx 1.7 \sin(188t) \text{ V}$$

Since we are asked for the maximum voltage, we really want $|\Delta V| = 1.7 \text{ V}$.

5. *Purcell 7.24.* A superconducting solenoid designed for whole-body imaging by nuclear magnetic resonance is 0.9 m in diameter and 2.2 m long. The field at the center is 0.4 T. Estimate roughly the energy stored in the field of this coil, in Joules.

A solenoid is really nothing more than a ginormous inductor, a big coil of wire. The energy stored in an inductor is $U = \frac{1}{2}LI^2$. For the specific case of a single coil - a solenoid - we know the inductance is:

$$L = \mu_0 n^2 V$$

Where n is the number of turns *per unit length* and V the volume of the inductor. Putting this in the energy equation:

$$U = \frac{1}{2} \mu_0 n^2 V I^2$$

Great. But we don't know n or I in this case directly. We do know the *magnetic field* for a solenoid, however, which will give us the product nI , really all we need:

$$B = \mu_0 nI$$

$$\implies nI = \frac{B}{\mu_0}$$

Putting it all together, and noting that the volume of a cylinder is $\pi r^2 l$, where r is the radius and l the length:

$$U = \frac{1}{2} \mu_0 n^2 V I^2 = \frac{1}{2} \mu_0 V (nI)^2 = \frac{1}{2} \mu_0 V \frac{B^2}{\mu_0^2} = \frac{\pi r^2 l B^2}{2 \mu_0} \approx 89100 \text{ J} = 89.1 \text{ kJ}$$

6. *Purcell 6.37.* Consider two solenoids, one of which is a tenth-scale model of the other. The larger solenoid is 2 m long, and 1 m in diameter, and is wound with 1 cm-diameter copper wire. When the coil is connected to a 120 V dc generator, the magnetic field at the center is exactly 0.1 T. The scaled-down version is exactly one-tenth the size in every linear dimension, including the diameter of the wire. The number of turns is the same in both coils, and both are designed to provide the same central field.

(a) Show that the voltage required is the same, namely, 120 V

(b) Compare the coils with respect to the power dissipated, and the difficulty of removing this heat by some cooling means.

This is basically a scaling problem: when everything is shrunk by 10 times, what happens to the required voltage for a given field? First, let's consider the large solenoid. Let's say it has length $L = 2$ m, radius $r = 0.5$ m, contains N turns of wire, and it provides a field $B = 0.1$ T with a current I . We know we can relate the field and the current:

$$B = \mu_0 \frac{N}{L} I$$

The solenoid is just a long single strand of wire wrapped around a cylinder. If we say that the total length of wire used to wrap the solenoid is l , and the wire's diameter is d , then we can calculate the resistance of the solenoid:

$$R = \frac{\rho l}{A} = \frac{\rho l}{\pi d^2/4}$$

Here we have used the wire's resistivity ρ , and its cross-sectional area $A = \pi r^2 = \pi d^2/4$. Given the resistance and voltage of $\Delta V = 120$ V, we can calculate the current:

$$I = \frac{\Delta V}{R} = \frac{\Delta V \pi d^2 / 4}{\rho l}$$

Now if we plug that into our first solenoid equation above, we can relate voltage and magnetic field:

$$B = \mu_0 \frac{N}{L} I = \mu_0 \frac{N}{L} \frac{\Delta V \pi d^2 / 4}{\rho l} = \frac{\mu_0 \pi N \Delta V d^2}{4 \rho L l}$$

Now, what about the small solenoid? Every dimension is a factor of 10 smaller. If *all* the dimensions are 10 times smaller, the number of turns that fit within 1/10 the length is the *same* as the big solenoid if the wire diameter is also 1/10 as large! In other words, both coils will have the same number of turns - the space for the wire is 10 times smaller, but so is the wire.

