UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

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Problem Set 5 Solutions

1. A tall, cylindrical chimney falls over when its base is ruptured. Treat the chimney as a thin rod of length l. At the instant it makes an angle θ with the vertical as it falls, what is the tangential acceleration of the top? The moment of inertia of a rod about its end point is $\frac{1}{3}ml^2$.

Solution: Though it may not seem obvious at first, an energy-based approach is somewhat easier in this case. First things first. When the chimney falls, any point along its length a distance r from the base will describe circular motion with radius r. Therefore, all we need to consider is circular motion, albeit with a varying angular acceleration.

We already know the radial (normal; a_r) and tangential (a_t) components of acceleration required for circular motion in terms of the linear velocity v, angular velocity $\omega = d\theta/dt = v/r$, angular acceleration $\alpha = d^2\theta/dt^2$, and distance from the circle's center:

$$a_t = \frac{d^2s}{dt^2} = r\alpha$$
$$a_r = \frac{v^2}{r} = r\omega^2$$

Here s is the length of the path covered by a particular point throughout the motion. For circular motion at radius r through an angle θ, this is just the arc length $s = \theta r$. We will need both α and ω to find a_t and a_r. We can find ω from conservation of energy, and differentiate it with respect to time to find α.

Let the chimney have length l, and define the $\hat{\mathbf{y}}$ direction to be vertical with the origin at the base of the chimney. If the chimney falls through an angle θ relative to the vertical, its center of mass will have gone from a height $y = l/2$ to $y = (l/2)\cos\theta$. The change in the center of mass height Δy_{com} gives a change in gravitational potential energy, which must be equal to the gain in rotational kinetic energy. With I as the moment of inertia of the chimney,

$$mg\Delta y_{com} = \frac{1}{2}mgl\,(1-\cos\theta) = \frac{1}{2}I\omega^2$$

The moment of inertia of the chimney is that of a thin rod of length l and mass m rotating a distance $l/2$ from its center of mass:

$$I = I_{com} + m\left(\frac{l}{2}\right)^2 = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2$$

Putting this all together, we can find ω^2, which will give us a_r.

$$\omega^2 = \frac{mgl}{I}(1-\cos\theta) = \frac{3g}{l}(1-\cos\theta)$$

We are interested in the accelerations at the very end of the chimney, which is thus rotating at a distance l from the base of the rod. The radial acceleration is then

$$a_r = r\omega^2 = l\omega^2 = 3g(1-\cos\theta) \approx 5.32\,\mathrm{m/s^2}$$

Since we know ω, we can straightforwardly find α and therefore a_t. It is somewhat easier to implicitly differentiate ω^2 as given above and apply the chain rule:

$$\begin{aligned}
\frac{d\left(\omega^2\right)}{dt} &= 2\omega\frac{d\omega}{dt} = 2\omega\alpha = \frac{3g}{l}\sin\theta\frac{d\theta}{dt} = \frac{3g\omega}{l}\sin\theta \\
2\omega\alpha &= \frac{3g\omega}{l}\sin\theta \\
\alpha &= \frac{3g}{2l}\sin\theta
\end{aligned}$$

Given α, we can find a_t:

$$a_t = l\alpha = \frac{3g}{2}\sin\theta \approx 8.44\,\mathrm{m/s^2}$$

Finally, we want to know the angle at which the tangential acceleration equals the gravitational acceleration at the end of the chimney, $a_t = g$. No problem:

$$a_t = g = \frac{3g}{2}\sin\theta \qquad \theta = \sin^{-1}\frac{2}{3} \approx 42^\circ$$

At this point, the end of the chimney is “falling faster than free-fall.” At this point, it would be safer to jump off of the chimney . . .

