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Motion in 1D

Our investigations into mechanics will start simply enough, trying to come up with a quantitative description of motion. Motion is already familiar to everyone, and it is true that many of the topics we will study are, at a *qualitative* level, obvious enough. A quantitative description, on the other hand, is less obvious. Our first task is then to consider some aspects of time and distance, the two ingredients required for describing motion. What we are after are quantitative observations, which can hopefully be turned in to more general quantitative relationships of phenomena allowing some predictive power. At that point, we're really doing science. Our end goal in this chapter is to come up with the theoretical formula for relating distance and time for a falling object, and build from there.

1 Types of Motion

Before we begin, we should probably lay down a few ground rules and definitions.

Our primary goal is to find the laws governing changes in the position of bodies as time goes on, and to do so we have to first describe these changes. We will – for now – ignore the agents causing changes in motion, and seek only to describe it. This is the study of *kinematics* that will occupy a great deal of our time this term. Generally speaking, there are three types of motion we will encounter, and we will deal with them in this order:

1. translational (e.g., car on the highway)
2. rotational (e.g., earth on its axis, tires on a moving car)
3. vibrational (e.g., pendulum, mass on a spring)

The first refers to movement from place to place without any sort of vibrating or spinning, the second to motion which is entirely spinning, and the third to motion which consists entirely of vibrating up and down or side to side. For the moment, we will deal only with the first type of motion, no spinning or vibrating. Later, we will deal with rotational and vibrational motion, and combinations of all three types of basic motions.

For most of what we are going to do, there is a nice simplification we can make known as the "particle model." Basically, the simplification is to ignore the spatial extent of objects all together, and treat them as point particles with a given mass. When the time comes, we will learn how and *when* to deal with the shape of a particular object, primarily by pretending that real objects are simply built up out of many of these little particles. Once we know how to analyze and predict the motion of a single little particle, figuring out what to do with a collection of them stuck together isn't much more difficult, as it turns out.

2 Describing motion mathematically

Much of what we're about to cover probably seems obvious, but it wasn't always so. Before Galileo, the study of motion was primarily in the domain of philosophy, based purely on argument and logic without appeal to experimental data. Galileo was skeptical of the whole endeavor, he was primarily interested in a *quantitative* explanation of motion, not merely logic and discourse.

Galileo's experiments dealt with the motion of falling objects. Rather than simply saying of the object "it falls to earth," he was more interested in the question of "how far did it go in a certain time interval?" For this, he observed the motion of an object, and measured the distance it had fallen after a certain amount of time.

These days, this sort of experiment is trivial. Even in Galileo's day, distance measurements were well known. Accurate time measurements, on the other hand, are a relatively recent invention - he didn't have a decent watch! What to do? First, he used his own pulse to provide a rudimentary time scale. Second, rather than simply dropping objects, he used a shallow inclined plane to slow them down a great deal. The whole experiment then consists of a long inclined plane marked off in even distance increments, and Galileo just needed to see how many units the object rolled past per pulse.

Now, we could perform a similar experiment, but with a great deal more accuracy, and without the need for the inclined plane to slow things down. In fact, one of your first laboratory experiments will be to track the position of a ball falling from rest as a function of time. Until then, here is an example of some data you might take, with the ball starting at rest at time $t=0$. The distance measured, x, is how far the ball fell from its starting position after the indicated amount of time t.

$\boldsymbol{t}$ (s)	$\boldsymbol{x}$ (m)
0	0
1	5
2	20
3	45
4	80
5	125

Table 1: *Position versus time for a falling ball. The clock was started ($t=0$) as soon as the object was released, and the object's position was measured relative to its starting position ($x=0$).*

Clearly this is not a real experiment - for one, the numbers are all a bit too 'round', and that's ignoring the fact that we don't have a 125 m (410 ft) hole to drop things into! That being said, what can we see from this data? How can we characterize the motion represented by this data? In order to characterize the motion properly, we need some further abstraction of motion to characterize how this ball falls. Not all motion is the same, qualitatively or quantitatively. We know that the ball fell 125 m in 5 s, but there are *zillions* of ways one can go 125 m in 5 s!