In order to find the relationship for the small solenoid, we will use the same symbols, but everything for the small solenoid will have a prime \prime . The number of turns in the small solenoid is N' , and in for the large solenoid it is just N . The voltage on the little solenoid is $\Delta V'$, and on the large one we have just ΔV . Using the results from above, magnetic field for the small solenoid is then easily found by substitution:

$$B' = \frac{\mu_0 \pi N' \Delta V' (d')^2}{4 \rho L' l'} = B$$

We don't have to bother with a prime on the resistivity, both coils have the same sort of wire. Remember, our desired condition is that $B' = B$. We know that $N' = N$, and all the dimensions are 10 times smaller - the length of the solenoid, the wire diameter, and therefore also the length of wire required. We have the same number of *turns* in each coil, but in the smaller coil the circumference of each turn is 10 times smaller, which means overall, the total length of wire required l is 10 times smaller. Thus:

$$\begin{aligned} B' &= \frac{\mu_0 \pi N' \Delta V' (d')^2}{4 \rho L' l'} \\ &= \frac{\mu_0 \pi N \Delta V' (d')^2}{4 \rho L' l'} && \text{note that } N' = N \\ &= \frac{\mu_0 \pi N \Delta V' (\frac{d}{10})^2}{4 \rho \frac{L}{10} \frac{l}{10}} && \text{scale all dimensions by } \frac{1}{10} \\ &= \frac{\mu_0 \pi N \Delta V' d^2}{4 \rho L l} \end{aligned}$$

Now, we want to enforce the condition that the field is the same in both solenoids:

$$\begin{aligned}
& B' = B \\
\Rightarrow & \frac{\mu_0 \pi N \Delta V' d^2}{4 \rho L l} = \frac{\mu_0 \pi N \Delta V d^2}{4 \rho L l} \\
& \Rightarrow \Delta V' = \Delta V
\end{aligned}$$

Thus, a solenoid shrunk by 10 times in every dimension will require the same applied voltage for the same magnetic field. What about the power consumption? The current in the large solenoid was

$$I = \frac{\Delta V}{R} = \frac{\Delta V \pi d^2 / 4}{\rho l}$$

In the small solenoid, we now know that the voltage is the same, but the resistance is not, so we should have:

$$I' = \frac{\Delta V}{R'} = \frac{\Delta V \pi (d')^2 / 4}{\rho l'} = \frac{\Delta V \pi (\frac{d}{10})^2 / 4}{\rho \frac{l}{10}} = \frac{1}{10} \frac{\Delta V \pi d^2 / 4}{\rho l} = \frac{1}{10} I$$

The current in the little solenoid is 10 times less - sensible, since the total length of wire is 10 times smaller, but the area of the wire is 100 times smaller. The power required for each is the product of current and voltage:

$$\begin{aligned}
\mathcal{P}_{\text{big}} &= I \Delta V \\
\mathcal{P}_{\text{small}} &= I' \Delta V = \frac{1}{10} I \Delta V = \frac{1}{10} \mathcal{P}_{\text{big}}
\end{aligned}$$

Not only is the larger solenoid ten times larger, it requires ten times more power, and therefore dissipates ten times more heat. The cooling requirements will be far more formidable for the larger solenoid. For instance, if we decide to use water cooling, the flow rate will need to be at least 10 times larger for the large solenoid to extract a heat load ten times larger. Not to mention the fact that we have to acquire a much larger power supply in the first place - practically speaking, the difference between a 5 A current source and a 50 A current source is significant. Keep in mind that your normal household outlets deliver 120 V at a maximum of ~15 A.

7. A long coaxial cable consists of two concentric conductors. The inner conductor is a cylinder of radius a , and it carries a current I uniformly distributed over its cross section. The outer conductor is a cylindrical shell with inner radius $b > a$ and outer radius c . It carries a current I that is also uniformly distributed over its cross section, and that is opposite in direction to the current on the inner conductor. Calculate the magnetic field \vec{B} as a function of the distance r from the axis, and plot the field strength as a function of r .

First, look at the diagram below, a cross-section through the cable. Assume a current $+I_0$ flows out of the page through the central conductor, and the same current returns through the outer conductor, for a current $-I_0$ into the page. Based on the cylindrical symmetry of the coax, we can already tell that the magnetic field will be constant on circles centered around the axis of the cable, with the field strength depending only on the distance r from the cable axis. This leads us to a straightforward application of Ampere's law.

