Problem Set 9: Solutions

1. *Purcell 7.15* A taut wire passes through the gap of a small magnet, where the field strength is 0.5 T. The length of the wire within the gap is 1.8 cm. Calculate the amplitude of the induced alternating voltage when the wire is vibrating at its fundamental frequency of 2000 Hz with an amplitude of 0.03 cm.

We know that a vibrating string can be described by a sine wave, and at its fundamental mode, the string has the shape of one-half cycle of a sine wave – it is pinned at two end points, and gradually reaches a maximum deflection halfway between the ends. In this case, the amplitude of the vibration $A = 3 \times 10^{-4}$ m is very small compared to the length of the wire in the gap, $l = 0.018$ m. Even if the wire were exactly 0.018 m long and no longer, the deflection of the wire would still only be about 2%. The curvature of the wire inside the magnet is *tiny* compared to even its minimum length, which means we may treat the wire as still essentially straight, rather than a sine wave. As the wire vibrates, we will treat the segment within the magnet as simply a straight segment which oscillates up and down, ignoring the slight curvature.

A given point on a vibrating wire can be described by $y = A \cos(\omega t)$, where y is the height of the string above or below its mean position, A is the amplitude of vibration, and $\omega_o = 2\pi f_o$ is the frequency of vibration. The velocity of the wire at a given point is easily found:

$$v_y = \frac{dy}{dt} = \frac{d}{dt}\left[A\cos\left(2\pi f_o t\right)\right] = -2\pi f_o A \sin\left(2\pi f_o t\right)$$

If we treat the portion of the vibrating wire inside the magnet as still essentially straight, what we have is a straight segment of wire of length l moving at velocity v_y perpendicularly to a magnetic field B. This means we must have an induced voltage, as we do any time we have a conductor moving in a magnetic field. Presume that the wire is oscillating in a direction perpendicular to the magnetic field. In that case, the induced voltage on our "straight" segment vibrating up and down is

$$\Delta V = -Blv_y = 2\pi f_o BlA \sin\left(2\pi f_o t\right)$$

So, indeed there is an induced alternating voltage, which has the same frequency as the vibrational frequency of the wire (though it has a $\pi/2$ phase shift). We are asked to find only the amplitude of the induced voltage, which means just the pre-factor of the sine term above:

$$|\Delta V| = 2\pi f_o BlA = (2\pi)\left(2 \times 10^3 \, \text{s}^{-1}\right)(0.5\,\text{T})\left(1.8 \times 10^{-2}\,\text{m}\right)\left(3 \times 10^{-4}\,\text{m}\right) \approx 0.034\,\text{V}$$

In order for the units to work out correctly, remember that $1\,\text{T} = 1\,\text{V·s/m}^2$.

2. *Serway 30.71* A sphere of radius R has a uniform volume charge density ρ. Determine the magnetic field at the center of the sphere when it rotates as a rigid object with angular speed ω about an axis through its center.

Coming soon, we hope ... there will not be questions this hard on the final.

3. *Serway 32.75* The lead-in wires from a television antenna are often constructed in the form of two parallel wires. **(a)** Why does this configuration of conductors have an inductance? **(b)** What constitutes the flux loop for this configuration? **(c)** Ignoring any magnetic flux inside the wires, show that the inductance of a length x of this type of lead-in is

$$L = \frac{\mu_o x}{\pi} \ln \left(\frac{w - a}{a} \right)$$

where a is the radius of the wires and w is their center-to-center separation.

Our antenna lead-in can be modeled as two wires of thickness a and center-to-center separation w:
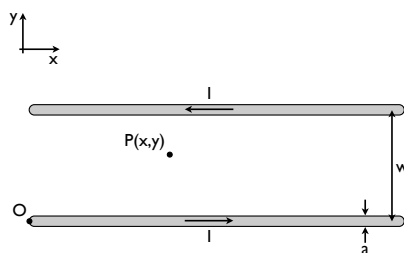
This configuration will have an inductance, because there is a magnetic field present due to the current loop, and thus it stores energy. A current through the conductors will create a magnetic flux through the rectangular region between the two conductors, which constitutes a flux loop. If the total flux of the configuration depends on the current through the configuration, then there is a magnetic flux by definition: $L = \Phi_B / I$. If we can calculate the magnetic flux for a given current, we can find the inductance.