We should be able to solve the problem with torque as well. It is really not that much harder in the end, and arguably just as straightforward. First: the only torque is due to the weight of the chimney acting at a distance $l/2$ from the center of mass. The force is $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{y}}$, and a vector pointing from the center of rotation to the point of its application is $\vec{\mathbf{r}} = (l/2)\sin\theta\,\hat{\boldsymbol{\imath}} + (l/2)\cos\theta\,\hat{\boldsymbol{\jmath}}$. Or, if you like, the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ is θ, $|\vec{\mathbf{r}}| = l/2$, and $|\vec{\mathbf{F}}| = mg$ since we only need magnitudes. In any case: this torque must give $I\alpha$:

$$\begin{aligned}
I\alpha = \tau_{\text{net}} &= \vec{\mathbf{r}}\times\vec{\mathbf{F}} = |\vec{\mathbf{r}}||\vec{\mathbf{F}}|\sin\theta = -\frac{l}{2}mg\sin\theta \\
\alpha &= -\frac{l}{2I}mg\sin\theta = -\frac{3g}{2l}\sin\theta
\end{aligned}$$

The torque is negative, consistent with its tending to cause a clockwise rotation of the chimney. Using the torque method, you arrive at an expression for α identical to the one above derived using energy. At this point, you might still want to find ω. You can do that by equating the work done by the torque τ acting through an angle θ to the increase in kinetic energy:

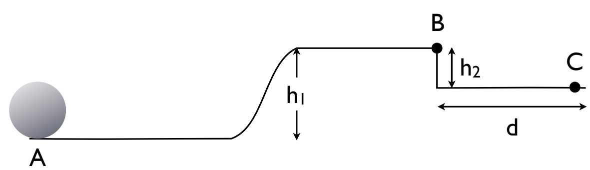
$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta = \frac{1}{2} I \omega^2$$

The integral is simple, and its limits are from the initial vertical configuration ($\theta_i = 0$) to the final angle of interest (θ_f). You can verify that this yields the same expression for ω^2 as above.

Two different approaches, same answer, as it has to be. It is sometimes just a matter of taste, memory, and minimizing pain that determines which method to use.

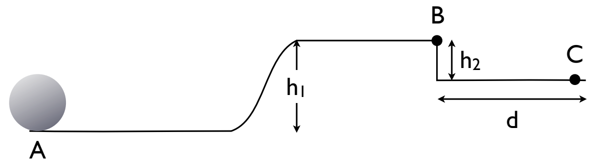
2. A wheel is rotating freely at angular speed 800 rev/min on a shaft. The shaft has negligible rotational inertia. A second wheel, initially at rest and with twice the rotational inertia of the first wheel, is suddenly coupled to the same shaft. **(a)** What is the angular speed of the resultant combination of the shaft and the two wheels? **(b)** What fraction of the original rotational kinetic energy is lost?

3. In the figure below, a small, solid, uniform ball is to be shot from point A so that it rolls smoothly along a horizontal path, up a ramp, and onto a plateau. Then it leaves the plateau horizontally to land on a game board, a horizontal distance d from the right edge of the plateau. The vertical heights are $h_1 = 5.00$ cm and $h_2 = 1.60$ cm. With what speed must the ball be shot at point A for it to land at $d = 6.00$ cm? The moment of inertia of a solid sphere is $\frac{2}{5} m R^2$.



Solution: **Solution:** Our sphere starts out at point A in the sketch below already undergoing smooth rolling motion, with center of mass velocity v_i. Since the sphere rolls without slipping, its angular and linear velocities must be related by the sphere's radius R, $v_i = R\omega$. We can apply conservation of mechanical energy to find the sphere's velocity at point B. Let the zero of gravitational potential energy be the lowest level in the diagram (the height of point A). At A, the total mechanical energy is purely kinetic, with both linear and rotational terms:

$$K_A + U_A = \frac{1}{2} m v_i + \frac{1}{2} I \omega_i^2 = \frac{1}{2} m v_i + \frac{1}{2} I \frac{v_i^2}{R} = \frac{1}{2} v_i^2 \left(m + \frac{I}{R^2} \right)$$



