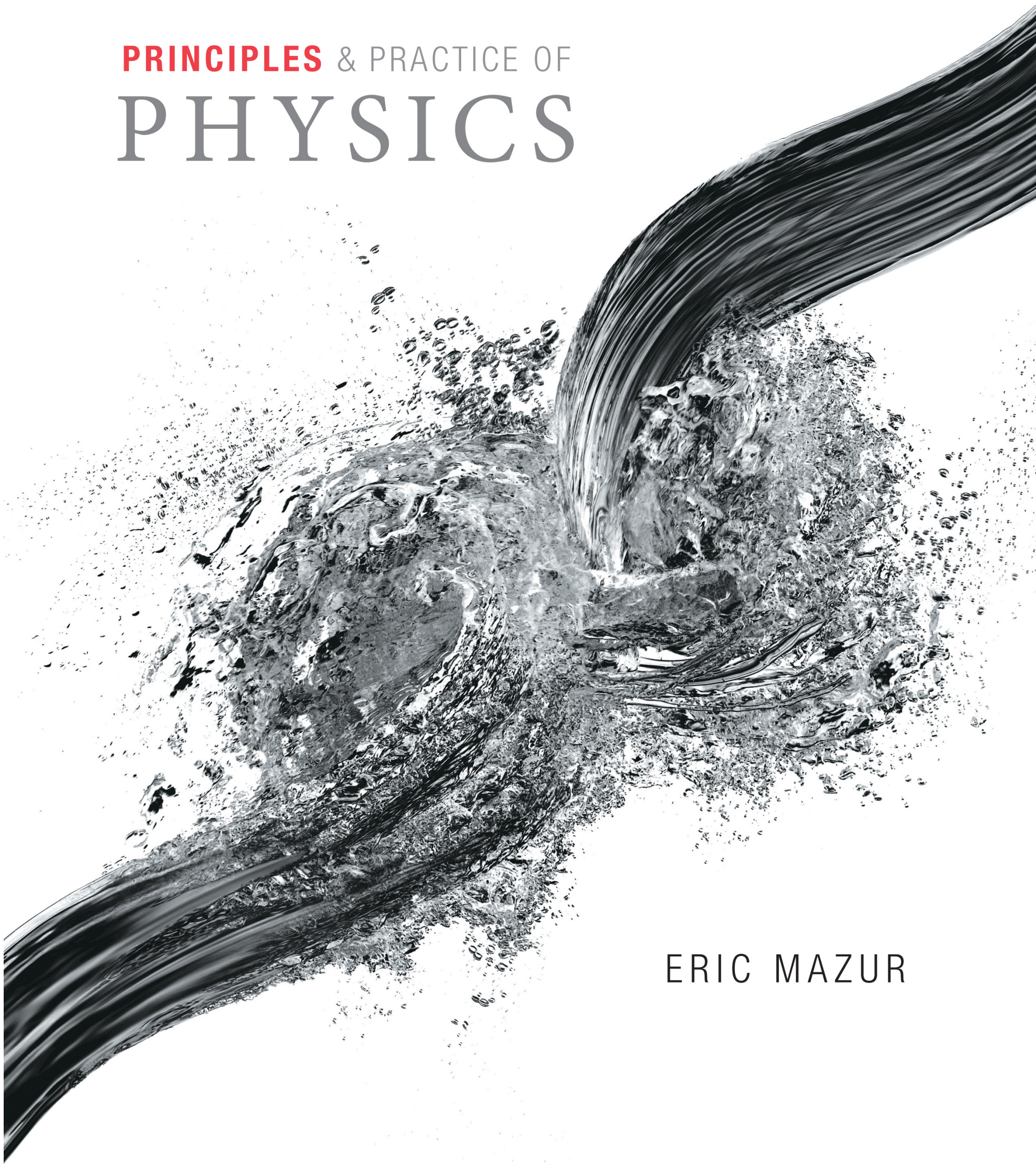
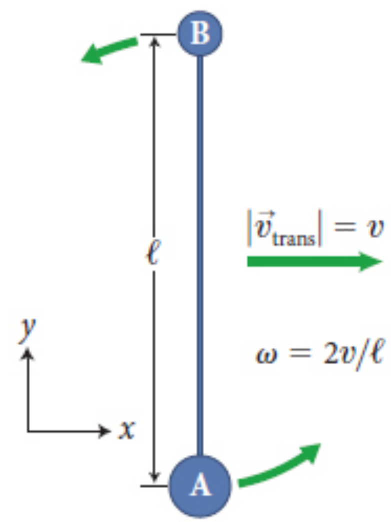
Chapter 13
Gravity



12.12

- The system shown below consists of two balls *A* and *B* connected by a thin rod of negligible mass. Ball *A* has five times the inertia of ball *B* and the distance between the two balls is ℓ. The system has a translational velocity of v in the x direction and is spinning counterclockwise at an angular speed of $\omega = 2v/\ell$.

- Determine the ratio of the instantaneous speeds of the two balls v_A/v_B at the moment shown.



12.12

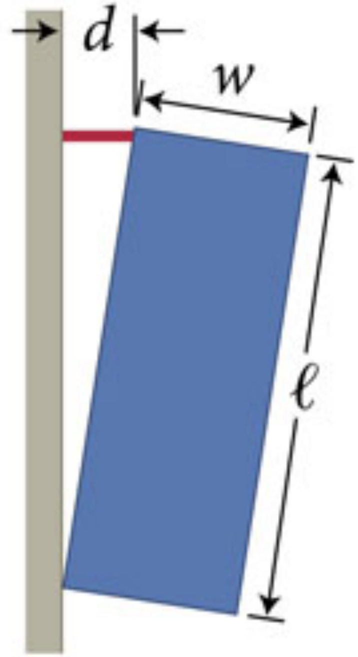
- First, find the center of mass – balls will rotate about it

$$x_{com} = \frac{m_A x_A + m_B x_B}{m_A + m_B}$$

- Each ball has a tangential speed *in addition to* the overall velocity of the dumbbell: $v_t = R\omega$
- *R* is the distance from a given ball to the COM
- *Add or subtract* tangential speed to overall speed of the dumbbell

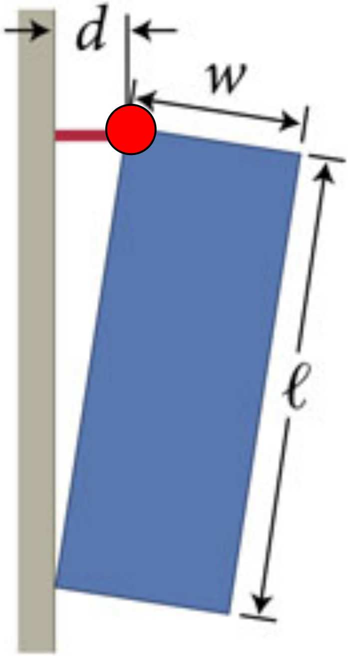
12.22

- The one about the clock.
- The geometry is the hard part.

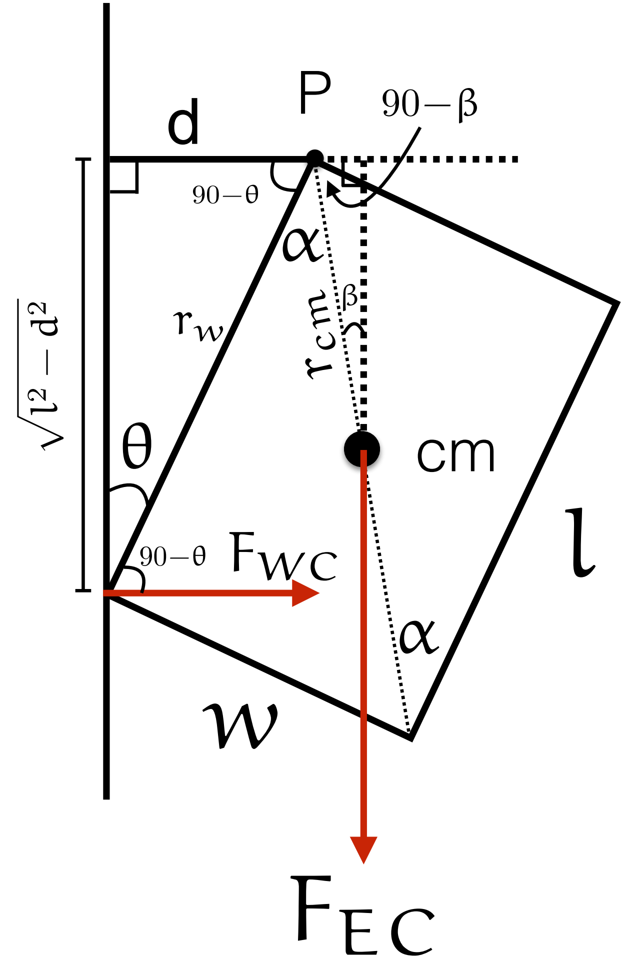


12.22

- For the clock to stay still, no net torque
- Where to find torques?
- Can’t be where clock meets wall.
- Choose point where nail attaches, don’t need to worry about force of nail then.
- Real problem? Geometry.



12.22



$$(90 - \theta) + \alpha + (90 - \beta) = 180$$

$$\beta = \alpha - \theta$$

$$r_{cm} = \frac{1}{2}\sqrt{l^2 + w^2}$$

$$r_{wc} = l$$

$$\angle(\vec{F}_{wc}, \vec{r}_w) = 90 - \theta$$

$$\angle(\vec{F}_{EC}, \vec{r}_{cm}) = \alpha - \theta$$

sin's, cos's, etc from right triangles …

$$\tau_{EC} = F_{EC} r_{cm} \sin\beta, \text{etc.}$$

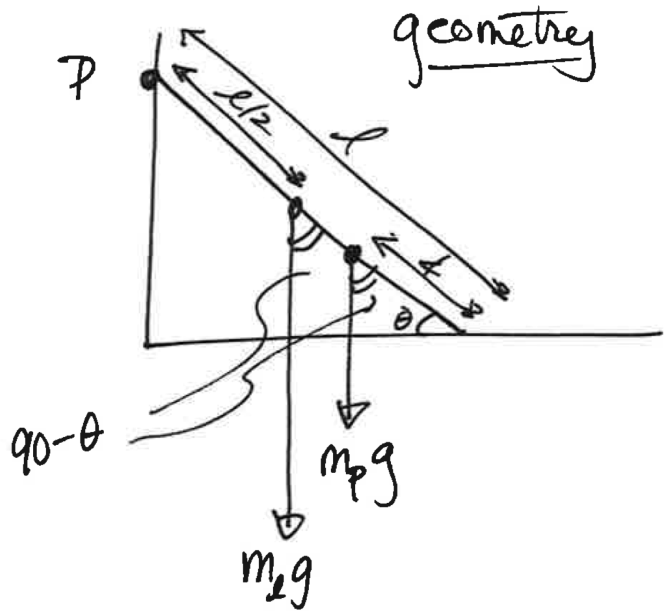
$$\sin(\alpha - \theta) = \sin\alpha\cos\theta - \cos\alpha\sin\theta$$

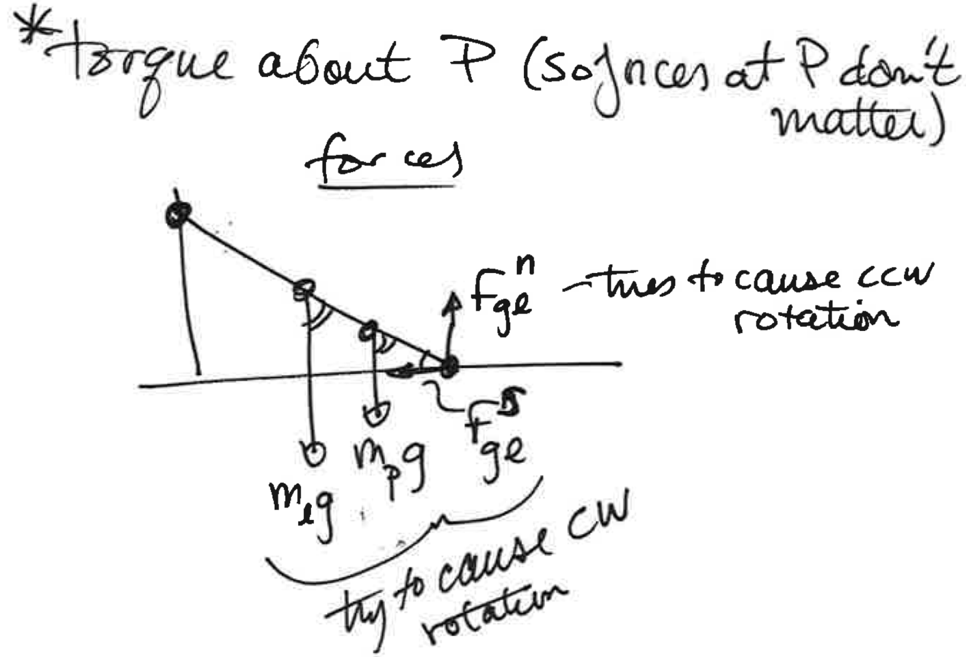
12.48

- A 24-kg ladder of length 5.5 m leans against a smooth wall and makes an angle of 50° with the ground. A 75-kg man starts to climb the ladder.
- If the coefficient of static friction between ground and ladder is 0.50, what distance along the ladder can the man climb before the ladder starts to slip?