Let the origin be at the center of the bottom lower wire, at the lower left. The wires run in the x axis, with the y axis perpendicular to the wire axes. Take an arbitrary point $P(x, y)$ between the wires. This point is a vertical distance y from the center of the bottom wire, and a distance $w - y$ from the center of the top wire. Recall the length of the wires is x.

In order to find the inductance of this configuration, we need to calculate the magnetic flux. We will assume that the wires are very long compared to their separation ($x \gg w$), such that we may treat the them as infinitely long. In this case, the total field at an arbitrary point P between the wires is just

$$B_{\text{tot}} = B_{\text{upper}} + B_{\text{lower}} = \frac{\mu_o I}{2\pi y} + \frac{\mu_o I}{2\pi (w - y)}$$

Keep in mind that the field is calculated based on the distance from the *center* of the wire – Ampere's law tells us that the magnetic field outside of the wire behaves as if the current were concentrated on an infinitesimally thin region at the center of the wire. We can now calculate the magnetic flux easily.

A differential unit of area between the two conductors is $dx\,dy$, and x runs from 0 to l while y runs from a to $w-a$.

$$\Phi_B = \int\limits_0^x dx \int\limits_a^{w-a} \frac{\mu_o I}{2\pi y} + \frac{\mu_o I}{2\pi(w-y)}\,dy$$

We can make one more simplification: integrated over the whole rectangular region between the two wires, symmetry dictates that both wires will give the same contribution to the flux. Therefore, we can simply integrate the field of one wire (say, the bottom one) and double the result:

$$\Phi_B = 2\int\limits_0^x dx \int\limits_a^{w-a} \frac{\mu_o I}{2\pi y}\,dy = 2x \int\limits_a^{w-a} \frac{\mu_o I}{2\pi y}\,dy = \frac{\mu_o x I}{\pi} \int\limits_a^{w-a} \frac{dy}{y}$$

$$= \frac{\mu_o x I}{\pi}\Big[\ln y\Big]_a^{w-a} = \frac{\mu_o x I}{\pi}\Big[\ln(w-a) - \ln a\Big]$$

$$= \frac{\mu_o x I}{\pi}\ln\left[\frac{w-a}{a}\right]$$

The inductance is just the flux per unit current, which gives us the desired result:

$$L = \frac{\Phi_B}{I} = \frac{\mu_o x}{\pi}\ln\left[\frac{w-a}{a}\right]$$

4. *Serway 35.62* As light from the Sun enters the atmosphere, it refracts due to the small difference between the speeds of light in air and in vacuum. The optical length of the day is defined as the time interval between the instant when the top of the Sun is just visibly observed above the horizon, to the instant at which the top of the Sun just disappears below the horizon. The geometric length of the day is defined as the time interval between the instant when a geometric straight line drawn from the observer to the top of the Sun just clears the horizon, to the instant at which this line just dips below the horizon. The day's optical length is slightly larger than its geometric length.

By how much does the duration of an optical day exceed that of a geometric day? Model the Earth's atmosphere as uniform, with index of refraction $n = 1.000293$, a sharply defined upper surface, and depth 8767 m. Assume that the observer is at the Earth's equator so that the apparent path of the rising and setting Sun is perpendicular to the horizon. You may take the radius of the earth to be 6.378×10^6 m. Express your answer to the nearest hundredth of a second.

First, we need to draw a little picture. This is the situation we have been given:
We presume that some human is standing at point A on the earth's surface, looking straight out toward the horizon. This line of sight intersects the boundary between the atmosphere and space (which we are told to assume is a sharp one) at point B. Light rays from the sun, which is slightly below the horizon, are refracted toward the earth's surface at point B, and continue on along the line of sight from B to A. We know the index of refraction of vacuum is just unity ($n_{\text{vacuum}} = 1$), while
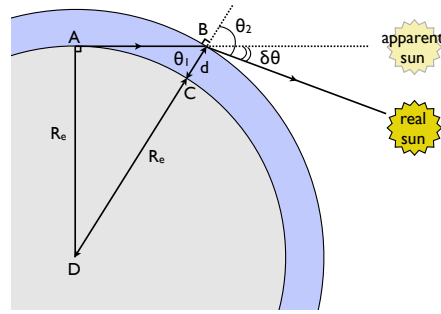
that of the atmosphere is $n = 1.000293$. The day appears to be slightly longer because we see the sun even after it has gone through an extra angle of rotation $\delta\theta$ due to atmospheric refraction.