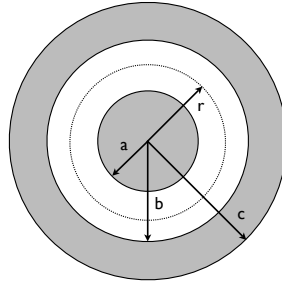


Figure 1: Problem 7

Ampere's law tells us that the magnetic field can be found from the amount of current enclosed by a surface alone. If we draw circles centered on the cable axis of radius r , the magnetic field will depend only on how much current passes through our circle. In the special case that we have a circular Amperian path, \vec{B} is constant in magnitude everywhere along our path, and parallel to the displacement along the path. This makes our integration around the circular path trivial:

$$\oint \vec{B} \cdot d\vec{l} = \oint B dl = B \oint dl = B (2\pi r) \mu_0 I_{\text{encl}}$$

$$\implies B = \frac{\mu_0 I_{\text{encl}}}{2\pi r}$$

Thus, the problem of finding the magnetic field anywhere has been reduced to finding how much current passes through arbitrary circles of radius r centered on the cable axis. In order to concretely apply Ampere's law, we can logically break our problem up into four different regions, distinguished by their distance r from the cable axis: region I, inside the inner conductor $r < a$; region II, between the two conductors $a \leq r < b$; region III, inside the outer conductor $b \leq r < c$; and region IV, outside both conductors $r \geq c$.

First, look at region 1. For this region, we can draw a circle of radius $r < a$, and it will contain a current determined only by the fraction of the conductor enclosed, since the current is distributed uniformly. The current density over the inner conductor is just the total current divided by the area of the conductor, $J = I_0/\pi a^2$, so the current contained in a circle of radius $r \leq a$ is

$$I_{\text{encl}} = JA = \frac{I_o}{\pi a^2} (\pi r^2) = I_o \frac{r^2}{a^2} \quad (r \leq a)$$

Now we can readily find the field for region I:

$$B(r \leq a) = \frac{\mu_o I_{\text{encl}}}{2\pi r} = \frac{\mu_o I_o r^2}{2\pi r a^2} = \frac{\mu_o I_o r}{2\pi a^2} \quad (\text{region I})$$

We can check that our expression has the proper limits by noticing that the field goes to zero as $r \rightarrow 0$, and approaches the value for a plain current-carrying wire a distance a away as $r \rightarrow a$. What about region II? In between the two conductors, we have an enclosed current of I_o , since we enclose the entire inner conductor. Thus:

$$B(a < r \leq b) = \frac{\mu_o I_o}{2\pi r} \quad (\text{region II})$$

Ok, on to region III. Now we enclose a current $+I_o$ from the inner conductor, *plus* a fraction of the current $-I_o$ on the outer conductor. In order to find the fraction of the outer current enclosed, we again calculate the current density on the outer conductor and multiply that by the enclosed area . . . remembering that the area of an annulus is just the difference in area between two circles.

$$I_{\text{outer, encl}} = JA = \left[\frac{-I_o}{\pi c^2 - \pi b^2} \right] [\pi r^2 - \pi b^2] = -I_o \left[\frac{r^2 - b^2}{c^2 - b^2} \right] \quad (\text{region III})$$

Thus, the total enclosed current, taking into account both conductors, is

$$I_{\text{encl}} = I_o - I_o \left[\frac{r^2 - b^2}{c^2 - b^2} \right] = I_o \left[1 - \frac{r^2 - b^2}{c^2 - b^2} \right] = I_o \left[\frac{c^2 - r^2}{c^2 - b^2} \right] \quad (\text{region III})$$

The magnetic field is then easily found:

$$B(b < r \leq c) = \frac{\mu_o I}{2\pi r} \left[\frac{c^2 - r^2}{c^2 - b^2} \right] \quad (\text{region III})$$

Again, our expression has the proper limits: as $r \rightarrow b$ we recover our previous expression for the field from a single wire carrying a current I_o , and for $r \rightarrow c$ the field goes to zero – there is no *net* current enclosed! That brings us to region IV. No problem: outside the coax, the total enclosed current zero, and so is the field. Putting everything together so far:

$$B(r) = \begin{cases} \frac{\mu_0 I_0 r}{2\pi a^2} & (0 \leq r \leq a) \\ \frac{\mu_0 I_0}{2\pi r} & (a < r \leq b) \\ \frac{\mu_0 I}{2\pi r} \left[\frac{c^2 - r^2}{c^2 - b^2} \right] & (b < r \leq c) \\ 0 & (c < r) \end{cases} \quad (1)$$

Finally, check out the plot below, where we have taken $a = 1$, $b = 2$, and $c = 3$ and put the magnetic field in somewhat arbitrary units where $\mu_0 I / 2\pi \equiv 1$. Note that $B(r)$ really is continuous everywhere – we can only have a discontinuity in B where we have a sheet of current, just as discontinuities in E can come only from sheets of charge.