So, how do we figure out the nature of the ball's motion? Philosophically, one could come up with various arguments or hypotheses for how the ball ought to fall, and then test them against the data we have. The idea of *testability* is crucial - if it can't be tested or disproven, what's the point? One possible hypothesis (which turns out to be false) might be:

Possible hypothesis for motion of a falling object:
For every increment of time Δt, the ball falls the same distance Δx, or $\Delta x \propto \Delta t$. (False)

This basically says that during every 1 s interval, the ball should fall the same distance further. Mathematically, it means that distance fallen x is proportional to time t, so a plot of $x(t)$ should give a straight line. This is a beautifully simple idea, logical enough ... and spectacularly wrong! Let's test it with the data we have. All we need to do is figure out the differences in the distances between successive measurements Δx, since all the measurements are separated by 1 s. Generally speaking, when we write a something like Δx, the Δ implies a *change* in the quantity x:

$$\Delta x = \text{change in } x = x(t_2) - x(t_1) \tag{1}$$

where the notation $x(t_1)$ indicates position x measured at time t_1 and $x(t_2)$ indicates a measurement at time t_2 so it is clear that (1) x is a function of time t, (2) t_2 and t_1 represent two different times, and (3) the position is measured at these two different times.

In fact, this change of position Δx per time interval Δt would be the slope of the $x(t)$ graph. Specifying this slope or rate of change $\Delta x/\Delta t$, as opposed to just the distance covered Δx, has the advantage of not tying us to any particular time interval Δt, the rate of change already accounts for how long one waits between measurements.

time t (s)	distance x (m)	rate distance changed $\Delta x/\Delta t$ (m/s)
0	0	–
1	5	5
2	20	15
3	45	25
4	80	35
5	125	45

Table 2: *The rate at which the ball's position changes during every increment of time.*

Clearly, our hypothesis is not true – the ball falls even farther in a given second the longer it has been falling! On the other hand, we notice something interesting now, the *rate of change* is increasing at a steady rate. In fact, we could describe the increase in the rate of change perfectly with the equation

$$\frac{\Delta x}{\Delta t} = 10t - 5 \tag{2}$$

If we were to look even more carefully, we would find that the position versus time data increases *parabolically*, according to $x(t) = 5t^2$. But more on that later.

Evidently, for our falling object, the rate at which the position changes increases linearly in time. What is this quantity? It has units of distance over time, or meters per second, and it is what we would usually call the object's *speed*! In physics, words like 'speed' and 'velocity' have particular meaning, and cannot be used interchangeably . . . but we will come to that in time. Another important point is that what we have calculated is speed or velocity over a whole interval of time, not just at one particular instant. We will make a distinction between *average* velocities like this one, where we subtract two positions and divide by the difference of two corresponding times, and *instantaneous* velocities, specified at a single instant in time.

In any event, what we have calculated is a suitable definition for average velocity. Averaging will usually be denoted by a horizontal line over the symbol in question.

$$\overline{v} = \text{average velocity} = \frac{\Delta x}{\Delta t} \tag{3}$$

Ideally, we would want to have enough data that we can make Δt as small as possible, in order to specify velocity over increasingly fine intervals of time. The ultimate limit would be letting the time interval go to zero, so we could speak of velocity at a precise instant in time. Mathematically, this is no more than the definition of a derivative, and the derivative of position with respect to time gives us the *instantaneous velocity* v:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt} \tag{4}$$

Of course, no real data set has $\Delta t \to 0$, but for the purpose of building a mathematical model, it is much more convenient to use the *continuous time* derivative formulation rather than the *discrete time* finite difference formulation.

We can take our analysis one step further and ask how the object's *average velocity* changes in time. We do this in the same way, calculating the successive differences in average velocity between measurements and dividing by the time interval, thereby finding the rate at which the velocity is changing while the object falls:

time t (s)	distance x (m)	average velocity $\overline{v}$ (m/s)	rate velocity changes $\Delta\overline{v}/\Delta t$ (m/s^2)
0	0	–	–
1	5	5	–
2	20	15	10
3	45	25	10
4	80	35	10
5	125	45	10

Table 3: *The rate at which the ball's average velocity changes during every increment of time.*

Now we are on to something interesting: the rate at which the *velocity* changes is indeed a constant! We have discovered what we might call a constant of the motion, and this leads us to a second hypothesis:

Possible hypothesis for motion of a falling object, #2:
The rate at which the velocity changes for a falling object is a constant. (True)

Equivalent hypothesis for motion of a falling object, #2:
The acceleration for a falling object is a constant. (True)

The rate of average velocity change with time has units of $(m/s)/s$, or m/s^2, and this quantity is what we usually call average acceleration $\overline{a}$. For a falling object, it is about $10\,m/s^2$, and apparently a constant for the entire motion.

$$\overline{a} = \frac{\Delta\overline{v}}{\Delta t} \tag{5}$$

Again, the "average" here means that we have calculated the rate of change over some discrete interval in time, rather than at a single instant. As we did with velocity, we can take the limit $\Delta t \to 0$ to find an *instantaneous* acceleration a, rather than using discrete time differences:

$$a = \lim_{\Delta t \to 0} \frac{\Delta\overline{v}}{\Delta t} \equiv \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2} \tag{6}$$

Thus, acceleration is the time rate of change of velocity, which is itself the time rate of change of position. Note that if we had started with the assumption of constant acceleration (for some reason), we are now able to derive the velocity and position as a function of time, simply by integrating. First, we can integrate acceleration to find velocity. Since $a = dv/dt$, we may write $dv = a\,dt$ and integrate. Let us assume we want to know the velocity at time t having started at time 0.

$$v(t) = \int_0^t dv = \int_0^t a\,dt = at \tag{7}$$

We can similarly integrate the velocity to find position, since $v = dx/dt$, then $dx = v\,dt$. We will further assume that the object of interest started out at position 0 at time 0.

$$x(t) = \int_0^t dx = \int_0^t v\,dt = \int_0^t at\,dt = \frac{1}{2}at^2 \tag{8}$$

Had we know that falling motion was characterized by constant acceleration, we could have come up with the correct forms for $x(t)$ and $v(t)$. Note that we needed two additional experimental inputs - the initial velocity and position. This is generally the case; since we integrated twice, there are two necessary constants of integration to completely specify the motion, and these constants (or *boundary conditions*) can only come from additional knowledge about the problem at hand. Later, we will derive the results above more carefully, properly taking into account these boundary conditions.

Beyond position and time, we have come up with two further abstractions, velocity and acceleration, which give us a better understanding of the character of motion. In particular, we now know that acceleration is a constant for free-fall motion. We will consider both more detail shortly, and apply our new techniques to more general types of motion..

3 Position, velocity, & acceleration

Let us take a step back and look at time, position, velocity, and acceleration again in more detail.

3.1 Time

So far as we know, and for all practical purposes, time runs continuously. For our discussions, we will assume that there are tiny stopwatches everywhere that can be started, stopped, and zeroed at will to measure the passage of time during the events of interest. Generally, we will refer to two special times denoted by subscripts:

$$t_i = \text{initial time, event starts} \tag{9}$$
$$t_f = \text{final time, event stops} \tag{10}$$
$$\Delta t = t_f - t_i = \text{elapsed time} \tag{11}$$

The 'zero' for time is wherever we find it convenient, so long as we are clear what instant we define to be $t = 0$

3.2 Position

Position is a quantity everyone is familiar with. We might state a position '227 Gallalee Hall', 'the corner of University and Hackberry', or '1 meter above my head.' What is common to describing position in any of these examples is an explicit or implied coordinate system and origin. As evidenced by these examples, we are more used to dealing with *relative positions* with an implied origin or point of reference.

In physics, we generally need to be explicit about not only the point of reference but the coordinate system we are using to describe positions. Neither changes the fundamental physics we are describing, and the choice of either is usually based on convenience more than anything. None the less, without being clear about these choices, only confusion can result.

A chosen coordinate system might be cartesian or circular in two dimensions, and we will usually pick the coordinate system based on the symmetry of the problem. For instance, for projectile motion, a cartesian $x-y$ coordinate system is fine, whereas for analyzing the motion of a child on a merry-go-round we might find circular coordinates easier to work with. The choice of origin is equally arbitrary, but will usually be chosen at some convenient location, such as the starting or ending point of the motion we are analyzing.