- Again, geometry is crucial

12.48



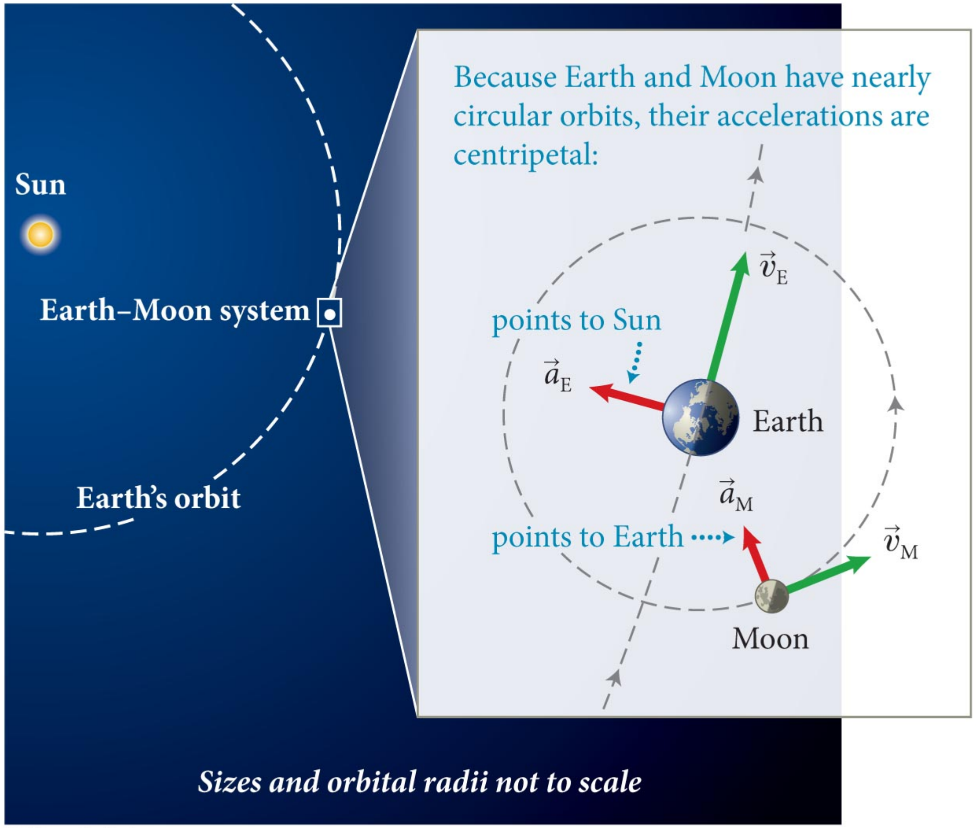


- Weights, normal force make angles of (90-θ) with vector to pivot point
- Friction makes angle θ

Concepts

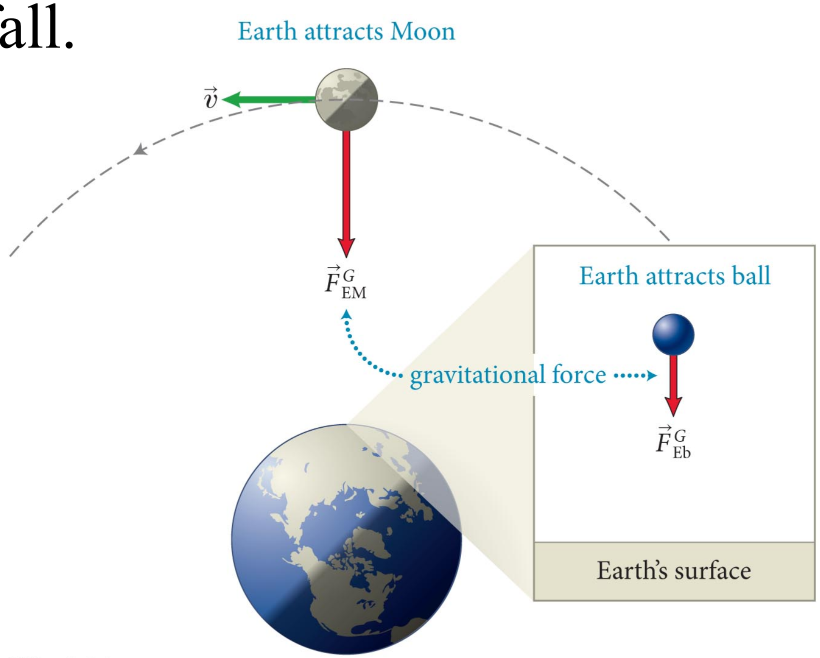
Section 13.1: Universal gravity

- Astronomical observations show that the Moon revolves around Earth and that Earth and other planets revolve around the Sun in roughly circular orbits and at roughly constant speeds.
- As we saw in Chapter 11, any object that is in circular motion requires some force to supply a centripetal acceleration.
- The force must be *central* – toward the center of the orbit



Section 13.1: Universal gravity

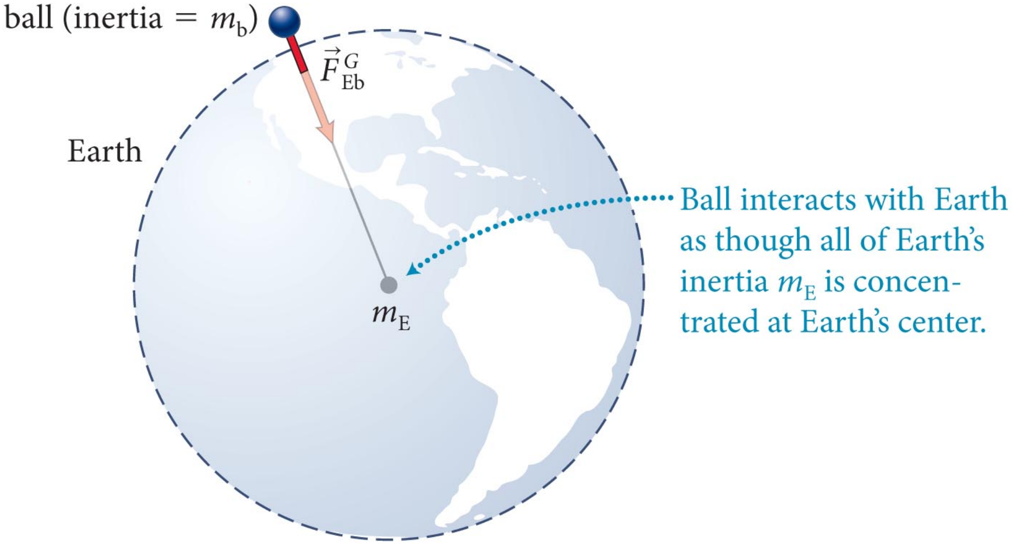
- In the late 17th century, Isaac Newton posited that the gravitation force is a *universal* attractive force between all objects in the universe.
  - The force that holds celestial bodies in orbit is the gravitational force.
  - This is the same force that causes objects near Earth's surface to fall.



Section 13.1: Universal gravity

- Newton further postulated:
  1. The effect of the gravitational force weakens with distance:
     - **A uniform solid sphere exerts a gravitational force outside the sphere with a $1/r^2$ depends as if all the matter in the sphere were concentrated at its center.**

(*a*) Gravitational interaction between Earth and ball near Earth's surface



Checkpoint 13.2

13.2 If the force of gravity decreases with the inverse square of the distance, why were we allowed, in all our earlier work on the gravitational force, to say that an object sitting on the ground, an object sitting in a tree 10 m above the ground, and an object flying at an altitude of 10 km all experience the same 9.8-m/s^2 acceleration due to gravity?

These distances are tiny compared to the earth's radius, ~6400 km

The relevant distance for a sphere is to its center, since we can treat it as though all mass is concentrated there.

Section 13.1: Universal gravity

- Newton further postulated:
  2. The strength of the gravitational force on an object is proportional to the quantity of material in it, a quantity called the **mass** of the object:
     - In everyday situations we can say:
       - **The mass of an object is equal to the object's inertia.**
- Relativity: this equality breaks down for motion at very high speeds.
- We can denote both quantities by the same symbol m and express each in kilograms.

Section 13.1: Universal gravity

- Combining these conclusions, we can write Newton's **law of gravity** for two objects of mass m_1 and m_2 separated by a distance r:

$$F^G_{21} = F^G_{12} \propto m_1 m_2 / r^2$$

Why $m_1 m_2$?

doubling either mass should double the force

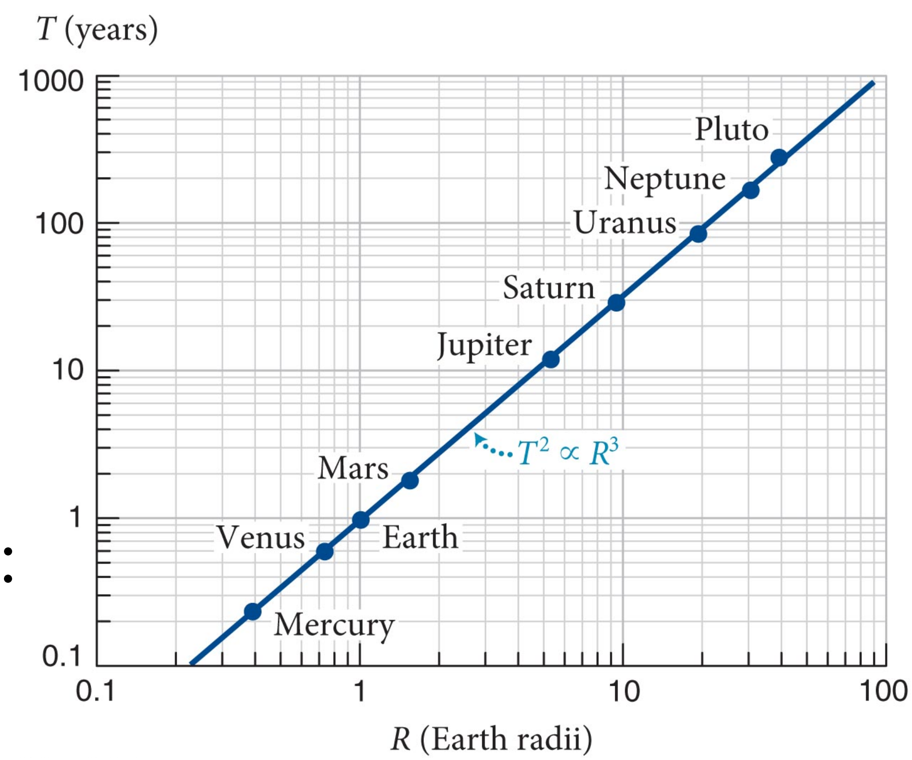
Why $1/r^2$?

interaction spreads out over surface of a sphere (like waves)

Other laws *possible*, but not consistent with observations

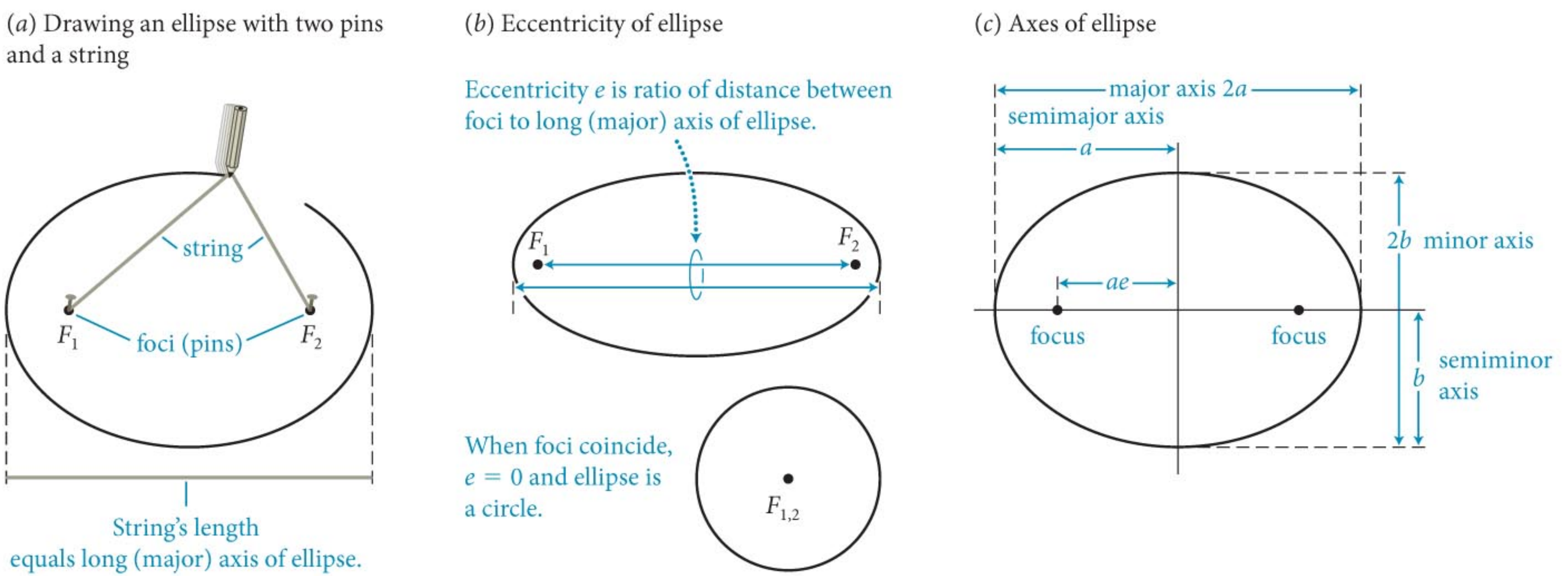
Section 13.1: Universal gravity

- Evidence for the $1/r^2$ dependence of the gravitational force is provided by the following observed relationship between the radii R and the period T of planetary orbits:
  - **The square of the period of a planetary orbit is proportional to the cube of the orbit's radius.**



Section 13.1: Universal gravity

- The $1/r^2$ dependence also explains the shape of the planetary orbits.
- Using conservation of energy and momentum, we can show that the orbit of a body moving under the influence of gravity must be an ellipse, a circle, a parabola, or a hyperbola.