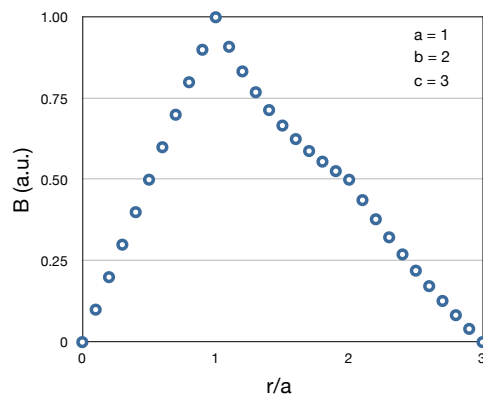


Figure 2: Plot of the problem 7 solution

8. A long wire is bent into a hairpin, like the shape shown below. find an exact expression for the magnetic field at point P which lies at the center of the half-circle.

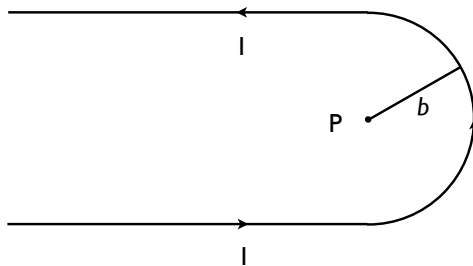


Figure 3: Problem 8

There is a sneaky way to solve this problem using symmetry. Qualitatively, we can immediately observe that the field must point out of the plane of the page. Think of the hairpin as being broken up into three sections: an upper semi-infinite wire, a half circle, and a lower semi-infinite wire. All three segments give

the same direction of field at P. Further, if we were to rotate the entire hairpin in the plane of the page, this will not change. Do that in your head once . . . rotate the entire setup, say, 90° clockwise, and you will find that the magnetic field at P will not change. If this is the case, let us consider the particular case where we have the same arrangement of wires rotated a full 180°, and add this to the existing setup, as shown below:

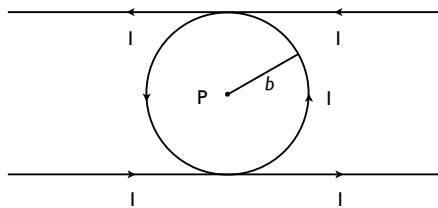


Figure 4: Problem 8 solution

Since both the “normal” and “rotated” hairpins give the same field, this arrangement just gives us twice the field of the original arrangement. With this arrangement, we can break the problem down into two infinite wires and a single circular loop of radius r . We have already calculated the field due to straight wires and loops; at point P, these three fields superimpose, and the calculation is trivial:

$$2\vec{B} = \frac{\mu_0 I}{2\pi b} \hat{z} + \frac{\mu_0 I}{2\pi b} \hat{z} + \frac{\mu_0 I}{2b} \hat{z} \quad \implies \quad \vec{B} = \frac{\mu_0 I}{4\pi b} \left(1 + \frac{2}{\pi} \right) \hat{z}$$

How about the hard way? In order to use the Biot-Savart law and perform the requisite integration, we need to break up the hairpin into its constituent regions: two semi-infinite lines, and a semicircle. We have already calculated the fields from semi-infinite lines and semicircles in class; you should be able to superimpose those solutions to reproduce the total field we found above.

9. What is the induced voltage between the ends of the wingtips of a Boeing 737 when it is flying over the magnetic south pole? Use your google-fu for the numbers you require.

The induced voltageⁱ can be found by considering the motion of the conducting metal plane in a perpendicular magnetic field, and making a few seemingly outlandish (but justifiable) assumptions.

First, at the south magnetic pole, the magnetic field will be essentially straight down. If the 737 is flying level over the ground, this means that its metal (conducting) skin is in motion relative to a magnetic field.

ⁱIn the original problem set, I mistakenly said “EMF” instead of voltage. I find EMF to be an antiquated and unnecessary term; if I slip up like that again, read “EMF” as “potential difference” or “voltage.”