At point B, we also have translational and rotational kinetic energy, characterized by linear and angular velocities v_b and ω_b, respectively. We still have $v_b = R\omega_b$, since the motion is purely rolling without slipping. We also have now a gravitational potential energy mgh_1, and

$$K_B + U_B = \frac{1}{2}v_b^2\left(m + \frac{I}{R^2}\right) + mgh_1$$

Applying conservation of energy between A and B, we can solve for v_i:

$$\begin{aligned} K_A + U_A &= K_B + U_B \\ \frac{1}{2}v_i^2\left(m + \frac{I}{R^2}\right) &= \frac{1}{2}v_b^2\left(m + \frac{I}{R^2}\right) + mgh_1 \\ v_i^2 &= v_b^2 + \frac{2mgh_1}{m + I/R^2} \end{aligned}$$

We need only an expression for v_b. At point B, the sphere is launched from height h_2 above the far right platform, and it behaves just as any other projectile. In the absence of air resistance, the rate of rotation ω will not change from B to C, and we can therefore ignore the rotational motion. The sphere covers a horizontal distance d in a time t after being launched horizontally at v_b, and it covers a vertical distance h_2 in the same time t under the influence of gravity. Thus,

$$\begin{aligned} d &= v_b t \\ -h_2 &= -\frac{1}{2}gt^2 \\ \Longrightarrow \quad v_b &= d\sqrt{\frac{g}{2h_2}} \end{aligned}$$

Using this result in our expression above, and noting $I = \frac{2}{5}mr^2$ for a solid sphere,

$$\begin{aligned} v_i^2 &= v_b^2 + \frac{2mgh_1}{m + I/R^2} = \frac{d^2 g}{2h_2} + \frac{2mgh_1}{m + I/R^2} \\ v_i^2 &= \frac{d^2 g}{2h_2} + \frac{2mgh_1}{m + \frac{2}{5}m} = \frac{d^2 g}{2h_2} + \frac{2gh_1}{\frac{7}{5}} = \frac{d^2 g}{2h_2} + \frac{10}{7}gh_1 \\ v_i &= \sqrt{\frac{d^2 g}{2h_2} + \frac{10}{7}gh_1} \approx 1.34\,\mathrm{m/s} \end{aligned}$$

4. A (spherical) star of radius $R = 5 \times 10^5$ km has a rotational period of 60 days. Later in its life, its radius expands to $R = 5 \times 10^6$ km, though its mass M remains constant. What is the new rotational period after expansion? Presume the star's moment of inertial is kMR^2 at all times.

Solution: We can't readily use conservation of energy here, because to do so we'd have to account for the energy it took to accomplish the expansion. One thing we can rely on, however, is conservation of angular momentum. Both before and after, the angular momentum $L = I\omega$ should be the same. We only need to know the relationship between angular velocity ω and period T. In one period T, the star rotates through 2π radians at angular velocity ω, so it must be the case that

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \tag{1}$$

Presuming the star's mass remains constant and only its radius changes, conservation of angular momentum gives

$$I_i\omega_i = I_f\omega_f \tag{2}$$
$$kMR_i^2\omega_i = kMR_f^2\omega_f \tag{3}$$
$$R_i^2\left(\frac{2\pi}{T_i}\right) = R_f^2\left(\frac{2\pi}{T_f}\right) \tag{4}$$
$$T_f = \left(\frac{R_f}{R_i}\right)^2 T_i = 6000\,\text{days} \tag{5}$$

5. A small body of mass m hangs in equilibrium at one end of a light string of length l, the upper end of which is fixed. A small body of mass m moving horizontally with velocity $2\sqrt{gl}$ strikes the former body and adheres to it. Find:

(a) the velocity with which the combined bodies begin to move,
(b) the angle through which the string turns before coming to rest for an instant,

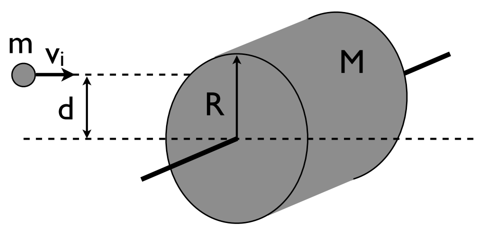
Solution: Conservation of momentum will get us the velocity after the collision.