To set up the geometry, we first draw a radial line from point B to the center of the earth. This line, $\overline{BC}$, will intersect the boundary of the atmosphere at point B, and will be normal to the atmospheric boundary. This defines the angle of incidence θ_2 and the angle of refraction θ_1 for light coming from the sun. The difference between these two angles, $\delta\theta$, is how much the light is bent downward upon being refracted from the atmosphere. How do we relate this to the extra length of the day one would observe? We know that the earth revolves on its axis at a constant angular speed - one revolution in 24 hours. Thus, we can easily find the angular speed of the earth:

$$\text{earth's angular speed} = \omega = \frac{\text{one revolution}}{1\text{day}} = \frac{360°}{86400\,\text{s}}$$

Here we used the fact that there are $24 \cdot 60 \cdot 60 = 86400$ seconds in one day. Given the angular velocity of the earth, we know exactly how long it will take for the earth to rotate through the "extra" angle $\delta\theta$ due to refraction:

$$\delta\theta = \omega\delta t$$

We only need one last bit: the atmospheric refraction occurs *twice per day* – once at sun-up and once at sun-down. The total "extra" length of the day is then $2\delta t$. Thus, if we can find $\delta\theta$, we can figure out how much longer the day seems to be due to atmospheric refraction. In order to find it, we need to use the law of refraction and a bit of geometry. First, from the law of refraction and the fact that $\delta\theta = \theta_2 - \theta_1$, we can state the following:

$$\theta_2 - \theta_1 = \delta\theta$$
$$n \sin\theta_1 = \sin\theta_2 = \sin(\theta_1 + \delta\theta)$$

In order to proceed further, we draw a line from point A to the center of the earth, point D. This forms a triangle, $\triangle ABD$. Because line $\overline{AD}$ is a radius of the earth, by construction, it must intersect line $\overline{AB}$ at a right angle, since the latter is by construction a tangent to the earth's surface. Thus, $\triangle ABD$ is a right triangle, and

$$\sin\theta_1 = \frac{\overline{AD}}{\overline{BD}} = \frac{R_e}{R_e + d}$$

Plugging this into the previous equation,

$$n \sin \theta_1 = \sin \theta_2 = \sin(\theta_1 + \delta\theta) = n\frac{R_e}{R_e + d}$$

In principle, we are done at this point. The previous expression allows one to calculate θ_1, while the present one allows one to find $\delta\theta$ if θ_1 is known. From that, one only needs the angular speed of the earth.

$$\theta_2 = \theta_1 + \delta\theta = \sin^{-1}\left[\frac{nR_e}{R_e + d}\right]$$

$$\delta\theta = \sin^{-1}\left[\frac{nR_e}{R_e + d}\right] - \theta_1 = \sin^{-1}\left[\frac{nR_e}{R_e + d}\right] - \sin^{-1}\left[\frac{R_e}{R_e + d}\right] = \omega\delta t$$

$$2\delta t = \frac{2\delta\theta}{\omega} \approx 163.82\,\mathrm{s}$$

Of course, it is more satisfying to have an analytic approximation. We will leave that as an exercise to the reader for now.

5. *Frank 16.1* What is the apparent depth of a swimming pool in which there is water of depth $3\,\mathrm{m}$, **(a)** When viewed from normal incidence? **(b)** When viewed at an angle of $60°$ with respect to the surface? The refractive index of water is 1.33.

