Name & ID ____

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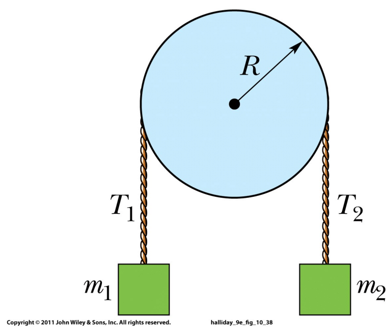
Exam III Solutions

1. In the figure below, block 1 has mass m_1, block 2 has mass m_2 (with $m_2 > m_1$), and the pulley (a solid disc), which is mounted on a horizontal axle with negligible friction, has radius R and mass M. When released from rest, block 2 falls a distance d in t seconds without the cord slipping on the pulley. **(a)** What are the magnitude of the accelerations of the blocks? **(b)** What is T_1? **(c)** What is T_2? **(d)** What is the pulley's angular acceleration? The moment of inertia of a solid disc is $I = \frac{1}{2}MR^2$.



Solution: Give $m_2 > m_1$, we expect a clockwise rotation. Taking the positive y direction as upward, that makes the acceleration of mass 2 negative and that of mass 1 positive. We need to do two thing: first, balance the forces on the hanging masses, and two, analyze the torque on the disc.

With the sign conventions noted above, the forces are

$$T_2 - m_2 g = -m_2 a \tag{1}$$
$$T_1 - m_1 g = m_1 a \tag{2}$$

What we must be careful about now are the facts that the tension in each side of the rope is *not* just the weight of the hanging mass (this can't e true if the masses are accelerating, as the equations above indicate), and we should not assume that $T_1 = T_2$ when we have the pully's moment of inertia to consider. That means we have three unknowns (T_1, T_2, and a) but only two equations. Adding the torque analysis gets us the last equation we need.

$$\sum \tau = RT_2 - RT_1 = R(T_2 - T_1) = I\alpha \tag{3}$$

Noting that $\alpha = a/R$, one can solve the resulting equations for T_1, T_2, and a. The angular acceleration is also readily found. I'll assume you can work out the details:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M}\right) g \tag{4}$$

$$\alpha = \left(\frac{m_2 - m_1}{m_1 + m_2 + \frac{1}{2}M}\right) \frac{g}{R} \tag{5}$$

$$T_1 = \left(\frac{2m_1m_2 + \frac{1}{2}Mm_1}{m_1 + m_2 + \frac{1}{2}M}\right) g \tag{6}$$

$$T_2 = \left(\frac{2m_1m_2 + \frac{1}{2}Mm_2}{m_1 + m_2 + \frac{1}{2}M}\right) g \tag{7}$$

How do we know this is plausible? We can set $I=0$ to ignore the effect of the pulley, which reduces the system to the simple case of two masses on an ideal massless pulley that we've already studied. With $I=0$:

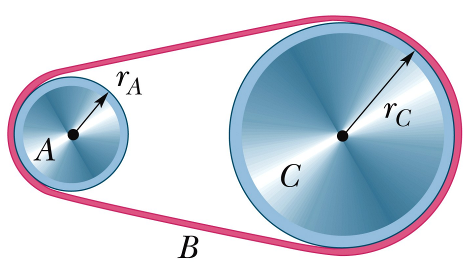
$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g \tag{8}$$

$$T_1 = \left(\frac{2m_1m_2}{m_1 + m_2}\right) g \tag{9}$$

$$T_2 = \left(\frac{2m_1m_2}{m_1 + m_2}\right) g \tag{10}$$

Now we see $T_1=T_2$, and the tensions and acceleration are just what we found before.

2. A flywheel rotating freely on a shaft is suddenly coupled by means of a drive belt to a second flywheel sitting on a parallel shaft (see figure below). The initial angular velocity of the first flywheel is ω, that of the second is zero. The flywheels are uniform discs of masses M_a and M_c with radii R_a and R_c respectively. The moment of inertia of a solid disc is $I=\frac{1}{2}MR^2$. The drive belt is massless and the shafts are frictionless. **(a)** Calculate the final angular velocity of each flywheel. **(b)** Calculate the kinetic energy lost during the coupling process. *Hint: if the belt does not slip, the* **linear** *speeds of the two rims must be equal.*



Solution: If the belt doesn't slip, the linear velocity of the wheels must be the same at their outer rim when the final state is reached. That implies

$$v_a = v_c \tag{11}$$

$$R_a\omega_a = R_c\omega_c \tag{12}$$

$$\omega_c = \frac{R_a}{R_c}\omega_a \tag{13}$$

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The sudden coupling of the second flywheel is basically a collision, and as is usually the case with collisions, conservation of energy is not a viable approach (how would you figure out how much energy the collision cost?). Conservation of momentum, or angular momentum when we have a rotation problem, is the way to go. Initially we have only the first flywheel rotating at ω, after the fact both are rotating. Conservation of angular momentum, combined with the relationship between ω_a and ω_c gives:

$$\begin{aligned}
&& L_i &= L_f && (14)\\
&& I_a\omega &= \omega_a I_a + \omega_c I_c = \omega_a I_a + \frac{R_a}{R_c} I_c && (15)\\
\Longrightarrow && \omega_a &= \frac{\omega}{1 + R_a I_c / R_c I_a} && (16)\\
\Longrightarrow && \omega_c &= \frac{\omega}{R_c/R_a + I_c/I_a} && (17)
\end{aligned}$$