$$mv_i = 2m\sqrt{gl} = 2mv_f \qquad \Longrightarrow \qquad v_f = \sqrt{gl} \tag{6}$$

If the string turns through an angle θ, its endpoint is now a distance $l\cos\theta$ from the vertical plane from which it hangs, rather than a distance l away as it is before the collision. The mass has then risen by an amount $l - l\cos\theta$. The maximum angle will be determined by the balance of the potential energy gained by moving through this height and the original kinetic energy after the collision.

$$\begin{aligned}
\frac{1}{2}(2m)\,v_f^2 &= 2mg\Delta y && (7)\\
mgl &= 2mgl\,(1-\cos\theta) && (8)\\
2-2\cos\theta &= 1 && (9)\\
\cos\theta &= \frac{1}{2} && (10)\\
\theta &= 60^\circ && (11)
\end{aligned}$$

6. A wad of sticky clay with mass m and velocity v_i is fired at a solid cylinder of mass M and radius R as shown below. The cylinder is initially at rest and mounted on a fixed horizontal axle that runs through its center of mass. The line of motion of the projectile is perpendicular to the axis and at a distance $d<R$ from the center. Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. The moment of inertia of a solid cylinder is $I=\frac{1}{2}MR^2$, the moment of inertia of a point particle mass m a distance R from an axis of rotation is $I=mR^2$.



Solution: Clearly this is an inelastic collision, since the wad of clay sticks to the cylinder, and conservation of energy is right out. Conservation of momentum is fine, but the cylinder rotates after the collision and makes things difficult. Conservation of angular momentum will be a whole lot easier. Since we must eventually find the angular velocity of the cylinder, it is natural to consider angular momentum about the center of the cylinder.[i]

Consider first the moment just before the wad hits the cylinder. The wad has momentum $p=mv$, acting at a distance R from the center of the cylinder. The angular momentum about the center of the cylinder is the wad's linear momentum times the perpendicular distance to the axis of rotation, which is just d. The angular momentum about the center of the cylinder is thus $L_i=mvd$.

After the wad hits the cylinder and both begin to rotate with angular velocity ω, the angular momentum can be found from the moments of inertia and ω. We have the wad, with $I_w=mR^2$, and the cylinder, with $I_c=\frac{1}{2}MR^2$, both rotating with angular velocity ω, and thus the total angular

[i] It does not really matter, angular momentum will be conserved regardless of our choice of origin. Choosing the center of the cylinder is simply more convenient.

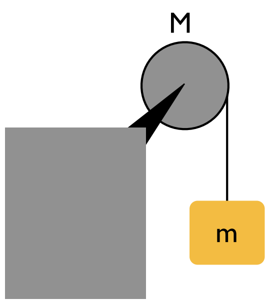
momentum after the collision is

$$L_f = I_w\omega + I_c\omega = mR^2\omega + \frac{1}{2}MR^2\omega = R^2\omega\left(m + \frac{1}{2}M\right)$$

Equating initial and final angular momentum, we can easily find ω:

$$R^2\omega\left(m + \frac{1}{2}M\right) = mvd$$
$$\omega = \frac{mvd}{R^2\left(m + \frac{1}{2}M\right)}$$

7. A uniform disk with mass $M = 2.5\,\mathrm{kg}$ and radius $R = 20\,\mathrm{cm}$ is mounted on a fixed horizontal axle, as shown below. A block of mass $m = 1.2\,\mathrm{kg}$ hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. *Note: the moment of inertia of a disk about its center of mass is* $I = \frac{1}{2}MR^2$.