Position is interesting to us because changes in position constitute motion, and that is what we seek to understand first. For now, we will start with motion along a straight line (one dimensional motion), and simple numbers are all we need to describe position. When we move on to motion in a plane (two dimensional motion), we will need at least two numbers to describe position - either two coordinates and an origin, or the two components of a vector.

When talking about motion in one dimension, we wish to quantify changes in position. First, there is the *relative* change in position, which we'll usually call *displacement*. This is nothing more than the difference between the starting and ending position:

$$\Delta x = \text{displacement} = x_{\text{final}} - x_{\text{initial}} \equiv x_f - x_i \tag{12}$$

Displacement is the distance 'as the crow flies,' from point to point. If you drive to Birmingham and back again, your net displacement is zero because *in net* you haven't gone anywhere! This of course misses something important, the actual *distance* traveled. Distance is the entire length of the path traveled, so while your displacement for your drip to Birmingham might be zero, the distance is still 118 mi round trip.

3.3 Speed & velocity

Speed and velocity both represent the time rate of change of position, and in straight-line motion (motion in one dimension) they are essentially the same. They are not the same in general, however, a distinction which will be more apparent when we move on to motion in a plane (motion in two dimensions). For the moment, suffice it to say that *speed* is the rate at which distance changes, while *velocity* is the rate at which displacement changes, and includes the direction in which position is changing. Thus, a proper speed might be something like 2 m/s, whereas a proper velocity is something like 2 m/s *east*.

If we have measurements of position $x(t_f)$ and $x(t_i)$ at times t_f and t_i, respectively, we can define an *average* velocity over the time interval t_i to t_f:

$$\text{average velocity} = \overline{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t} = \frac{\text{displacement}}{\text{time}} \tag{13}$$

The average velocity is just the displacement over the elapsed time. For your trip to Birmingham and back, the average velocity is in fact zero, since your net displacement is zero! The average *speed*, on the other hand, is the total distance covered divided by the elapsed time:

$$\text{average speed} = \frac{\text{total distance covered}}{\text{total time}} \tag{14}$$

The average speed for your trip to Birmingham will not be zero, it will be the 118 mi round trip distance divided by the total elapsed time. With rare exception, we will deal only with velocities.

3.3.1 Instantaneous velocity

The average velocity as defined above represents the slope of a line drawn between points (t_i, x_i) and (t_f, x_f), and by its nature ignores any variation of velocity in the time interval between t_i and t_f. Obviously this is not so good in general, so we seek to make the time difference $\Delta t = t_f - t_i$ as small as possible. What happens if we keep making it smaller and smaller? Long story short, if Δt becomes infinitesimally small, what we end up with is the slope of the *tangent* line to the position-time graph at the point of interest, and the average velocity over a fleeing instant of time. Taken to the extreme of $\Delta t \to 0$, we find that the instantaneous velocity is the time derivative of the position.

$$\text{instantaneous velocity} = v(t_1) = \text{slope of tangent at } x(t_1) = \left.\frac{dx}{dt}\right|_{t_1} \tag{15}$$

Table 4: *Characterizing velocity*

$v \neq 0$	object is moving, position changes
$v < 0$	position decreasing
$v > 0$	position increasing
v changes sign	object changing direction

3.4 Acceleration

Acceleration represents the time rate of change of velocity, and as we will see soon, is closely tied to our notion of force. As with velocity, we can define both an average and instantaneous acceleration,

$$\text{average acceleration} = \overline{a} = \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t} = \frac{\text{change in velocity}}{\text{time}} \tag{16}$$

$$\text{instantaneous accel.} = a(t_1) = \text{slope of tangent at } v(t_1) = \left.\frac{dv}{dt}\right|_{t_1} = \left.\frac{d^2x}{dt^2}\right|_{t_1} \tag{17}$$

For our falling object, our hypothesis was that acceleration is a constant, so it makes no difference whether we use the average or instantaneous definition. In fact, for the bulk of this course, we will deal with constant (or zero) accelerations, and as such using the instantaneous acceleration will be sufficient.

Table 5: *Characterizing acceleration*

$a \neq 0$	velocity is changing
$a < 0$	velocity decreasing
$a > 0$	velocity increasing
a changes sign	object changing from speeding up to slowing down (or vice versa)

4 Motion with constant acceleration

In cases where we have a constant acceleration, we can come up with some relatively simple formulas for describing motion by thinking about what the resulting velocity and position must be.