As always, we first need to draw a little picture of the situation at hand.
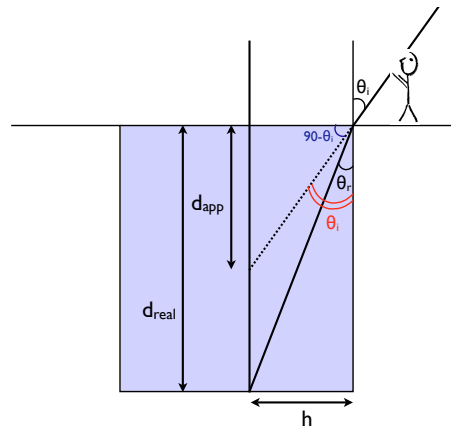
It is slightly more convenient to redefine the angle of incidence θ_i to be with respect to the *normal* of the water's surface itself, rather than with respect to the surface, since that is our usual convention. That means we are interested in incident angles for the observer of $90°$ and $30°$. The depth of the pool will be $d_{\mathrm{real}} = 3\,\mathrm{m}$. If an observer views the bottom of the pool with an angle θ_i with respect to the surface normal, refracted rays from the bottom of the pool will be bent away from the surface normal on the way to their eyes. That is, rays emanating from the bottom of the pool will make an angle $\theta_r < \theta_i$ with respect to the surface normal, and rays exiting the pool will make an angle θ_i with the surface normal. This is owing to the fact that the light will be bent *toward* the normal in the faster medium, the air, on exiting the water.

What depth does the observer actually see? They see what light would do in the absence of refraction, the path that light rays would appear to take if the rays were not "bent" by the water. In this case, that means that the observer standing next to the pool would think they saw the light rays coming from an angle θ_i with respect to the surface normal (dotted line in the pool). The *lateral* position of the bottom of the pool would remain unchanged. If the real light rays intersect the bottom of the pool a distance h from the edge, then the apparent bottom of the pool is also a distance h from the edge of the pool. Try demonstrating this with a drinking straw in a glass of water!

So what to do? First off, we can apply Snell's law. If the index of refraction of air is 1, and the water has an index of refraction n, then

$$n \sin \theta_r = \sin \theta_i$$

We can also use the triangle defined by d_{real} and h:

$$\tan \theta_r = \frac{h}{d_{\text{real}}}$$

as well as the triangle defined by d_{real} and h^{i}:

$$\tan (90 - \theta_i) = \frac{d_{\text{app}}}{h} = \frac{1}{\tan \theta_i}$$

Solving the last two equations for h,

$$h = d_{\text{real}} \tan \theta_r = d_{\text{app}} \tan \theta_i$$
$$\implies \quad d_{\text{app}} = d_{\text{real}} \left[\frac{\tan \theta_r}{\tan \theta_i} \right]$$

From Snell's law, we have a relationship between θ_r and θ_i already:

$$\theta_r = \sin^{-1} \left[\frac{\sin \theta_i}{n} \right]$$

Putting everything together,

$$d_{\text{app}} = \frac{d_{\text{real}}}{\tan \theta_i} \tan \theta_r = \frac{d_{\text{real}}}{\tan \theta_i} \left[\tan \left(\sin^{-1} \left[\frac{\sin \theta_i}{n} \right] \right) \right]$$

If you just plug in the numbers at this point, you have a problem. One of the angles is $\theta_i = 0$, normal incidence, which means we have to divide by zero in the expression above. Dividing by zero is worse than drowning kittens, far worse. Thankfully, we know enough trigonometry to save the poor kittens.

We can save the kittens by remembering an identity for $\tan \left[\sin^{-1} x \right]$. If we have an equation like $y = \sin^{-1} x$, it implies $\sin y = x$. This means y is an angle whose sine is x. If y is an angle in a right

triangle, then it has an opposite side x and a hypotenuse 1, making the adjacent side $\sqrt{1-x^2}$. The tangent of angle y must then be $x/\sqrt{1-x^2}$. Thus,

$$\tan\left[\sin^{-1} x\right] = \frac{x}{\sqrt{1-x^2}}$$

Using this identity in our equation for d_{app},

$$d_{\text{app}} = \frac{d_{\text{real}}}{\tan\theta_i}\left[\frac{\sin\theta_i}{n\sqrt{1 - \left[\frac{\sin\theta_i}{n}\right]^2}}\right] = \frac{d_{\text{real}}}{\tan\theta_i}\left[\frac{\sin\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}\right] = \frac{d_{\text{real}}\cos\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}$$