Solution: In order to get acceleration and angular acceleration, we'll need to use force and torque, respectively. Start with the pulley. The tension T in the rope pulls on the edge of the disk a distance R from the center of rotation at an angle of $\theta_{RT} = 90°$, which causes a torque τ. This torque must equal the disk's moment of inertia times the angular acceleration.

$$\tau = RT\sin\theta_{RT} = RT = I\alpha = \frac{1}{2}MR^2\alpha \tag{12}$$
$$\alpha = \frac{2T}{MR} \tag{13}$$

We can get the tension by considering the force balance for the hanging mass. We have the tension in the tope pulling up, the weight of the mass pulling down, and an overall acceleration downward. Thus

$$\sum F = T - mg = -ma \tag{14}$$

Noting that $a=R\alpha$, this gives $T=mg-MR\alpha$. Now we've got two equations for α, which we can combine.

$$\alpha = \frac{2T}{MR} = \frac{2}{MR}\left(mg - mR\alpha\right) = \frac{2mg}{MR} - \frac{2m}{M}\alpha \tag{15}$$

$$\frac{2mg}{MR} = \alpha\left(1+\frac{2m}{M}\right) \tag{16}$$

$$\alpha = \frac{2mg}{R\left(M+2m\right)} \approx -24\,\mathrm{rad/s^2} \tag{17}$$

Given α, we can find a and T.

$$a = R\alpha = \frac{2mg}{M+2m} \approx -4.8\,\mathrm{m/s^2} \tag{18}$$

$$T = mg - MR\alpha = mg - \frac{2m^2g}{M+2m} = mg\left(1-\frac{2m}{M+2m}\right) = g\left(\frac{mM}{M+2m}\right) \approx 6.0\,\mathrm{N} \tag{19}$$

8. A bowler throws a bowling ball of radius R along a lane. The ball slides on the lane with initial speed v_o and initial angular speed $\omega_o=0$. The coefficient of kinetic friction between the ball and the lane is μ_k. The kinetic frictional force $\vec{\mathbf{f}}_k$ acting on the ball causes a linear acceleration of the ball while producing a torque that causes an angular acceleration of the ball. When the center of mass speed v_{cm} has decreased enough and angular speed ω has increased enough, the ball stops sliding and then rolls smoothly. **(a)** What then is the center of mass speed v_{cm} in terms of ω? During the sliding, what are the ball's **(b)** linear acceleration and **(c)** angular acceleration? **(d)** How long does the ball slide? **(e)** How far does the ball slide? **(f)** What is the linear speed of the ball when smooth rolling begins?

Solution: **(a)** Initially, the bowling ball is purely sliding, and as friction takes hold, the ball begins to roll. During the pure sliding phase, the ball rotates about its center of mass, independent of the overall center of mass motion.

After sufficient time, the rolling motion "catches up" with the sliding motion, and the ball begins to roll - it is no longer spinning about its center of mass, rolling smoothly. This smooth rolling is equivalent to a rotation about a point on the *surface* of the ball (not the center of mass), and as we derived earlier, this means that at the point we have smooth rolling motion, center of mass velocity and angular velocity are simply related:

$$v_{com} = r\omega$$

Here r is the given radius of the ball. During the sliding phase, we should write $v_{com} > r\omega$. The angular velocity is not high enough for the ball to "catch" on the lane. [ii]

(b) During the sliding phase, rotation is irrelevant to the dynamics - it is just like any other sliding

[ii] My parents used to own a bowling alley. I can go into much more detail on this problem for the curious.

object we have analyzed. A force of kinetic friction acts at the interface between the ball and the lane, which is equal in magnitude to $f_k = \mu_k F_N$, where μ_k is the coefficient of kinetic friction and $F_N = mg$ the normal force. Since this is the only force acting, we can easily apply Newton's law:

$$\begin{aligned}\sum F &= ma = -f_k \\ a &= -f_k/m = -\mu_k g\end{aligned}$$

(c) The angular acceleration α during the sliding phase is also provided by the friction force f_k. The friction force acts at a distance r from the center of mass, and at a right angle to a radius drawn from the center of mass to the intersection between the ball and lane. Thus, f_k also provides a torque τ, and as the only torque present, it must equal the moment of inertia of the ball times the angular acceleration. Noting $I = \frac{2}{5}mr^2$ for a solid sphere,

$$\begin{aligned}\tau_{\text{net}} &= rf_k = I\alpha = \frac{2}{5}mr^2\alpha \\ \alpha &= \frac{rf_k}{\frac{2}{5}mr^2} = \frac{5\mu_k mg}{2mr} = \frac{5\mu_k g}{2r}\end{aligned}$$