Example13.1 Comparing gravitational pulls

Compare the gravitational force exerted by Earth on you with (*a*) that exerted by a person standing 1 m away from you and (*b*) that exerted by Earth on Pluto.

Example13.1 Comparing gravitational pulls (cont.)

❶ GETTING STARTED Although I don’t know yet how to calculate a numerical value for the gravitational force, I know that the gravitational force between two objects with masses m_1 and m_2 separated by a distance r is proportional to the factor m_1m_2/r^2.

To compare gravitational forces, I should therefore compare these factors for each given situation.

Example13.1 Comparing gravitational pulls (cont.)

❷ DEVISE PLAN My mass is about 70 kg, and I can get the masses of Earth and Pluto from Google. When I stand on the surface of Earth, the distance between me and the center of the planet is its radius, which I can also Google.

To do part *b*, I need to know the distance between Pluto and Earth. Planetary orbits are very nearly circular, however, so I can consider the semimajor axis *a* to be the radius of each (nearly) circular orbit, and … Google.

Example13.1 Comparing gravitational pulls (cont.)

❷ DEVISE PLAN Google shows that the "radius" of Pluto's orbit is about 40 times greater than that of Earth's orbit, and so I make only a small error by taking the Sun-Pluto distance as a measure of the (average) Earth-Pluto distance.

Armed with this information, I'll calculate the factor m_1m_2/r^2 first for Earth and me, then for the two situations described in the two parts of this problem.

Example13.1 Comparing gravitational pulls (cont.)

❸ EXECUTE PLAN When I stand on the surface of Earth, $r = R_E$, and so the factor $m_1 m_2/r^2$ is

$$\frac{m_1 m_E}{R_E^2} = \frac{(70\ \text{kg})(5.97 \times 10^{24}\ \text{kg})}{(6.38 \times 10^6 \text{m})^2} = 1.0 \times 10^{13}\ \text{kg}^2/\text{m}^2.$$

Example13.1 Comparing gravitational pulls (cont.)

❸ EXECUTE PLAN (*a*) For two 70-kg people separated by 1 m, I get

$$\frac{m_1 m_2}{r_{12}^2} = \frac{(70\text{ kg})^2}{(1\text{ m})^2} = 4.9 \times 10^3 \text{ kg}^2/\text{m}^2.$$

Thus the gravitational force exerted by a person standing 1 m from me is $(4.9 \times 10^3 \text{ kg}^2/\text{m}^2)/(1.0 \times 10^{13} \text{ kg}^2/\text{m}^2) = 4.9 \times 10^{-10}$ times the gravitational force exerted by Earth on me.✔

Example13.1 Comparing gravitational pulls (cont.)

❸ EXECUTE PLAN (*b*) For Earth and Pluto, I have

$$\frac{m_E m_p}{r^2_{EP}} = \frac{(5.97 \times 10^{24} \text{ kg})(1.36 \times 10^{22} \text{ kg})}{(5.9 \times 10^{12} \text{m})^2}$$

$$= 2.3 \text{ x } 10^{21} \text{ kg}^2/\text{m}^2.$$

This is $(2.3 \text{ x } 10^{21} \text{ kg}^2/\text{m}^2)/(1.0 \text{ x } 10^{13} \text{ kg}^2/\text{m}^2) = 2.3 \text{ x } 10^8$ times greater than the attraction between Earth and me. ✔

Example13.1 Comparing gravitational pulls (cont.)

❹ EVALUATE RESULT That the gravitational force exerted by another person on me is more than a billion times smaller than that exerted by Earth on me makes sense: Only the gravitational attraction between Earth and objects on Earth is noticeable.

That the gravitational force exerted by Earth on Pluto is 200,000,000 times greater than that exerted by Earth on me, even though Pluto is 1,000,000 times farther from Earth's center than I am, is amazing. Of course, Pluto's mass is about 10^{20} times greater than mine, and that factor more than makes up for the large difference between the Earth-me distance and the Earth-Pluto distance.

How about you and the nearest star?

- Alpha Centari:
  - similar to solar mass, 2.19×10^{30} kg
  - Distance: 4.13×10^{16} m

- Exerts a force on you of $mM/R^2 \sim 0.1$
- *Over 50,000 times weaker than the person sitting next to you.*
- *This is relevant for astrology*

Section 13.2: Gravity and angular momentum

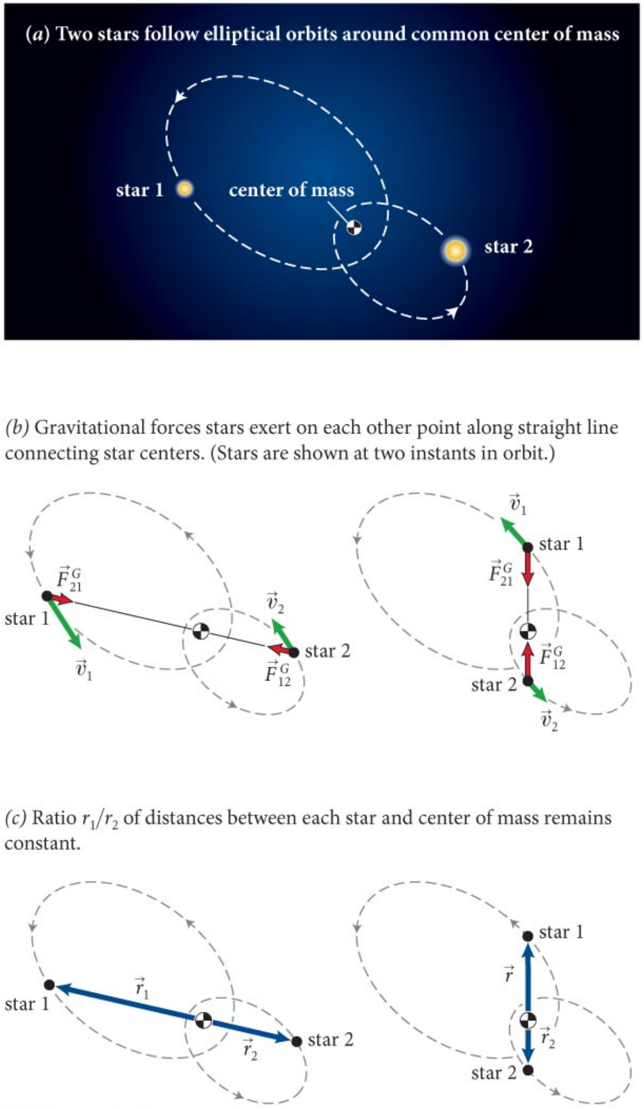
Section Goals

You will learn to

- Predict some consequences of the conservation of angular momentum for gravitationally interacting systems.
- Apply Kepler's laws to celestial objects.

Section 13.2: Gravity and angular momentum

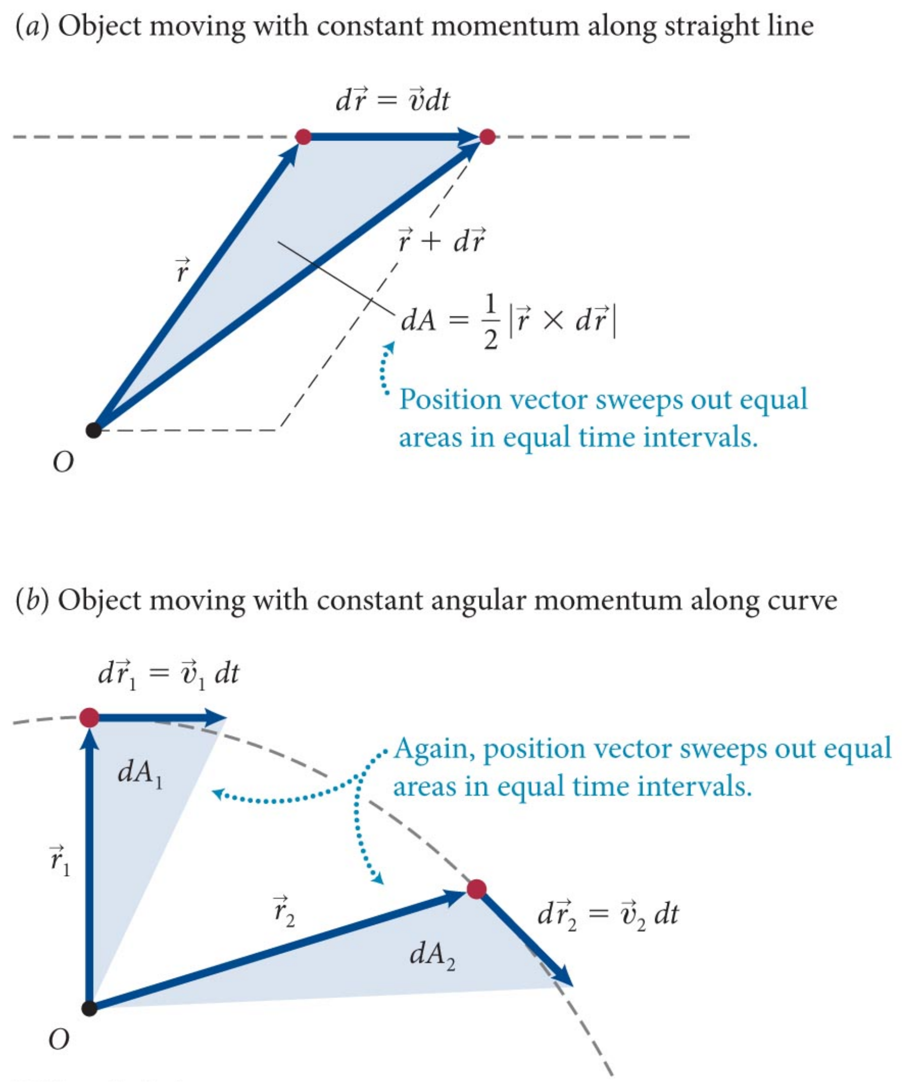
- The force of gravity is a **central force** (see figure):
  - The line of action of a central force lies along a straight line that connects the two interacting objects.
- **In an isolated system of two objects interacting through a central force, each object has a constant angular momentum about the center of mass.**



Section 13.2: Gravity and angular momentum

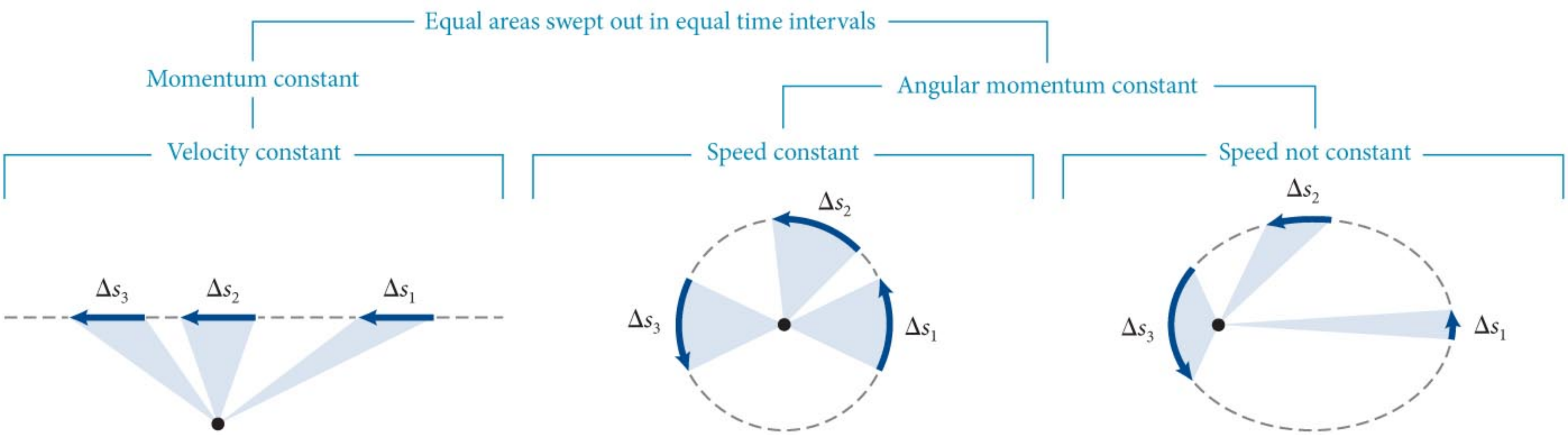
- As illustrated in the figure, we can show that
  - **The angular momentum of a particle about an origin is proportional to the rate at which area is swept out by the particle's vector position.**

$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}| = \frac{1}{2} r v_{\perp} = \frac{L}{2m}$$



Section 13.2: Gravity and angular momentum

- The figure shows three types of motion with constant angular momentum.