Viewed from normal incidence with respect to the surface means $\theta_i = 0$ – looking straight down at the surface of the water. In this case, $\sin\theta_i = 0$, and the result is simple:

$$d_{\text{app}} = \frac{d_{\text{real}}}{n} \approx 2.6\,\text{m}$$

Viewed from 60° with respect to the *surface* means 30° with respect to the normal, and thus

$$d_{\text{app}} = d_{\text{real}}\cos 30\left[\frac{1}{\sqrt{1.33^2 - \sin^2 30}}\right] \approx 2.1\,\text{m}$$

There are easier ways to solve the normal incidence problem, without endangering any kittens whatsoever. Solving that problem, however, is a special case, and of limited utility. You would still have to solve the case of 60° incidence separately. I wanted to show you here that solving the general problem just once is all you need to do, so long as you are careful enough.

6. *Serway 35.35* The index of refraction for **violet** light in silica flint glass is $n_{\text{violet}} = 1.66$, and for **red** light it is $n_{\text{red}} = 1.62$. In air, $n = 1$ for both colors of light.

What is the **angular dispersion** of visible light (the angle between red and violet) passing through an equilateral triangle prism of silica flint glass, if the angle of incidence is 50°? The angle of incidence is that between the ray and a line *perpendicular* to the surface of the prism. Recall that all angles in an equilateral triangle are 60°.
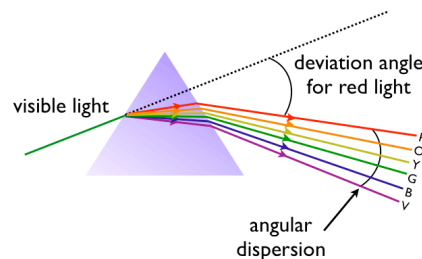
What we need to do is find the deviation angle for both red and violet light in terms of the incident angle and refractive index of the prism. The angular dispersion is just the difference between the deviation angles for the two colors. First, let us define some of the geometry a bit better, referring to the figure below.
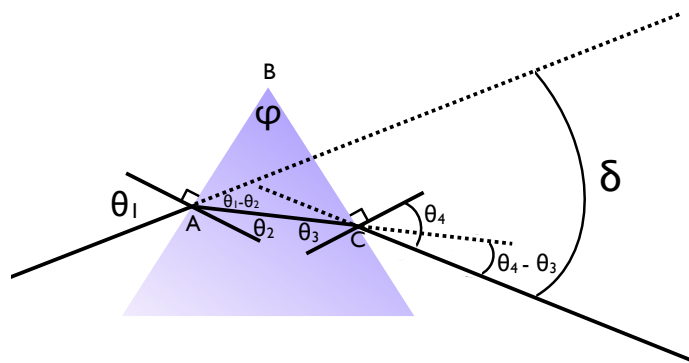
Let the angle of incidence be θ_1, and the refracted angle θ_2 at point A. The incident and refracted angles are defined with respect to a line *perpendicular* to the prism's surface. Similarly, when the light rays exit the prism, we will call the incident angle within the prism θ_3, and the refracted angle exiting the prism θ_4 at point C. If we call index of refraction of the prism n, and presume the surrounding material is just air with index of refraction 1.00, we can apply Snell's law at both interfaces:

$$n \sin \theta_2 = \sin \theta_1$$
$$n \sin \theta_3 = \sin \theta_4$$

Fair enough, but now we need to use some geometry to relate these four angles to each other, the deviation angle δ, and the prism's apex angle φ. Have a look at the triangle formed by points A, B, and C. All three angles in this triangle must add up to $180°$. At point A, the angle between the prism face and the line $\overline{AC}$ is $\angle BAC = 90 - \theta_2$ - the line we drew to define θ_1 and θ_2 is by construction perpendicular to the prism's face, and thus makes a $90°$ angle with respect to the face. The angle $\angle BAC$ is all of that $90°$ angle, *minus* the refracted angle θ_2. Similarly, we can find $\angle BCA$ at point C. We know the apex angle of the prism is φ, and for an equilateral triangle, we must have $\varphi = 60°$

$$(90° - \theta_2) + (90° - \theta_3) + \varphi = 180°$$
$$\implies \varphi = \theta_2 + \theta_3 = 60°$$