(d) During the sliding phase, the rotational and translational motion are essentially decoupled, and we can consider the center of mass motion from the point of view of standard kinematics. That is,

$$v_{com}(t) = v_{com}(0) + at = v_{com}(0) - \mu_k gt$$

Here $v_{com}(0)$ is the initial center of mass velocity, and we imply $t = 0$ at the moment the ball hits the lane. The same is true for the rotational motion, with the added simplification that the initial angular velocity is zero:

$$\omega(t) = \omega(0) + \alpha t = \left(\frac{5\mu_k g}{2r}\right) t$$

Say that the sliding stops at a time t_o. At the moment that sliding stops, we know that $v_{com}(t_o) = r\omega(t_o)$. This yields t_o, the time it takes to stop sliding, in terms of known quantities:

$$\begin{aligned}v_{com}(t_o) &= r\omega(t_o) \\ v_{com}(0) - \mu_k g t_o &= \frac{5}{2}\mu_k g t_o \\ t_o\left(\frac{5}{2}\mu_k g + \mu_k g\right) &= v_{com}(0) \\ t_o &= \frac{v_{com}(0)}{\frac{7}{2}\mu_k g} = \frac{2v_{com}(0)}{7\mu_k g}\end{aligned}$$

(e) Given the time to stop sliding, we can also find the distance covered d by standard kinematics:

$$
\begin{aligned}
d &= v_{com}(0)t_o + \frac{1}{2}at_o^2 \\
&= v_{com}(0)\left(\frac{2v_{com}(0)}{7\mu_k g}\right) - \frac{1}{2}\mu_k g\left(\frac{2v_{com}(0)}{7\mu_k g}\right)^2 \\
&= \frac{2\left[v_{com}(0)\right]^2}{7\mu_k g} - \frac{2\left[v_{com}(0)\right]^2}{49\mu_k g} \\
&= \frac{12\left[v_{com}(0)\right]^2}{49\mu_k g}
\end{aligned}
$$

(f) The linear (i.e., center of mass) speed at the moment sliding stops is also just kinematics:

$$v_{com}(t_o) = v_{com}(0)+at = v_{com}(0)-\mu_k gt = v_{com}(0)-\mu_k g\frac{2v_{com}(0)}{7\mu_k g} = v_{com}(0)-\frac{2}{7}v_{com}(0) = \frac{5}{7}v_{com}(0)$$

The linear velocity at the onset of rolling motion is 5/7 the initial velocity, independent of the coefficient of friction - only the moment of inertia comes into play.

9. A torque τ acts on a body and rotates it about a fixed axis from angle θ_i to angle θ_f. **(a)** Prove that the work done is

$$W = \int_{\theta_i}^{\theta_f} \tau\, d\theta \tag{20}$$

(b) Show that the rate at which work is done, power, is

$$P = \frac{dW}{dt} = \tau\omega \tag{21}$$

Solution: It is easiest to start with part (b), and we can use that result to prove (a). First, we note that since $\Delta K = W$, it is true that $dW/dt = dK/dt$. We also know what the rotational kinetic energy is, $K = \frac{1}{2}I\omega^2$. We will presume the moment of inertia I is constant.

$$\frac{dW}{dt} = \frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = I\omega\frac{d\omega}{dt} = I\omega\alpha = \omega\tau \tag{22}$$

Given that,

$$\frac{dW}{dt} = \omega\tau = \tau\frac{d\theta}{dt} \tag{23}$$

Integrating both sides with respect to t produces the desired result.

10. **(a)** Starting from $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$, show that $\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\boldsymbol{\tau}}$. **(b)** Show that if there is no external force, angular momentum is conserved.