(*a*) Motion at constant velocity along straight line

(*b*) Motion at constant speed along circular path

(*c*) Motion about one focus of an ellipse in response to central force

Kepler's laws of planetary motion

Kepler's first law describes the shape of the planetary orbits:

> **All planets move in elliptical orbits with the Sun at one focus.**

Although the deviation from circular orbits is small, this statement was a radical departure from the accepted wisdom, dating back to Plato, that the planets, being heavenly bodies, were perfect and therefore could move in only perfect circles or combinations of circles.

Elliptical orbits follows from $F \sim 1/r^2$

Kepler's laws of planetary motion

Kepler's second law reveals that, even if planets are not in circular motion at constant speed, their motions obey the following requirement:

> **The straight line from any planet to the Sun sweeps out equal areas in equal time intervals.**

- Follows from the fact that gravity is a central force - geometry!
- Can also say this follows from conservation of L.

Kepler's laws of planetary motion

Kepler's third law relates the planetary orbits to one another:

> **The squares of the periods of the planets are proportional to the cubes of the semimajor axes of their orbits around the Sun.**

Kepler discovered this third law by painstakingly examining, over a period of many years, countless combinations of planetary data. **Follows from $F \sim 1/r^2$**

Kepler's laws of planetary motion

In keeping with Aristotelian notions, Kepler believed that a force was necessary to drive the planets along their orbits, not to keep them in orbit. Consequently, Kepler was unable to provide a correct explanation for these three laws. It was not until Newton that the single unifying reason for these laws was established.

Kepler's three laws follow directly from the law of gravity: The 1st and 3rd laws are a consequence of the $1/r^2$ dependence, and the 2nd law reflects the central nature of the gravitational force.

Section 13.2
Question 2

Which has greater acceleration in its orbit around Earth, the Moon or the International Space Station?

1. Both have the same acceleration
2. The Moon
3. The International Space Station

Section 13.2
Question 2

Which has greater acceleration in its orbit around Earth, the Moon or the International Space Station?

1. Both have the same acceleration
2. The Moon
3. ✔ The International Space Station – $a = F/m \sim M_e/r^2$, ISS has smaller orbital radius

Checkpoint 13.7

13.7 The space shuttle (use to) orbit Earth at an altitude of about 300 km.

(a) By what factor is the shuttle's distance to the center of Earth increased over that of an object on the ground?

(b) The gravitational force exerted by Earth on an object in the orbiting shuttle is how much smaller than the gravitational force exerted by Earth on the same object when it is sitting on the ground?

(c) What is the acceleration due to gravity at an altitude of about 300 km?

(d) While in orbit, the shuttle's engines are off. Why doesn't the shuttle fall to Earth?

Checkpoint 13.7

By what factor is the shuttle's distance to the center of Earth increased over that of an object on the ground?

- The radius of the earth is about 6400 km, so the distance to earth's center is increased by
  - $(6400 + 300)/(6400) \sim 1.05$

- Seriously: giant rockets only get us 5% further away from Earth's surface. The next 5% is easier though …

Checkpoint 13.7

What is the acceleration due to gravity at an altitude of about 300 km?

- Since the force goes as $1/r^2$, it decreases by a factor of $1/(1.05)^2 \sim 0.91$
- This gives $a \sim (0.91)(9.8\ \text{m/s}^2) \sim 8.9\ \text{m/s}^2$

Checkpoint 13.7

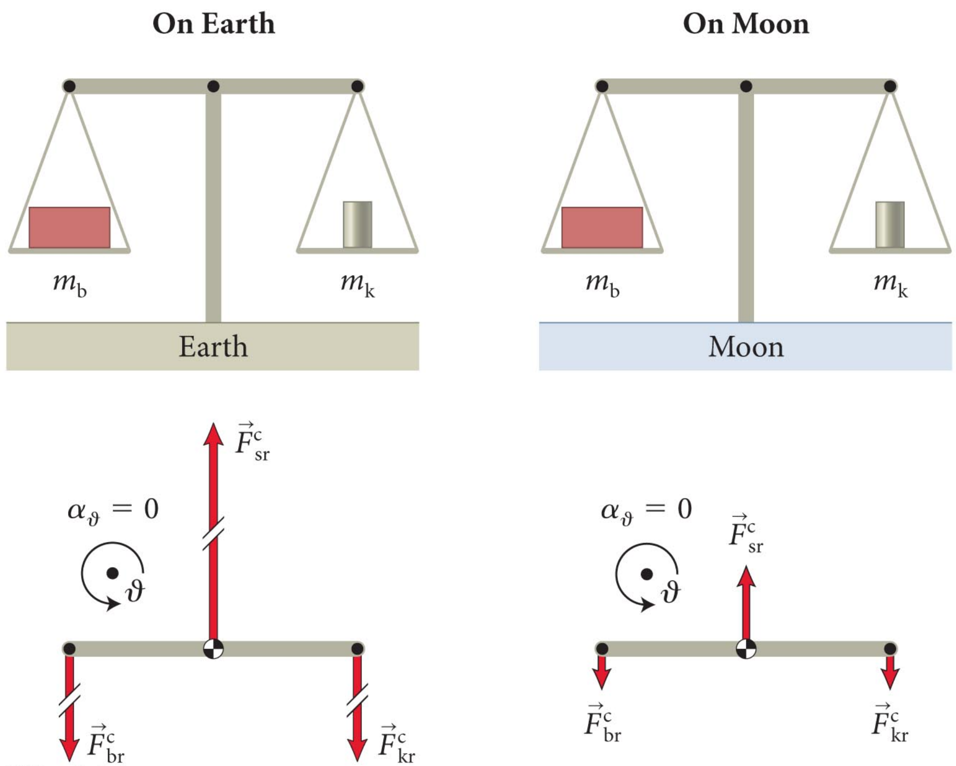
While in orbit, the shuttle's engines are off. Why doesn't the shuttle fall to Earth?

- The shuttle travels at such a high speed that the gravitational force exerted by earth is just enough to provide centripetal acceleration, $g = v^2/r$ (with r the distance from the shuttle to earth's center)
- Basically, it *is* falling, it just never manages to hit the earth …

- *"There is an art to flying, or rather a knack. Its knack lies in learning to throw yourself at the ground and miss. ... Clearly, it is this second part, the missing, that presents the difficulties."*

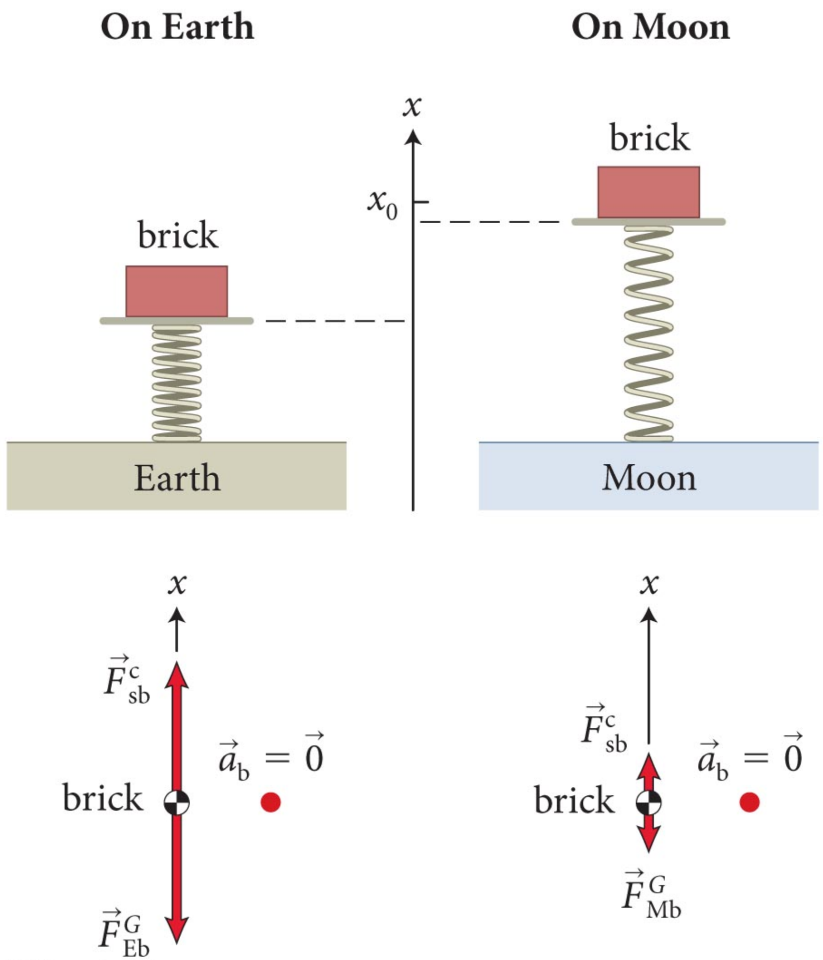
Section 13.3: Weight

- The figure illustrates the use of a **balance** to weigh an object:
  - An object whose mass (m_b) is to be determined is compared to a known mass (m_k).
  - As the free-body diagrams show, when the balance is in equilibrium, we obtain $m_b = m_k$ on the Moon just as we do on Earth.



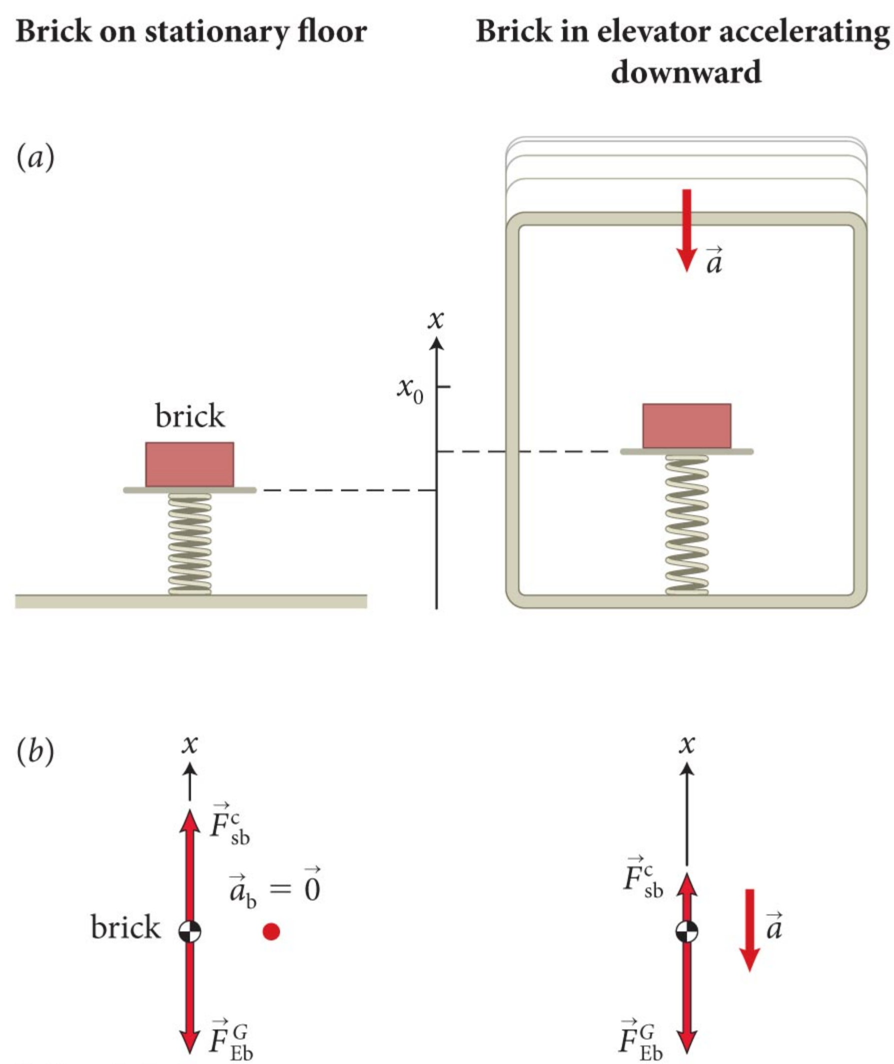
Section 13.3: Weight

- Another type of weighing device is the **spring scale**:
  - As illustrated by the figure, the spring scale tells us that the object weighs less on the Moon than on Earth.
  - The difference between the balance and the spring scale arises because
- **A spring scale measures the downward force exerted on it by its load, but a balance compares gravitational forces and measures mass.**



