University of Alabama
Department of Physics and Astronomy

PH 125 / LeClair Fall 2014

Problem Set 1 Solutions

1. Water is poured into a container that has a leak. The mass m of the water is as a function of time t is

$$m = 5.00t^{0.8} - 3.00t + 20.00$$

with $t \geq 0$, m in grams, and t in seconds. At what time is the water mass greatest?

Solution: **Given:** Water mass versus time $m(t)$.

Find: The time t at which the water mass m is greatest. This can be accomplished by finding the time derivative of $m(t)$ and setting it equal to zero, followed by checking the second derivative to be sure we have found a maximum.

Sketch: It is useful to plot the function $m(t)$ and graphically estimate about where the maximum should be, roughly.[i]

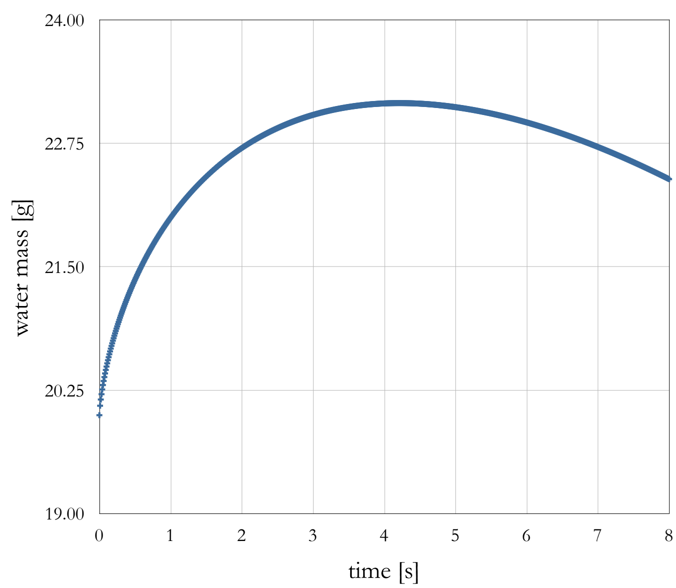

Figure 1: *Water mass versus time, problem 1. Note the rather expanded vertical axis, with offset origin.*

It is clear that there is indeed a maximum water mass, and it occurs just after $t = 4$ s.

[i]It is relatively easy to do this on a graphing calculator, which can be found online these days: `http://www.coolmath.com/graphit/`.

Relevant equations: We need to find the maximum of $m(t)$. Therefore, we need to set the first derivative equal to zero. We must also check that the second derivative is negative to ensure that we have found a maximum, not a minimum. Therefore, only two equations are needed:

$$\frac{dm}{dt} = \frac{d}{dt}\left[m(t)\right] = 0 \qquad \text{and} \qquad \frac{d^2m}{dt^2} = \frac{d^2}{dt^2}\left[m(t)\right] < 0 \quad \Longrightarrow \quad \text{maximum in } m(t)$$

Symbolic solution:

$$\begin{aligned}
\frac{dm}{dt} &= \frac{d}{dt}\left[5t^{0.8} - 3t + 20\right] = 0.8\left(5t^{0.8-1}\right) - 3 = 4t^{-0.2} - 3 = 0 \\
4t^{-0.2} - 3 &= 0 \\
t^{-0.2} &= \frac{3}{4} \\
\Longrightarrow \quad t &= \left(\frac{3}{4}\right)^{-5} = \left(\frac{4}{3}\right)^{5}
\end{aligned}$$

Thus, $m(t)$ takes on an extreme value at $t = (4/3)^5$. We did not prove whether it is a maximum or a minimum however! This is important . . . so we should apply the *second derivative* test.

Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a} = 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a} < 0$, you have a maximum, if $d^2f/dx^2|_{x=a} > 0$ you have a minimum, and if $d^2f/dx^2|_{x=a} = 0$, the test basically wasted your time. Anyway:

$$\begin{aligned}
\frac{d^2m}{dt^2} &= \frac{d}{dt}\left[\frac{dm}{dt}\right] = \frac{d}{dt}\left[4t^{-0.2} - 3\right] = -0.2\left(4t^{-0.2-1}\right) = -0.8t^{-1.2} \\
\frac{d^2m}{dt^2} &< 0 \quad \forall \quad t > 0
\end{aligned}$$

Since $t^{-1.2}$ is always positive for $t > 0$, $\frac{d^2m}{dt^2}$ is always less than zero[ii], which means we have indeed found a maximum.