This in turn means that there will be a motionally-induced voltage. If the field is straight down, and the 737 travels straight forward, then positive charges will experience a force in the port (left) direction, and negative charges toward the starboard (right). This means that the wingtips will have a potential difference between them due to the magnetic force on the charges in the conducting skin. If the wingspan is l meters, the airplane's velocity v and the vertical magnetic field B , then we know the potential difference due to motion in a magnetic field is $\Delta V = Blv$.

The wingspan of a 737 is roughly 30 m, and its cruising speed is about 200 m/s.ⁱⁱ Currently, the earth's magnetic fieldⁱⁱⁱ at the south magnetic pole^{iv} is about $60 \mu\text{T}$. Putting this together,

$$\Delta V = Blv = (60 \mu\text{T}) (30 \text{ m}) (200 \text{ m/s}) \approx 0.36 \text{ V}$$

10. A flat circular disk with radius R carries a uniform surface charge density σ . It rotates with an angular velocity ω about the z -axis. Find the magnetic field $B(z)$ at any point z along the rotation axis.

First: see the schematic below.

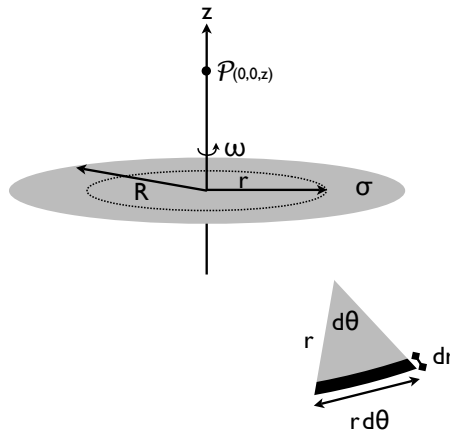


Figure 5: Problem 10 solution

Break up the disk into a series of loops of infinitesimal width. A given loop of radius r and thickness dr will have a total area $dA = 2\pi r dr$, and thus contain a total charge $dQ = 2\pi\sigma r dr$. This loop is rotating at an angular velocity ω , which means that the charge dQ on our loop makes a circuit around the axis every period of rotation, $T = 2\pi/\omega$ seconds. Since a charge dQ makes a circuit every T seconds, our infinitesimally thin ring is a current loop:

ⁱⁱhttp://en.wikipedia.org/wiki/Boeing_737

ⁱⁱⁱ<http://www.ngdc.noaa.gov/geomag/magfield.shtml>

^{iv}http://en.wikipedia.org/wiki/South_Magnetic_Pole

$$dI = \frac{dQ}{T} = \frac{2\pi\sigma r dr}{2\pi/\omega} = \sigma\omega r dr$$

This is just like the moving line charge of your previous problem set, where a line of linear charge density λ moved with velocity v and constituted a current $I = \lambda v$. Anyway: for this single loop, equivalent to a current dI , we can easily calculate the field a distance z above the axis. We did this in class, it is also an example problem in your textbook. Applied to the present case, the solution is:

$$dB(z) = \frac{\mu_0 r^2 dI}{2(r^2 + z^2)^{3/2}} = \frac{\mu_0 r^3 \sigma\omega dr}{2(r^2 + z^2)^{3/2}}$$

In order to find the total field, we have to integrate over all possible infinitesimal rings, from $r \rightarrow 0$ to $r = R$:

$$\begin{aligned} B(z) &= \int_0^R \frac{\mu_0 r^3 \sigma\omega dr}{2(r^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 \sigma\omega}{2} \int_0^R \frac{r^3 dr}{(r^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 \sigma\omega}{2} \left[\frac{2z^2 + r^2}{\sqrt{r^2 + z^2}} \right]_0^R \\ &= \frac{\mu_0 \sigma\omega}{2} \left[\frac{2z^2 + R^2}{\sqrt{R^2 + z^2}} - 2\frac{z^2}{z} \right] \\ &= \frac{\mu_0 \sigma\omega}{2} \left[\frac{2z^2 + R^2}{\sqrt{R^2 + z^2}} - 2|z| \right] \end{aligned}$$

The absolute value bit at the end is a bit sneaky, but necessary. Physically, we need it for the solution to be symmetric about $z=0$, which it must be. Also, without it, the field would diverge on one side of the disk, which is just silly.