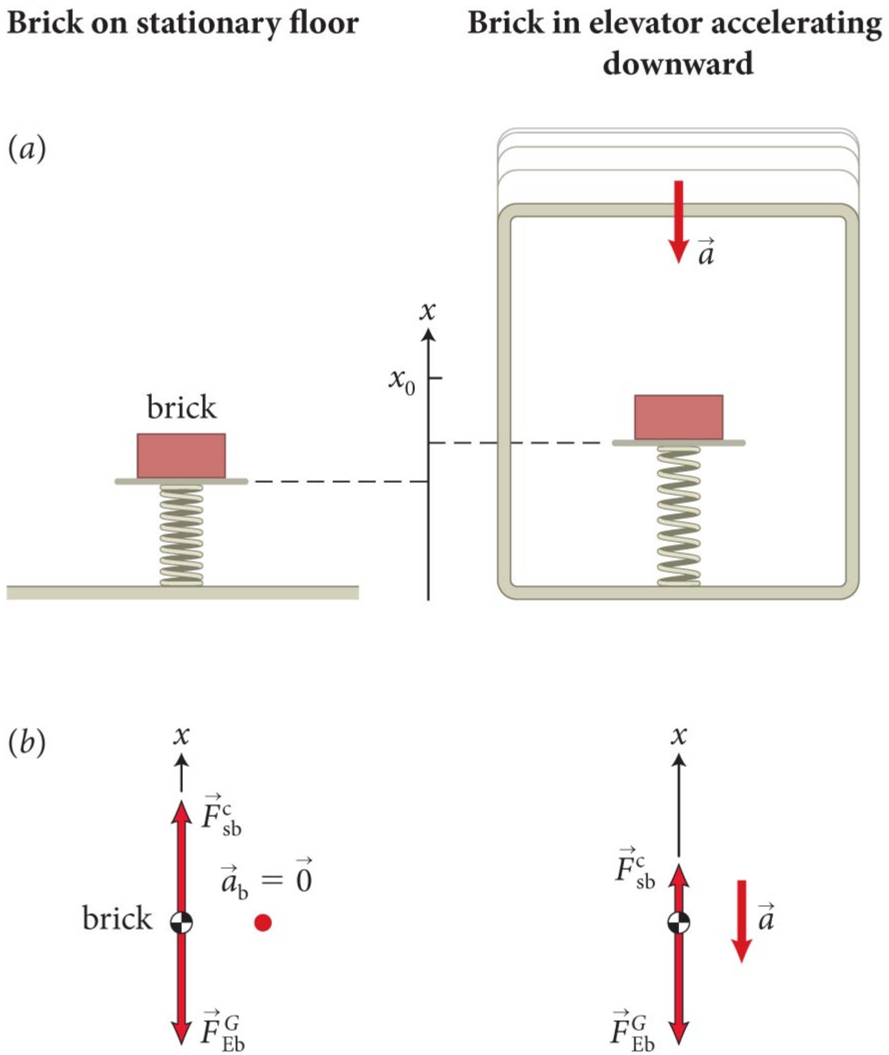
Section 13.3: Weight

- The reading on the spring scale depends not only on the gravitational pull but also on the acceleration of the scale.
- As illustrated by the free-body diagrams, a spring scale in a downward-accelerating elevator gives a smaller scale reading.



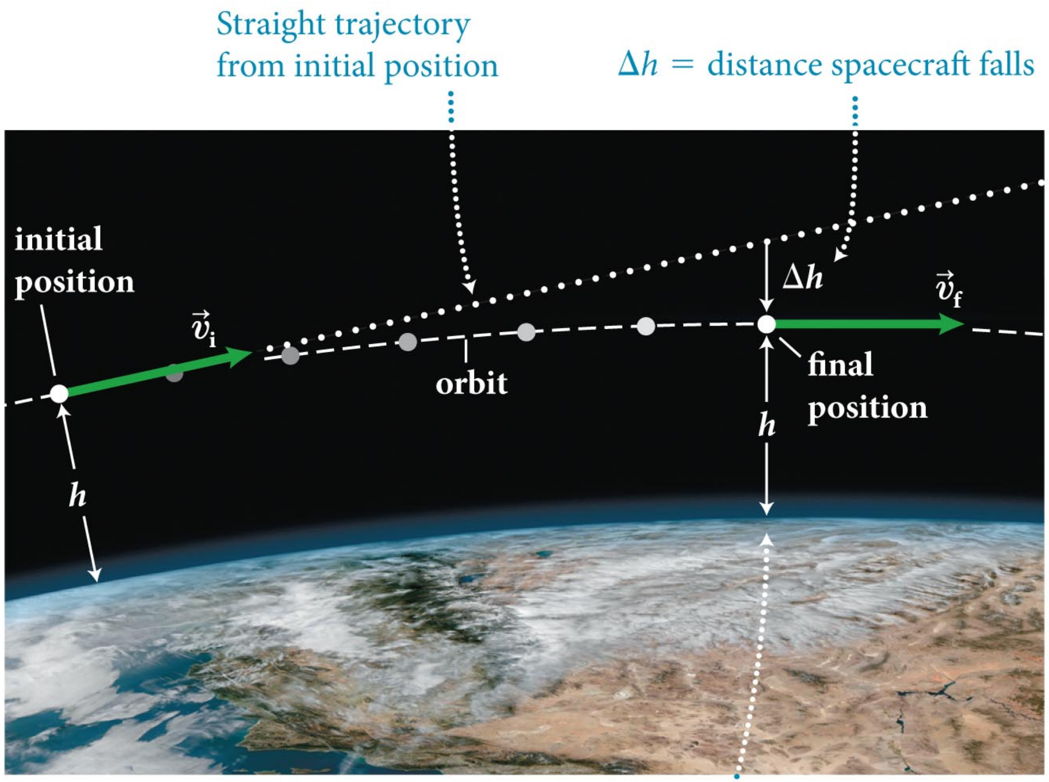
Section 13.3: Weight

- If the elevator has a downward acceleration of $a = g$ (in free fall) then the spring scale reading is zero:
  - **Any object in free fall—that is, any object subject to *only* a force of gravity—experiences weightlessness.**



Checkpoint 13.11

13.11 (*a*) How far would the space shuttle in Figure 13.20, 300 km above Earth, fall in 1.0 s? (*b*) If the radius of Earth is 6400 km, what would the shuttle's speed be?



Height h of orbit remains constant because of distance Δh spacecraft falls.

Checkpoint 13.11

- We already found $a \sim 8.9\ \text{m/s}^2$ at this altitude. The vertical distance h it falls in 1.0 s is

$$h = \tfrac{1}{2}at^2 \sim 4.5\ \text{m}$$

- If the shuttle remains 300 km above the earth it is in a circular orbit, so the acceleration is known: $a = v^2/r$
- Since in free fall $a = g$, we know $v^2/r = g$, or

$$v^2 = gr,\ \text{giving}\ v \sim 7.5 \times 10^3\ \text{m/s}$$

Section 13.3
Question 3

You stand on a spring scale placed on the ground and read your weight from the dial. You then take the scale into an elevator. Does the dial reading increase, decrease, or stay the same when the elevator **accelerates downward** as it moves upward?

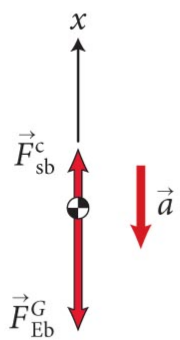
1. Increase
2. Decrease
3. Stay the same

Section 13.3
Question 3

You stand on a spring scale placed on the ground and read your weight from the dial. You then take the scale into an elevator. Does the dial reading increase, decrease, or stay the same when the elevator accelerates downward as it moves upward?

1. Increase
2. ✔ Decrease – free body diagram: $F_s - mg = -ma$

   so $F_s = mg - ma$
3. Stay the same

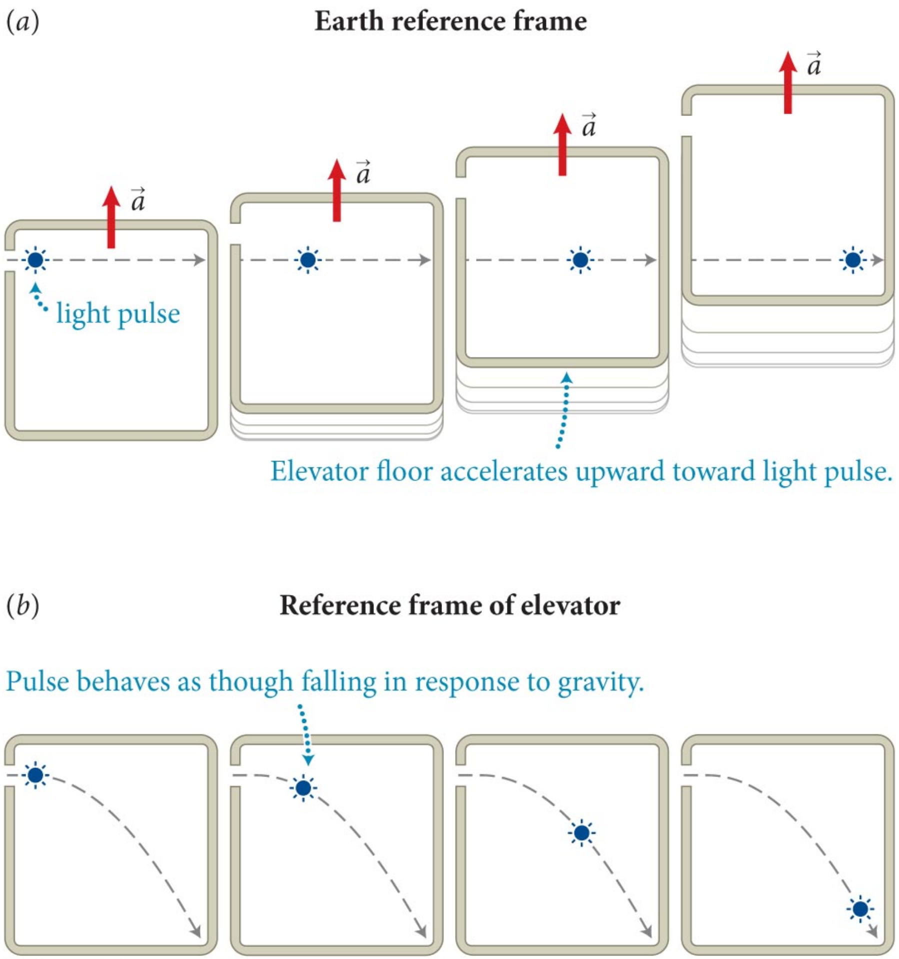


Section 13.4: Principle of equivalence

Section Goals

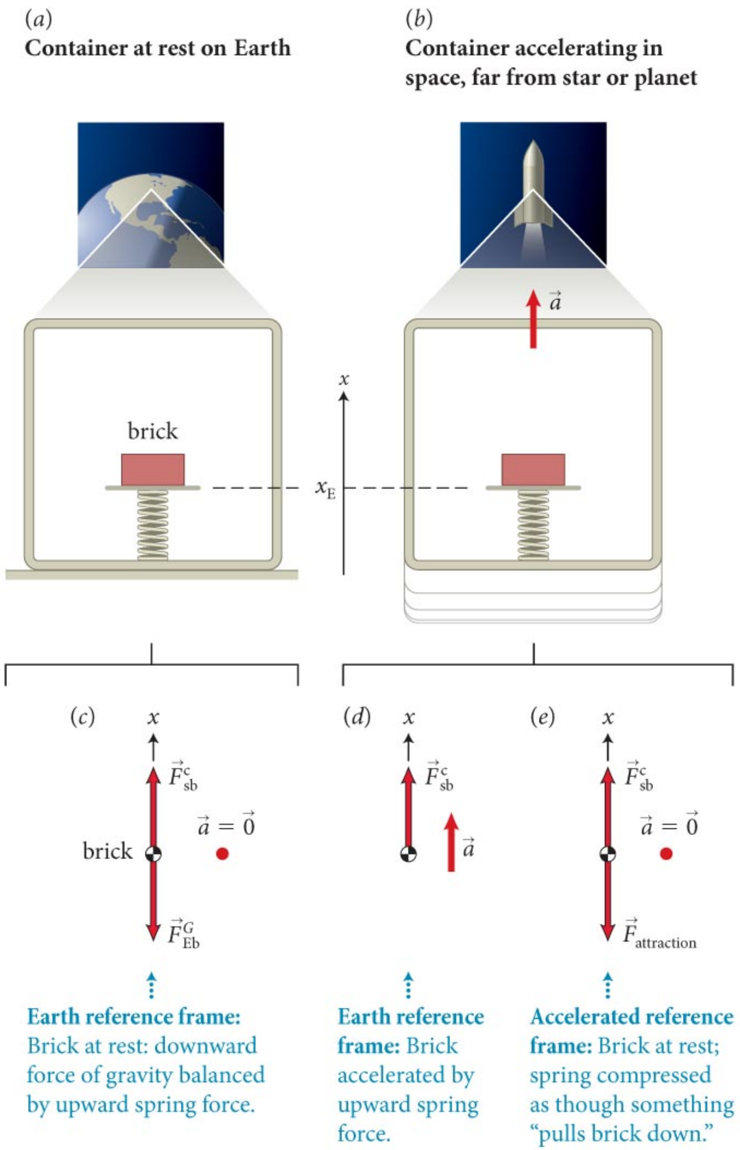
You will learn to

- Articulate the physical evidence for the principle of equivalence.
- Predict some consequences of the principle of equivalence for matter and light.