Numeric solution: Evaluating our answer numerically, remembering that t has units of seconds (s):

$$t = \left(\frac{4}{3}\right)^5 \approx 4.21399 \xrightarrow[\text{digits}]{\text{sign.}} 4.21\,\text{s}$$

The problem as stated has only three significant digits, so we round the final answer appropriately.

Double check: From the plot above, we can already graphically estimate that the maximum is somewhere around $4\frac{1}{4}$ s, which is consistent with our numerical solution to 2 significant figures.

[ii] You can read the symbol $\forall$ above as "for all." Thus, $\forall\ t > 0$ is read as "for all t greater than zero."

The dimensions of our answer are given in the problem, so we know that t is in seconds. Since we solved $dm/dt(t)$ for t, the units must be the same as those given, with t still in seconds – our units are correct.

2. **(a)** Find the separation vector $\Delta\vec{\mathbf{r}}=\vec{\mathbf{r}}-\vec{\mathbf{r}}'$ between the points $\vec{\mathbf{r}}'=(3,4,5)$ and $\vec{\mathbf{r}}=(7,2,17)$. **(b)** Determine its magnitude, and **(c)** construct the corresponding unit vector.

Solution:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}-\vec{\mathbf{r}}' = (7-3)\;\hat{\mathbf{x}} + (2-4)\;\hat{\mathbf{y}} + (17-5)\;\hat{\mathbf{z}} = 4\,\hat{\mathbf{x}} - 2\,\hat{\mathbf{y}} + 12\,\hat{\mathbf{z}}$$

$$|\Delta\vec{\mathbf{r}}| = \sqrt{4^2+(-2)^2+12^2} = 2\sqrt{41} \approx 12.8$$

$$\hat{\Delta\vec{\mathbf{r}}} = \frac{\vec{\mathbf{r}}-\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{4\,\hat{\mathbf{x}}-2\,\hat{\mathbf{y}}+12\,\hat{\mathbf{z}}}{2\sqrt{41}} = \frac{1}{\sqrt{41}}\,(2\,\hat{\mathbf{x}}-1\,\hat{\mathbf{y}}+6\,\hat{\mathbf{z}})$$

3. A ball is dropped from rest at height h. Directly below on the ground, a second ball is simultaneously thrown upward with speed v_0. **(a)** If the two balls collide at the moment the second ball is instantaneously at rest, what is the height of the collision? **(b)** What is the relative speed of the balls when they collide? Ignore air resistance.

Solution: This is a problem from "Problems and Solutions in Introductory Mechanics" by David Morin, a nice (and inexpensive) book you might find useful as a supplement.[iii] Let the dropped ball have position $x_1(t)$ and the thrown ball $x_2(t)$, with $t=0$ when the two balls are released. For simplicity, let $+x$ be upward, with the origin at ground level. This gives both balls an acceleration of $-g$. Now we can readily write down their positions at any time, given the starting height h of the first ball and the initial velocity v_0 of the second:

$$x_1(t) = h - \frac{1}{2}gt^2 \tag{1}$$

$$x_2(t) = v_0 t - \frac{1}{2}gt^2 \tag{2}$$

First we can find the time when the second ball is at rest

$$v_2(t) = \frac{dx_2}{dt} = v_0 - gt = 0 \tag{3}$$

$$\Longrightarrow \quad t = \frac{v_0}{g} \tag{4}$$

At this time, the position of both balls should be the same if they are to collide.

[iii] See `http://www.people.fas.harvard.edu/~djmorin/book.html`.

$$x_1(\frac{v_0}{g}) = h - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = h - \frac{v_0^2}{2g} \tag{5}$$

$$x_2(\frac{v_0}{g}) = v_o\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g} \tag{6}$$

$$\implies \quad h - \frac{v_0^2}{2g} = \frac{v_0^2}{2g} \tag{7}$$

$$\implies v_0 = \sqrt{gh} \tag{8}$$

This relates the initial velocity of the second ball to the starting height of the first ball. Using this in either $x_1(t)$ or $x_2(t)$ along with the previously found time gives us an expression for the height of the collision:

$$x_2(\frac{v_0}{g}) = \frac{v_0^2}{2g} = \frac{gh}{2g} = \frac{h}{2} \tag{9}$$

The balls collide at exactly half the starting height of the first ball, at a time $t = \frac{v_0}{g} = \sqrt{\frac{h}{g}}$. Their relative speed at the time of the collision is also readily found:

$$v_1(t) - v_2(t) = -gt - v_0 + gt = -v_0 \tag{10}$$

In fact, at *any* time the difference in the balls' speeds is v_0 - this is the relative speed they start out with, and since we have only the influence of gravity to worry about, their speeds at any later time changes by exactly the same amount, $-gt$.