Using the fact that the moments of inertia are $I = \frac{1}{2}MR^2$,

$$\omega_a = \frac{\omega}{1 + M_c R_c / M_a R_a} \tag{18}$$

$$\omega_c = \frac{R_a}{R_c} \frac{\omega}{1 + M_c R_c / M_a R_a} \tag{19}$$

The kinetic energy loss is straightforward to calculate, if messy.

$$K_f = \frac{1}{2} I_a \omega_a^2 + \frac{1}{2} I_c \omega_c^2 = \frac{1}{2} I_a \omega_a^2 \left(1 + \frac{I_c R_a^2}{I_a R_c^2}\right) \tag{20}$$

$$K_i = \frac{1}{2} I_a \omega^2 \tag{21}$$

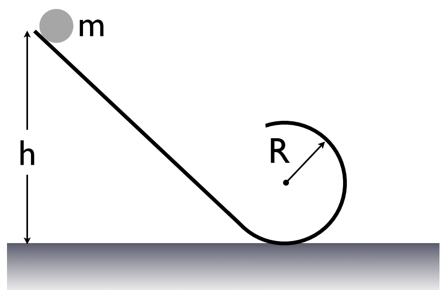
With a bit of algebra, you can work out the ratio

$$\frac{K_f}{K_i} = \frac{M_a R_a^2 \left(M_a + M_c\right)}{\left(M_a R_a + M_c R_c\right)^2} \tag{22}$$

3. A solid sphere, a solid cylinder, and a thin-walled pipe, all of mass m, roll smoothly along identical loop-the-loop tracks when released from rest along the straight section (see figure below). The circular loop has radius R, and the sphere, cylinder, and pipe have radius $r \ll R$ (i.e., the size of the objects may be neglected when compared to the other distances involved). If $h = 2.8R$, which of the objects will make it to the top of the loop? Justify your answer with an explicit calculation. The moments of inertia for the objects are listed below.

$$I = \begin{cases} \frac{2}{5}mr^2 & \text{sphere} \\ \frac{1}{2}mr^2 & \text{cylinder} \\ mr^2 & \text{pipe} \end{cases} \tag{23}$$

Hint: consider a single object with $I = kmr^2$ to solve the general problem, and evaluate these three special cases only at the end.



Solution: To start with, we just need to do conservation of energy. The object goes through a height $h - 2R$ to get to the top of the loop. Including both rotational and translational kinetic energy,

$$mg(h - 2R) = \frac{1}{2}mv^2 + \frac{1}{2}(kmr^2)\omega^2 = (1 + k)\left(\frac{1}{2}mv^2\right) \tag{24}$$

This doesn't tell us if the object actually makes it to the top of the loop or not. For that, we need to be sure that the velocity is high enough to be consistent with the required centripetal force. The centripetal force must be provided by the object's weight.

$$\frac{mv^2}{R} \geq mgv^2 \qquad \geq Rg \tag{25}$$

Using the energy equation, we have another equation for v^2. Combining:

$$v^2 = \frac{2g\,(h - 2R)}{1 + k} \geq Rg \tag{26}$$

$$k \leq \frac{h - 2R}{R} = \frac{h}{R} - 2 \tag{27}$$

Given $h = 2.8R$, our condition is that $k \leq 0.8$. This is true for the sphere ($k = 2/5$) and the cylinder ($k = 1/2$), but not for the pipe ($k = 1$). Thus, the sphere and cylinder make it, but the pipe does not.

4. The rotational inertia (moment of inertia) of a collapsing spinning star drops to $\frac{1}{3}$ its initial value. What is the ratio of the new rotational kinetic energy to the initial rotational kinetic energy?

Solution: If we need the rotational kinetic energy ratio, we'll have to get the relationship between the angular velocities first. For that all we need is conservation of angular momentum, noting that the final moment of inertia I_f is one third of the initial value I_i.

$$L_i = L_f \tag{28}$$

$$I_i\omega_i = I_f\omega_f = \frac{1}{3}I_i\omega_f \tag{29}$$

$$\omega_f = 3\omega_i \tag{30}$$

Makes sense: if the moment of inertia goes down three times, the rate of rotation has to go up three times to conserve angular momentum. That's all we need to get the kinetic energy ratio.

$$\frac{K_i}{K_f} = \frac{\frac{1}{2}I_i\omega_i^2}{\frac{1}{2}I_f\omega_f^2} = \frac{1}{3} \tag{31}$$