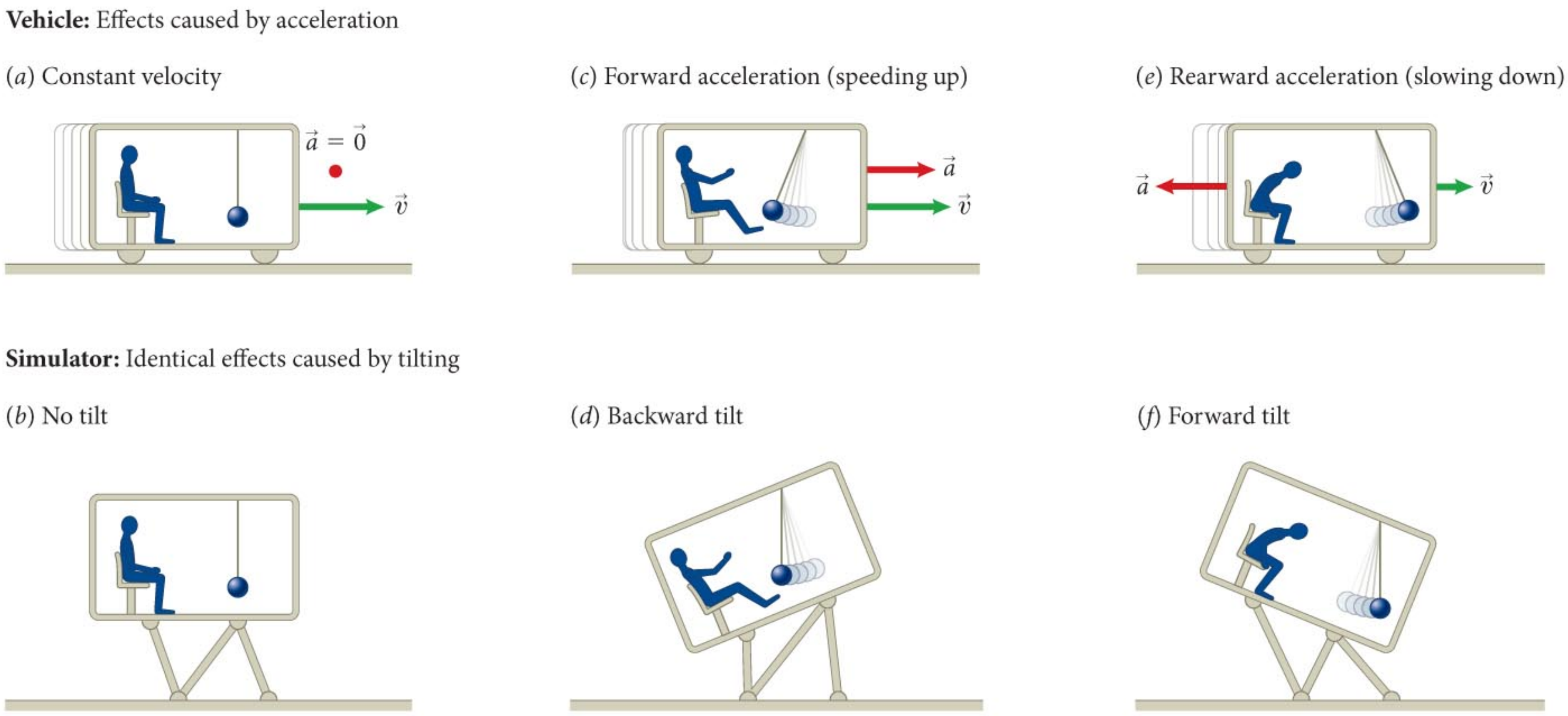
Section 13.4: Principle of equivalence

- Consider the experiment illustrated in the figure. We can conclude that
  - **One cannot distinguish locally between the effects of a constant gravitational acceleration of magnitude *g* and the effect of an acceleration of a reference frame of magnitude *g*.**
- This statement is called the **principle of equivalence.**



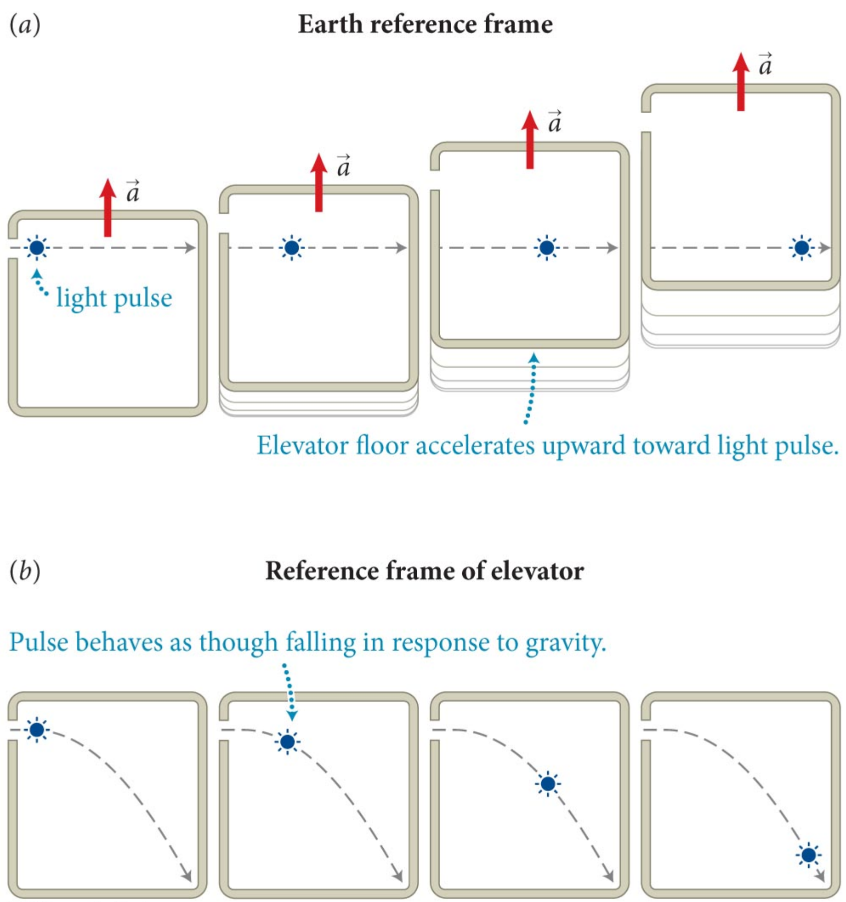
Section 13.4: Principle of equivalence

- Our inability to distinguish between gravity and acceleration is exploited in aircraft simulators and motion simulators at amusement parks.



Section 13.4: Principle of equivalence

- The figure shows a light pulse entering the accelerating elevator.
  - Light pulse travels in a straight line in the Earth's reference frame.
  - Viewed from inside the elevator, the light pulse travels along a curved path.
  - This effect is purely kinematic, and, given the principle of equivalence, the bending should also occur if the elevator was resting on a large mass.



Checkpoint 13.15

13.15 Light travels at approximately 3.0×10^8 m/s.

(a) How long does it take for a light pulse to cross an elevator 2.0 m wide?

(b) How great an acceleration is necessary to make the pulse deviate from a straight-line path by 1.0 mm? Is it likely that this effect can be observed?

(c) If light is bent by the gravitational pull of an object, light should "fall" when traveling parallel to the surface of Earth. How far does a beam of light travel in 0.0010 s, and how much does it fall over that distance? Is it likely that this effect can be observed?

Checkpoint 13.15

- How long does it take for a light pulse to cross an elevator $x = 2.0$ m wide?
  - The time is $t = x/v \sim 6.7 \times 10^{-9}$ s
- How great an acceleration is necessary to make the pulse deviate from a straight-line path by $y = 1.0$ mm? Is it likely that this effect can be observed?
  - $y = \frac{1}{2} a_y t^2 = \frac{1}{2} a_y (x/v)^2$ or $a_y = 2v^2 y/ax^2$
  - For $y = 1$ mm, need $a_y = 4.5 \times 10^{13}$ m/s^2
  - Not going to happen.

Checkpoint 13.15

- If light is bent by the gravitational pull of an object, light should "fall" when traveling parallel to the surface of Earth. How far does a beam of light travel in 0.0010 s, and how much does it fall over that distance? Is it likely that this effect can be observed?

- In $t = 0.0010$ s, a light beam travels $vt \sim 3 \times 10^5$ m
- In that time, it falls like an object in free fall, so
  - $y = \frac{1}{2}gt^2 \sim 4.9 \times 10^{-6}$ m
- Measuring this tiny displacement over a 300 km distance is beyond current measurement accuracy

Chapter 13: Self-Quiz #3

If the speed of a planet in its orbit around the Sun changes, how can the planet's angular momentum be constant?

Chapter 13: Self-Quiz #3

Answer

If the angular momentum is constant but the speed changes, the rotational inertia must change.

When a planet's speed increases, either its inertia must decrease or it must move nearer the object it orbits to keep the angular momentum constant.

Conversely, when a planet's speed decreases, either its inertia must increase or it must move away from the object it orbits.

Chapter 13: Self-Quiz #4

A brick of unknown mass is placed on a spring scale. When in equilibrium, the scale reads 13.2 N. Which of the following statements is/are true?

(*i*) Earth always exerts a gravitational force of 13.2 N on the brick.

(*ii*) The normal force the scale exerts on the brick is 13.2 N.

(*iii*) The brick has an inertia of 1.32 kg.

Chapter 13: Self-Quiz #4

Answer

Only statement (*ii*) is strictly true.

The scale reading may change from location to location, and the inertia of the brick is 1.32 kg only when the gravitational acceleration at the specific location is 10 m/s^2.

Chapter 13: Self-Quiz #6

If you were to travel in a vertical circle, at what point in the circle would you be most likely to experience weightlessness?

Chapter 13: Self-Quiz #6

Answer

To experience weightlessness, the only force acting on you is the gravitational force exerted by Earth, which points downward.

This downward force must provide the centripetal acceleration needed for traveling in a circle.

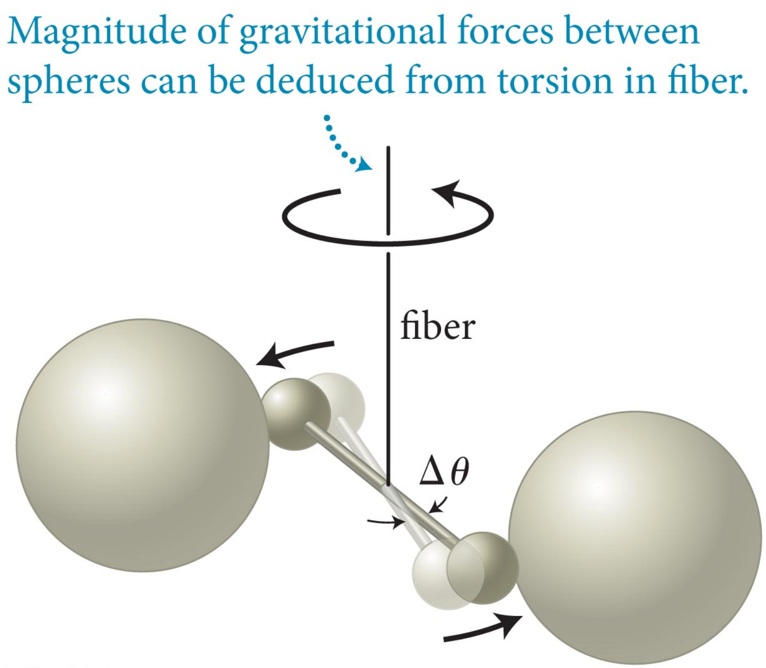
The only position from which the center of the circle is downward from you is at the top of the circle.