4. You throw a ball upward. After half of the time to the highest point, the ball has covered what fraction of its maximum height? Ignore air resistance.

Solution: Another problem from David Morin's book (see previous problem). Let the ground level be $x=0$, with $+x$ running upward. The ball's position at any time, assuming an initial velocity v_0, is then[iv]

$$x(t) = v_0 t - \frac{1}{2}gt^2 \tag{11}$$

The time to its highest point is found by maximizing $x(t)$, or equivalently, finding the time at which the velocity is zero.

[iv] We are not given the initial velocity, but we need it to work the problem. In most cases like this, quantities you had to introduce yourself will be part of the calculation and not the final answer. In this case, we introduced v_0 to solve the problem, but since it wasn't specified our final answer should be independent of v_0.

$$v(t) = \frac{dx}{dt} = v_0 - gt = 0 \tag{12}$$

$$\implies \quad t_{\text{max}} = \frac{v_0}{g} \tag{13}$$

At this time, we can find the ball's height:

$$x\left(t_{\text{max}}\right) = v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g} \tag{14}$$

At half this time, the ball's height is

$$x\left(\frac{1}{2}t_{\text{max}}\right) = v_0\left(\frac{v_0}{2g}\right) - \frac{1}{2}g\left(\frac{v_0}{2g}\right)^2 = \frac{3v_0^2}{8g} \tag{15}$$

The fraction of maximum height is then

$$\text{fraction of max height} = \frac{x(\frac{1}{2}t_{\text{max}})}{x(t_{\text{max}})} = \frac{\frac{3v_0^2}{8g}}{\frac{v_0^2}{2g}} = \frac{3}{4} \tag{16}$$

Since the ball is going much faster during the first half of its motion, it covers more distance. The last half of the ball's flight only covers 1/4 of the net vertical distance.

5. A ball is dropped, and then another ball is dropped from the same spot one second later. As time goes on while the balls are falling, what is the distance between them at any given time? (Ignoring air resistance, as usual.)

Solution: Let the starting point of the balls be $x=0$, with $+x$ upward. Let the time difference between dropping the two balls be $T=1\,\text{s}$ (so we can do this symbolically). The first ball falls a total of $t+T$ seconds after the second has fallen for t seconds, so the positions of the two balls can be written

$$x_1(t) = -\frac{1}{2}g\left(t+T\right)^2 \tag{17}$$

$$x_2(t) = -\frac{1}{2}gt^2 \tag{18}$$

Their difference is easily found:

$$\Delta x = x_2 - x_1 = -\frac{1}{2}gt^2 + \frac{1}{2}g\left(t+T\right)^2 = -\frac{1}{2}gt^2 + \frac{1}{2}gt^2 + gTt + \frac{1}{2}gT^2 = \frac{1}{2}gT^2 + gTt \tag{19}$$

The first term is constant, and represents how far the first ball falls before the second one is dropped. The second term increases linearly with time, representing the fact that while both balls have the same acceleration, the first ball has been accelerating a time T longer. While their relative velocity is constant (the two velocities will always differ by gT, the speed the first ball picks up before the second is dropped), the separation between the two balls increases linearly.

6. Find for me the numerical value I of the following integral, by any means necessary. No work need be shown for this problem, but do note how you obtained the answer.

$$I = \int\limits_0^{1.026955} 3\sin\left(x^2\right)dx$$

Solution: There is no analytic solution to this integral. You'll need a numerical technique, the simplest of which is to ask Wolfram Alpha. Try something like this (clickable link):

`http://www.wolframalpha.com/input/?i=integral+of+sin%28x%5E2%29+from+0+to+1.026955`

To 6 significant digits, the answer is 1.00000. This is one of the *Fresnel Integrals*, and they come up frequently in optics, among other things. `http://en.wikipedia.org/wiki/Fresnel_integral`