How do we find the deviation angle? Physically, the deviation angle is just how much in total the exit ray is "bent" relative to the incident ray. At the first interface, point A, the incident ray and reflected ray differ by an angle $\theta_1 - \theta_2$. At the second interface, point C, the ray inside the prism and the exit ray differ by an angle $\theta_4 - \theta_3$. These two differences *together* make up the total deviation - the deviation is nothing more than adding together the differences in angles at each interface due to refraction. Thus:

$$\delta = (\theta_1 - \theta_2) + (\theta_4 - \theta_3) = \theta_1 + \theta_4 - (\theta_2 + \theta_3)$$

Of course, one can prove this rigorously with quite a bit more geometry, but there is no need: we know physically what the deviation angle is, and can translate that to a nice mathematical formula.

Now we can use the expression for φ in our last equation:

$$\delta = \theta_1 + \theta_4 - \varphi$$

We were given $\theta_1 = 50°$, so now we really just need to find θ_4 and we are done. From Snell's law above, we can relate θ_4 to θ_3 easily. We can also relate θ_3 to θ_2 and the apex angle of the prism, φ. Finally, we can relate θ_2 back to θ_1 with Snell's law. First, let us write down all the separate relations:

$$\sin \theta_4 = n \sin \theta_3$$
$$\theta_3 = \varphi - \theta_2$$
$$n \sin \theta_2 = \sin \theta_1$$
$$\text{or} \quad \theta_2 = \sin^{-1}\left(\frac{\sin \theta_1}{n}\right)$$

If we put all these together (in the right order) we have θ_4 in terms of known quantities:

$$\sin \theta_4 = n \sin \theta_3$$
$$= n \sin (\varphi - \theta_2)$$
$$= n \sin \left[\varphi - \sin^{-1}\left(\frac{\sin \theta_1}{n}\right)\right]$$

With that, we can write the full expression for the deviation angle:

$$\delta = \theta_1 + \theta_4 - \varphi = \theta_1 + n \sin \left[\varphi - \sin^{-1}\left(\frac{\sin \theta_1}{n}\right)\right] - \varphi$$

Now we just need to calculate the deviation separately for red and violet light, using their different indices of refraction. You should find:

$$\delta_{\text{red}} = 48.56°$$
$$\delta_{\text{blue}} = 53.17°$$

The angular dispersion is just the difference between these two:

$$\text{angular dispersion} = \delta_{\text{blue}} - \delta_{\text{red}} = 4.62°$$

7. 10 points. Prove that any incoming ray that is parallel to the axis of a parabolic reflector will be reflected to a central point, or "focus."

In order to prove this without too much pain, it is useful to recall the geometric definition of the parabola. A parabola is the locus of points that is equidistant from a point (the focus) and a line (the directrix). This construction is shown below: every point on the parabola is the same distance from the focus $F(0, f)$ and the line $y = -f$.

We can derive the focus of the parabola fairly easily. We don't really need it, per se, but it is a useful bit of knowledge to have at our disposal. Take a parabola $y = ax^2$. This is perfectly general, as any
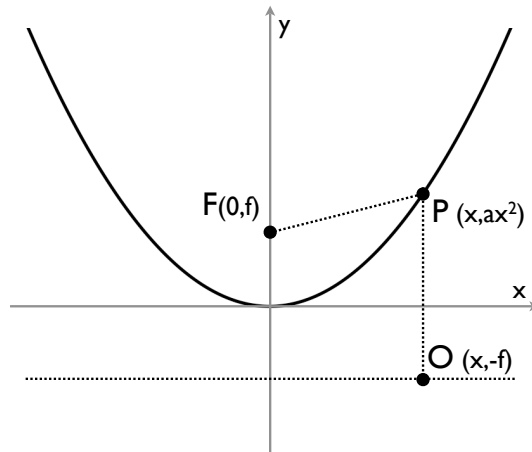
parabola $y = a_0 + a_1 x + a_2 x^2$ can be mapped onto the parabola $y' = a\,(x')^2$ with a suitable change of coordinates (namely, $x' \mapsto x + a_1/2a_2$ and $y' \mapsto y + a_1^2/4a_2 - a_0$). If we take any arbitrary point $P(x, y) = P(x, ax^2)$ on the parabola, by construction we have $||FP|| = ||QP||$ – the distance from P to the focus must be the same as the shortest distance from P to the directrix line. The latter distance must simply be a vertical line from $P(x, ax^2)$ to $O(x, -f)$. Using the distance formula,

$$||FP|| = \sqrt{x^2 + (y - f)^2} = ||QP|| = y + f$$
$$\sqrt{x^2 + (ax^2 - f)^2} = ax^2 + f$$