Formula sheet

$$g = 9.81\,\mathrm{m/s^2}$$
$$1\,\mathrm{N} = 1\,\mathrm{kg}\cdot\mathrm{m/s^2}$$
$$1\,\mathrm{J} = 1\,\mathrm{kg}\cdot\mathrm{m^2/s^2} = 1\,\mathrm{N}\cdot\mathrm{m}$$

Math:

$$ax^2 + bx^2 + c = 0 \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \mp \beta)$$
$$\cos\alpha \pm \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$$
$$\frac{d}{dx}\sin ax = a\cos ax \qquad \frac{d}{dx}\cos ax = -a\sin ax$$
$$\int \cos ax\,\mathrm{dx} = \frac{1}{a}\sin ax \qquad \int \sin ax\,\mathrm{dx} = -\frac{1}{a}\cos ax$$
$$\sin\theta \approx \theta \qquad \text{small } \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2$$

1-D motion:

$$v(t) = \frac{d}{dt}x(t)$$
$$a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t)$$
const. acc. $\downarrow$
$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$$
$$v_f^2 = v_i^2 + 2a_x\Delta x$$
$$v_f = v_i + at$$

Projectile motion:

$$v_x(t) = v_{ix} = |\vec{\mathbf{v}}_i|\cos\theta$$
$$v_y(t) = |\vec{\mathbf{v}}_i|\sin\theta - gt = v_{iy}\sin\theta - gt$$
$$x(t) = x_i + v_{ix}t$$
$$y(t) = y_i + v_{iy}t - \frac{1}{2}gt^2$$
over level ground:
$$\text{max height } = H = \frac{v_i^2\sin^2\theta_i}{2g}$$
$$\text{Range } = R = \frac{v_i^2\sin 2\theta_i}{g}$$

Force:

$$\sum\vec{\mathbf{F}} = \vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}} = \frac{d\vec{\mathbf{p}}}{dt}$$
$$\sum F_i = ma_i \quad \text{by component}$$
$$\vec{\mathbf{F}}_c = \sum F_{\mathrm{r}} = -\frac{mv^2}{r}\,\hat{\mathbf{r}}$$
$$f_k = \mu_k n$$
$$F_s = -kx$$
$$F_g = -mg$$

Work-Energy:

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$
$$\Delta K = K_f - K_i = W$$
$$W = \int F(x)\,dx = -\Delta U$$
$$U_g(y) = mgy$$
$$U_s(x) = \frac{1}{2}kx^2$$
$$F = -\frac{dU(x)}{dx}$$
$$K_i + U_i = K_f + U_f + W_{\text{ext}} = K_f + U_f + \int F_{\text{ext}}\,dx$$

Momentum, etc.:

$$x_{\text{com}} = \frac{1}{M_{\text{tot}}}\sum_{i=1}^{n} m_i x_i = \frac{m_1x_1 + m_2x_2 + \ldots m_nx_n}{m_1 + m_2 + \ldots m_n}$$
$$v_{\text{com}} = \frac{1}{M_{\text{tot}}}\sum_{i=1}^{n} m_i v_i = \frac{m_1v_1 + m_2v_2 + \ldots m_nv_n}{m_1 + m_2 + \ldots m_n}$$
$$F_{\text{net}} = M_{\text{tot}}a_{\text{com}} = \frac{dp}{dt} \qquad p_{\text{tot}} = M_{\text{tot}}v_{\text{com}}$$
$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \qquad \Delta p = p_f - p_i = F_{\text{avg}}\Delta t \qquad (\Delta p = 0 \text{ for isolated system})$$

Rotation: we use radians

$$s = \theta r \quad \leftarrow \text{arclength}$$
$$\omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d\omega}{dt}$$
$$a_t = \alpha r \quad \text{tangential} \qquad a_r = \frac{v^2}{r} = \omega^2 r \quad \text{radial}$$
$$I = \sum_i m_i r_i^2 \Rightarrow \int r^2\,dm = kmr^2$$
$$I_z = I_{com} + md^2 \quad \text{axis } z \text{ parallel, dist } d$$
$$\tau_{net} = \sum\vec{\boldsymbol{\tau}} = I\vec{\boldsymbol{\alpha}} = \frac{d\vec{\mathbf{L}}}{dt}$$
$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}}\times\vec{\mathbf{F}} \qquad |\vec{\boldsymbol{\tau}}| = rF\sin\theta_{rF}$$
$$\vec{\mathbf{L}} = \vec{\mathbf{r}}\times\vec{\mathbf{p}} = I\vec{\boldsymbol{\omega}}$$
$$K = \frac{1}{2}I\omega^2 = L^2/2I$$
$$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W = \int \tau\,d\theta$$
$$P = \frac{dW}{dt} = \tau\omega$$

Vectors:

$$|\vec{\mathbf{F}}| = \sqrt{F_x^2 + F_y^2} \quad \text{magnitude}$$
$$\theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \quad \text{direction}$$
$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_xb_x + a_yb_y + a_zb_z = \sum_{i=1}^{n} a_ib_i = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$$
$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad |\vec{\mathbf{a}}\times\vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta$$