Quantitative Tools

Section Goals

You will learn to

- Relate the gravitational constant to the acceleration of gravity near Earth's surface.
- Describe the Cavendish experiment for determining the gravitational constant.



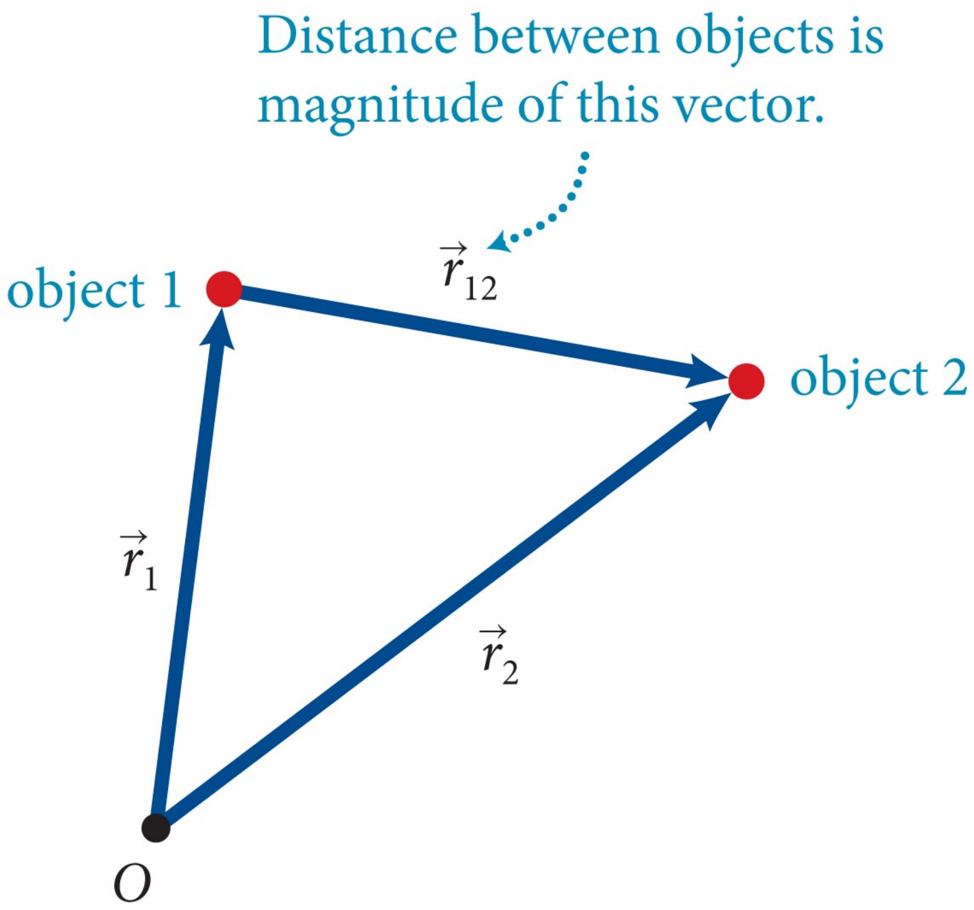
Section 13.5: Gravitational constant

- Using Newton's law of gravity, we can write the magnitude of the gravitational force as

$$F_{12}^{G} = G\frac{m_1 m_2}{r_{12}^2}$$

where the proportionality constant G is called the **gravitational constant**.
  - The magnitude of G was experimentally measured by Henry Cavendish in 1798 to be

$$G = 6.6738 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2$$



Section 13.5: Gravitational constant

- Now let us consider an object with mass m_o close to Earth's surface. If Earth's mass is m_E, then we can write

$$F^G_{Eo} = G\frac{m_E m_o}{r^2_{Eo}}$$

- For an object near Earth's surface, $r_{Eo} \sim R_E$, and the previous equation becomes

$$F^G_{Eo} = \frac{Gm_E m_o}{r^2_{Eo}} \approx G\frac{m_E m_o}{R^2_E} = m_o\left(\frac{Gm_E}{R^2_E}\right) \quad \text{(near Earth's surface)}$$

- However, we know from our study of gravitational force that $F^G_{mo} = mg$, and we get

$$g = \frac{Gm_E}{R^2_E} \quad \text{(near Earth's surface)}$$

  - Can now calculate "g" on other planets

Checkpoint 13.17

13.17 For an object released from a height $h \sim R_E$ above the ground, does the acceleration due to gravity decrease, increase, or stay the same as the object **falls** to Earth?

It increases because the gravitational force increases with decreasing distance. This effect is not *noticeable at* everyday distances, but it is *measureable.*

Example 13.3 Weighing Earth

Cavendish is said to have “weighed Earth” because his determination of G provided the first value for the planet’s mass m_E. Given that the radius of Earth is about 6400 km and given the value of G in Eq. 13.5, determine m_E.

Example 13.3 Weighing Earth (cont.)

❶ GETTING STARTED I am given Earth's radius and G and must use these values to determine Earth's mass. An expression containing G and the acceleration due to gravity g seems a good place to begin.

Example 13.3 Weighing Earth (cont.)

❷ DEVISE PLAN Equation 13.4 gives the acceleration due to gravity near Earth's surface, g, in terms of G, R_E, and m_E. Knowing the values of g, G, and R_E, I can determine m_E.

Example 13.3 Weighing Earth (cont.)

❸ EXECUTE PLAN From Eq. 13.4, I obtain $m_E = gR_E^2/G$,

so $$m_E = \frac{(9.8\ \text{m/s}^2)(6.4\times10^6\ \text{m})^2}{(6.6738\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)} = 6.0\times10^{24}\ \text{kg.}$$ ✔

Example 13.3 Weighing Earth (cont.)

❹ EVALUATE RESULT Table 13.1 gives 5.97 x 10^{24} kg, which agrees with my answer to within less than 1%.

Section 13.6: Gravitational potential energy

Section Goals

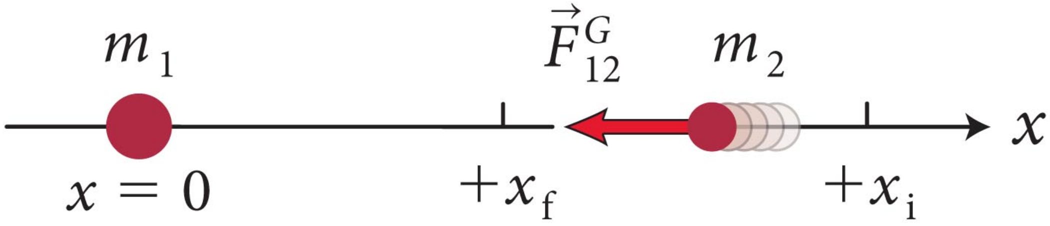
You will learn to

- Derive the expression for gravitational potential energy from the law of universal gravity.
- Represent gravitational potential energy on bar charts and graphs.
- Relate the path independence of the work done by gravity to the non-dissipative nature of the gravitational force.

Section 13.6: Gravitational potential energy

- Consider the gravitational force exerted on an object 2 of mass m_2 by object 1 of mass m_1, as shown. We can write

$$F^G_{12x} = -G\frac{m_1 m_2}{x^2}$$



- The work done by object 1 on object 2 can be written as

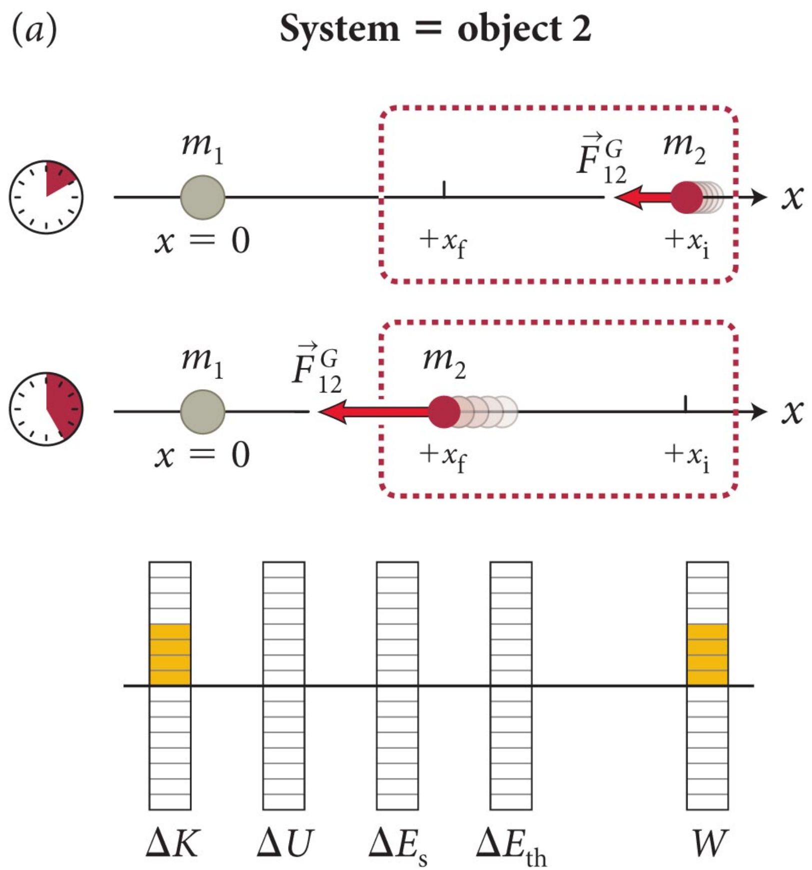
$$W = \int_{x_i}^{x_f} F_x(x)\,dx$$

- Combining the two equations, we get the work done by the gravitational force exerted by object 1 on the system consisting of object 2 only:

$$W = -Gm_1m_2\int_{x_i}^{x_f}\left(\frac{1}{x^2}\right)dx = Gm_1m_2\left[\frac{1}{x}\right]_{x_i}^{x_f} = Gm_1m_2\left(\frac{1}{x_f} - \frac{1}{x_i}\right)$$

Section 13.6: Gravitational potential energy

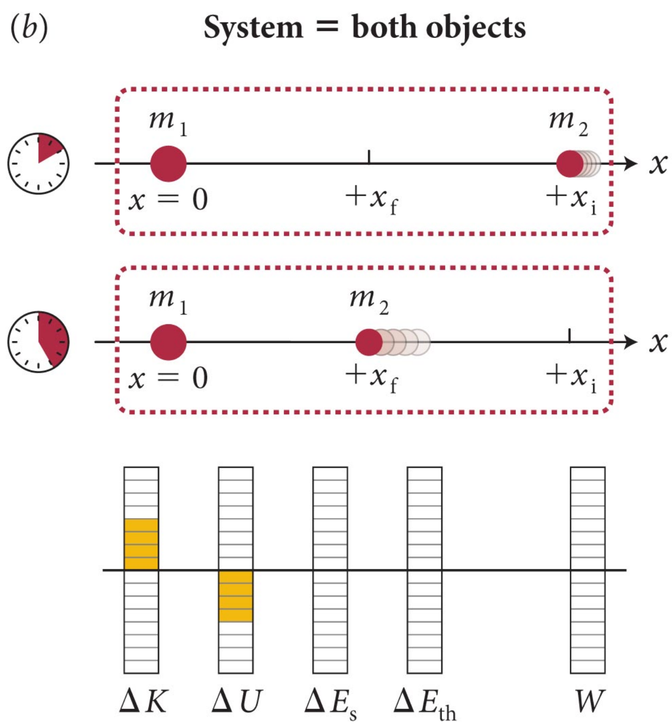
- Because no other energy associated with the system changes, energy conservation gives us, $\Delta K = W$, as illustrated in the figure.