Now square both sides, and solve for y:

$$x^2 + \left(ax^2 - f\right)^2 = \left(ax^2 + f\right)^2$$
$$x^2 + a^2 x^4 - 2ax^2 f + f^2 = a^2 x^4 + 2ax^2 f + f^2$$
$$x^2 = 4ax^2 f$$
$$4af = 1 \qquad (x \neq 0)$$
$$\implies \quad a = \frac{1}{4f}$$

Substituting this in our equation for the parabola, we have an equation for the parabola in terms of the focus distance f:
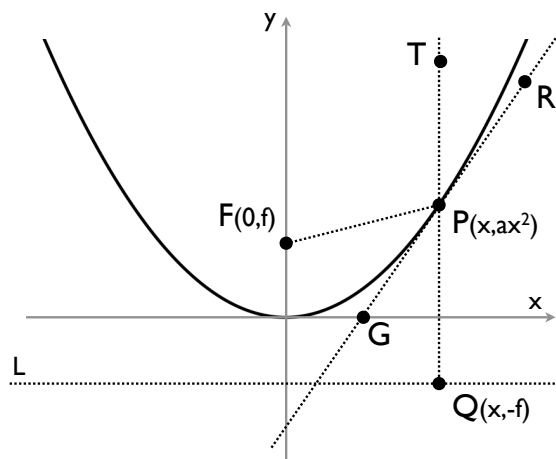
$$y = \frac{x^2}{4f}$$

Now, what about the reflective property? One can prove this in many different ways; we will illustrate one. First, begin by drawing a tangent to the parabola at point P, as shown below:[ii]
For our parabola $y = ax^2$, the tangent line has slope

$$\frac{dy}{dx} = 2ax = \frac{2y}{x}$$

[ii]This proof is basically the same as the one that can be found on the Wikipedia, under "Parabola."

Since the tangent line must also pass through point $P(x, ax^2)$ by construction, we can determine that it must intercept the y axis at the point $(0, -ax^2)$, and the x axis at $(x/2, 0)$. Let the x intercept be point $G(x/2, 0)$. This point G is the midpoint of F and Q:

$$F = (0, f)$$
$$Q = (x, -f)$$
$$\frac{F + Q}{2} = \frac{(0, f) + (x, -f)}{2} = \frac{(x, 0)}{2} = \left(\frac{x}{2}, 0\right) = G$$

If G is the midpoint of the segment FQ, this means that the segments FG and GQ must be of equal length: $||FG|| \cong ||GQ||$. We have already established that P is equidistant from F and Q, $||PF|| \cong ||PQ||$. This is sufficient to establish that triangles $\triangle FGP$ and $\triangle QGP$ are congruent, since they also share the side GP: $\triangle FGP \cong \triangle QGP$. If this is true, then it must also be true that $\angle FPG \cong \angle GPQ$.

Now consider a point T along a line which extends QP in the $+y$ direction. This line is also an incident ray. Also consider a point R further up the tangent line from P. Angles $\angle RPT$ and $\angle GPQ$ share the same vertex, are bounded by the same pair of lines, and are opposite to each other, hence $\angle RPT \cong GPQ$. Since we also know that $\angle FPG \cong \angle GPQ$, it follows that $\angle RPT \cong \angle FPG$. Since we have proven this for a general point P on the parabola, it is true for every point on the parabola.

Imagine now a light ray travels downward on the vertical line TP. It will bounce off of the parabola at point P, and will behave as though RG were a mirror. That is, applying the law of reflection locally at point P means that the incident angle $\angle RPT$ must be the same as the exit angle $\angle FPG$. We have just proven that this must be the case for a parabolic curve, and therefore the ray must bounce off of point P directed toward the focus at point F. Thus, any light beam moving vertically downward on a concave parabola, parallel to the axis of symmetry, will reflect off the parabola and move directly toward the focus.