Let us say we have a constant acceleration a_o. If acceleration is constant, what does that mean for velocity? Acceleration is the *slope* of the velocity versus time graph, so a constant acceleration means that we are looking for the sort of curve that is characterized by a constant slope – a line! Thus, the velocity versus time curve $v(t)$ is just a straight line of slope a_o, and must be of the form $v(t)=a_ot+b$.

What is the constant b? We don't know it yet, because acceleration only tells us the rate of change of velocity, we could add to that any *constant* velocity and not change the acceleration. Mathematically, it means that we know the slope of a line, but not its intercept. What we need to determine the intercept is a data point, or a *boundary condition*. Let us say that we know the initial velocity at time $t=0$ is v_i. Plugging in that condition,

$$v(0) = b + a_o\,(0) = v_i \qquad \Longrightarrow \qquad b = v_i \tag{18}$$

We could have done this much more quickly with a bit of calculus.

$$a = \text{const} \tag{19}$$

$$v(t) = \int a\,dt = at + C \tag{20}$$

Again, the arbitrary constant (C in this case, coming from the integration) is determined by the velocity at time zero, $v(0) = v_i$. In either case, the equation for velocity as a function of time under constant acceleration is completely known, provided we know the starting velocity (or the velocity at any single point in time):

Velocity versus time under constant acceleration a_o with starting velocity v_i:

$$v(t) = v_i + a_ot \tag{21}$$

Note that if the **acceleration is zero**, the velocity remains constant at v_i. What about the position versus time? We know that the velocity is increasing linearly with time according to the equation above, and velocity is the slope of the position versus time curve. What we seek is a curve whose slope increases linearly with time, or a curve whose tangent is a straight line of slope a_ot. Our particle starts out with velocity v_i at $t=0$ and undergoes acceleration a_o. Over this time, it moves from position x_i to position x at time t.

The average velocity over some time interval from 0 to t is just the average of the velocity at $t=0$ and the velocity at t,

$$\overline{v} = \frac{v_i + v(t)}{2} \tag{22}$$

Note that this equation only holds if the acceleration is constant. We know the expression for $v(t)$, however: $v(t)=v_i+a_ot$. Substituting this into the equation above,

$$\overline{v} = \frac{v_i + v_i + a_ot}{2} = v_i + \frac{1}{2}a_ot \tag{23}$$

On the other hand, we know that the average velocity is also the displacement divided by the elapsed time. If the particle starts at position x_i at time $t=0$ and moves to position x at time t,

$$\overline{v} = \frac{x - x_i}{t - 0} \tag{24}$$

Putting it all together and solving for x,

$$\overline{v} = \frac{x - x_i}{t} = v_i + \frac{1}{2}a_o t \tag{25}$$

$$x = x_i + v_i t + \frac{1}{2}a_o t^2 \tag{26}$$

Position versus time under constant acceleration a_o, starting velocity v_i and starting position x_i:

$$x(t) = x_i + v_i t + \frac{1}{2}a_o t^2 \tag{27}$$

We can solve equation 21 for time and substitute it into our position formula to come up with an expression relating only position, velocity, and acceleration without time at all:

$$t = \frac{v - v_i}{a_o}$$

$$x = x_i + v_i\left(\frac{v - v_i}{a_o}\right) + \frac{1}{2}a_o\left(\frac{v - v_i}{a_o}\right)^2 \quad \text{move } x_i\text{, multiply by } 2a_o$$

$$2a_o\left(x - x_i\right) = 2v_i v - 2v_i^2 + v^2 - 2v_i v + v_i^2$$

$$v^2 = v_i^2 + 2a_o\left(x - x_i\right) \tag{28}$$

This result is useful when you know everything except time.

Relating position, velocity, and acceleration when you don't know time:

$$v^2 = v_i^2 + 2a_o\left(x - x_i\right) \tag{29}$$

Here v_i and x_i are the starting velocity and position, a_o is the constant acceleration, and v and x are the velocity and position at another time.

5 Falling objects

Acceleration due to gravity: $g \approx 9.8\,\mathrm{m/s}$ toward the surface of the earth.