Section 13.6: Gravitational potential energy

- Now if we consider the closed system of the two interacting objects (now no external forces), then energy conservation gives us, $\Delta U^G = -\Delta K$.
- Because ΔK does not depend on the choice of system we get

$$\Delta U^G = Gm_1m_2\left(\frac{1}{x_i} - \frac{1}{x_f}\right)$$



- If we let object 2 move from $x = \infty$ (no interaction) to an arbitrary position x, we get

$$\Delta U^G = U^G(x) - U^G(\infty) = U^G(x) - 0 = Gm_1m_2\left(0 - \frac{1}{x}\right)$$

- So the **gravitational potential energy** is

$$U^G(x) = -G\frac{m_1m_2}{x}$$

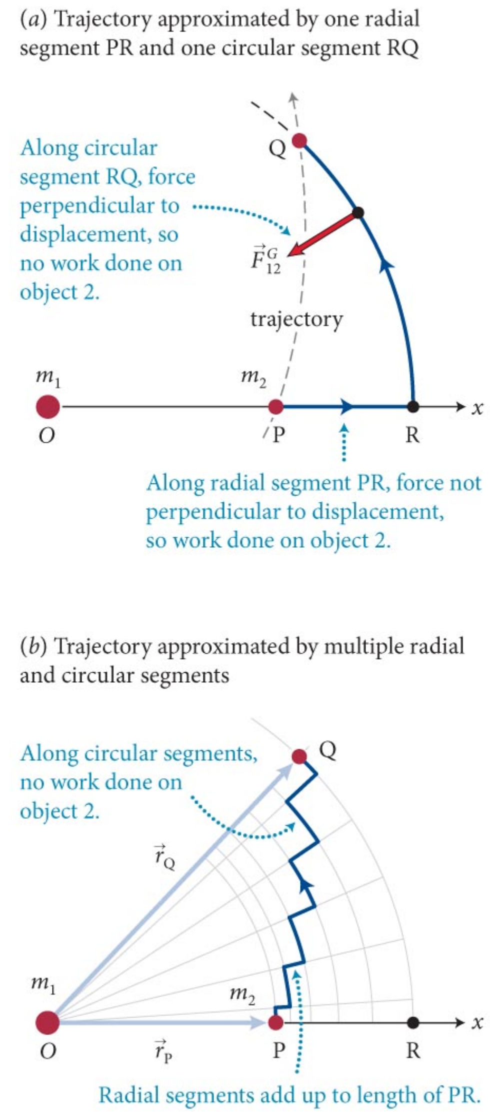
where is U zero?

Section 13.6: Gravitational potential energy

- Because the force of gravity is a central force, we can generalize Equation 13.11 to more than one dimension:

$$U^G(\vec{r}) = -G\frac{m_1 m_2}{r} \quad \text{(zero at infinite } r\text{)}$$

- The work done by gravitational force depends only on the positions of the endpoints relative to m_1, **not on the path taken**.
- This means the force is conservative, and it is valid to assign U^G



Section 13.6: Gravitational potential energy

- Consider the paths 1 and 2 from P to Q, as shown. We can write

$$W_{\mathrm{P}\rightarrow\mathrm{Q,\ path\ 1}} = W_{\mathrm{P}\rightarrow\mathrm{Q,\ path\ 2}}$$

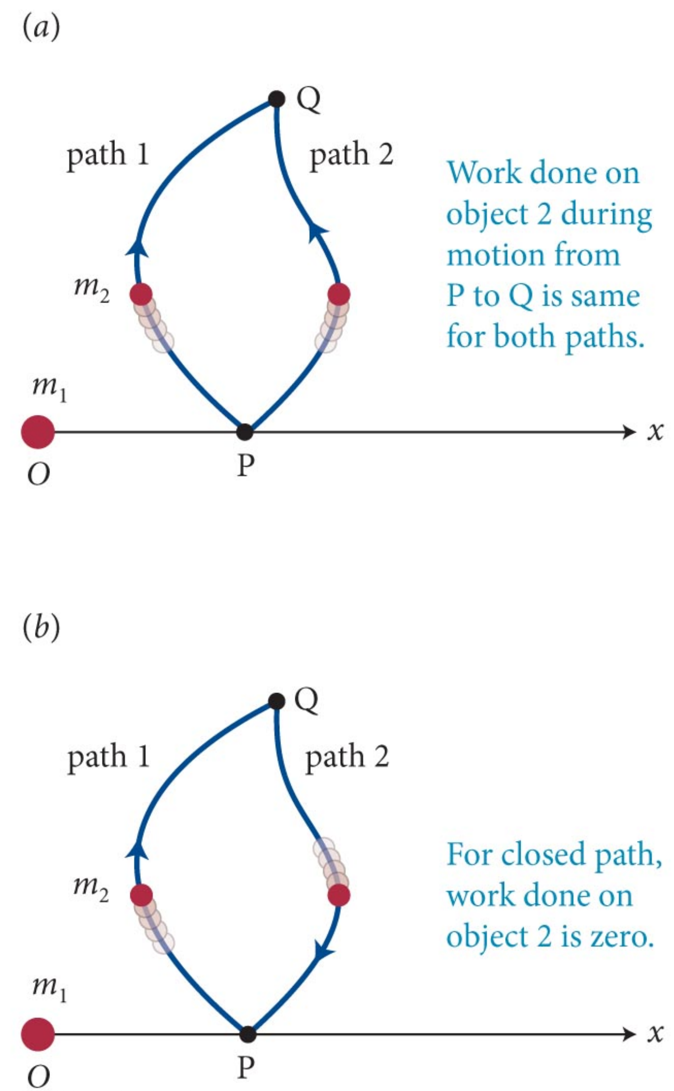
- We also know that

$$W_{\mathrm{Q}\leftarrow\mathrm{P,\ path\ 2}} = -W_{\mathrm{P}\rightarrow\mathrm{Q,\ path\ 2}}$$

- So, if an object moves from P to Q along path 1 and back to P along path 2, we have

$$W = W_{\mathrm{P}\rightarrow\mathrm{Q,\ path\ 1}} + W_{\mathrm{Q}\leftarrow\mathrm{P,\ path\ 2}} = W_{\mathrm{P}\rightarrow\mathrm{Q,\ path\ 2}} + (-W_{\mathrm{P}\rightarrow\mathrm{Q,\ path\ 2}}) = 0$$

- For a **closed path**, work done by gravity on an object is zero.

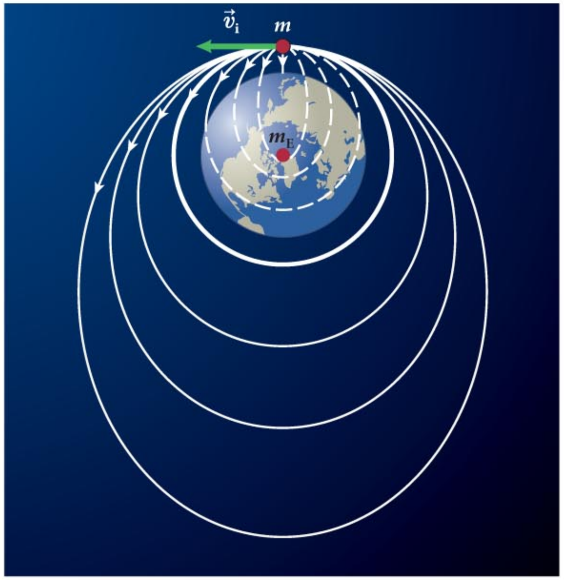


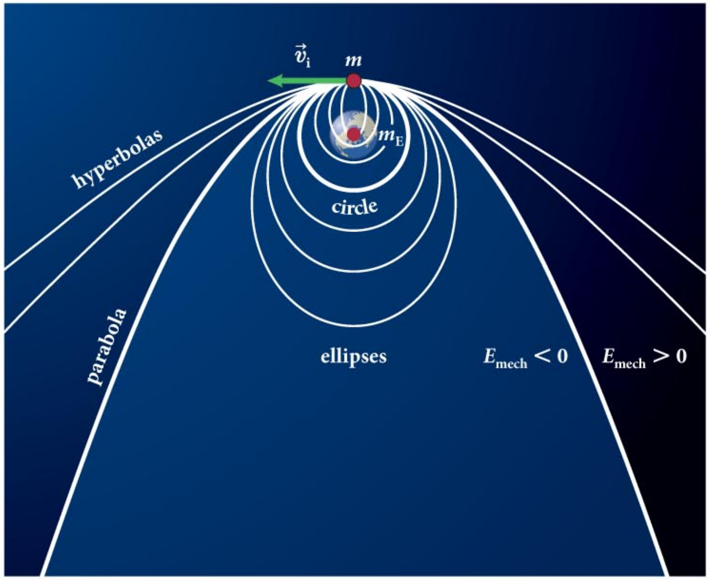
Section 13.7: Celestial mechanics

Section Goal

You will learn to

- Relate the geometry of the orbits of celestial objects to the energetics of the situation.

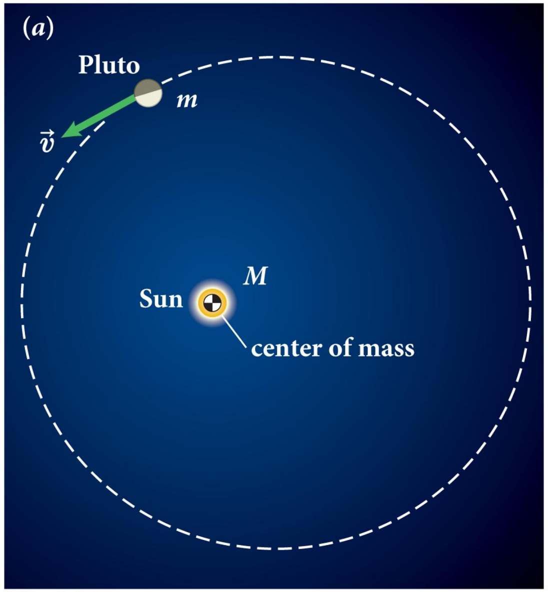


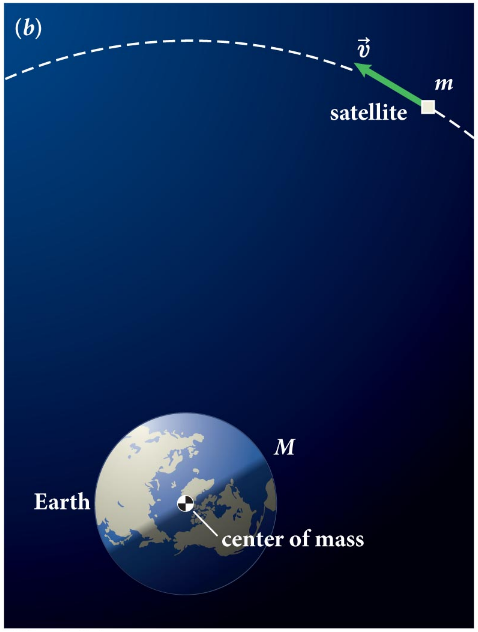


Section 13.7: Celestial mechanics

- Consider a system consisting of two objects (star and a satellite as shown in figure) with masses M and m, where $M >> m$. We can consider the center of mass of the system to be fixed at the center of the large object.
- If the system is closed and isolated

$$\Delta E = 0 \quad \text{and} \quad \Delta\vec{L} = \vec{0}$$





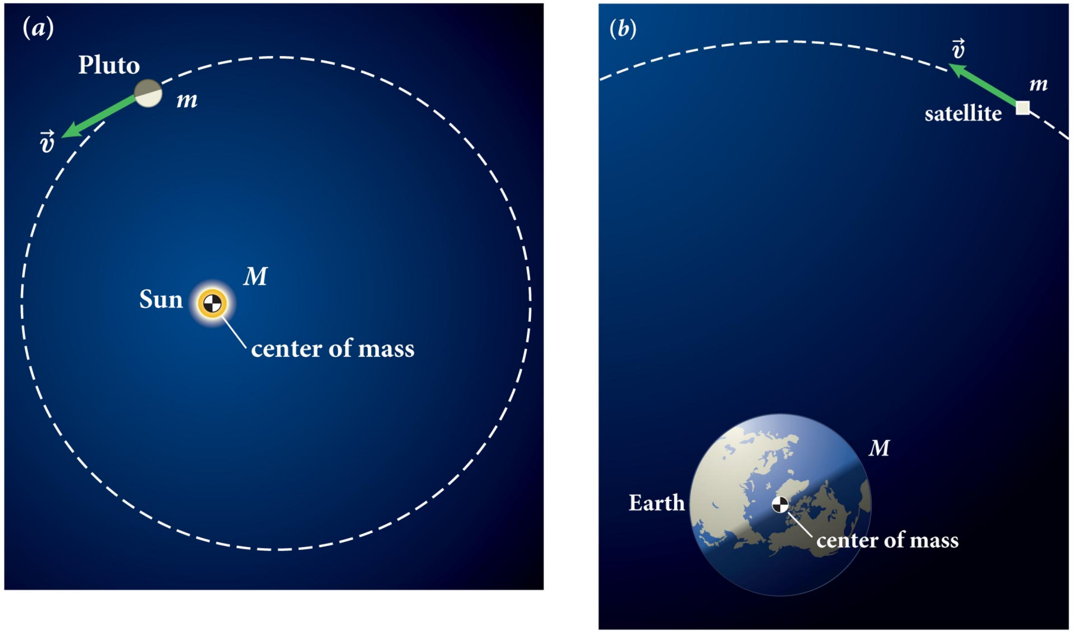
Section 13.7: Celestial mechanics

- Because the satellite is in motion

$$E = K + U^G = \tfrac{1}{2}mv^2 - \frac{GMm}{r}$$

- Because the force of gravity is central, we can write for the satellite

$$L = r_\perp mv$$



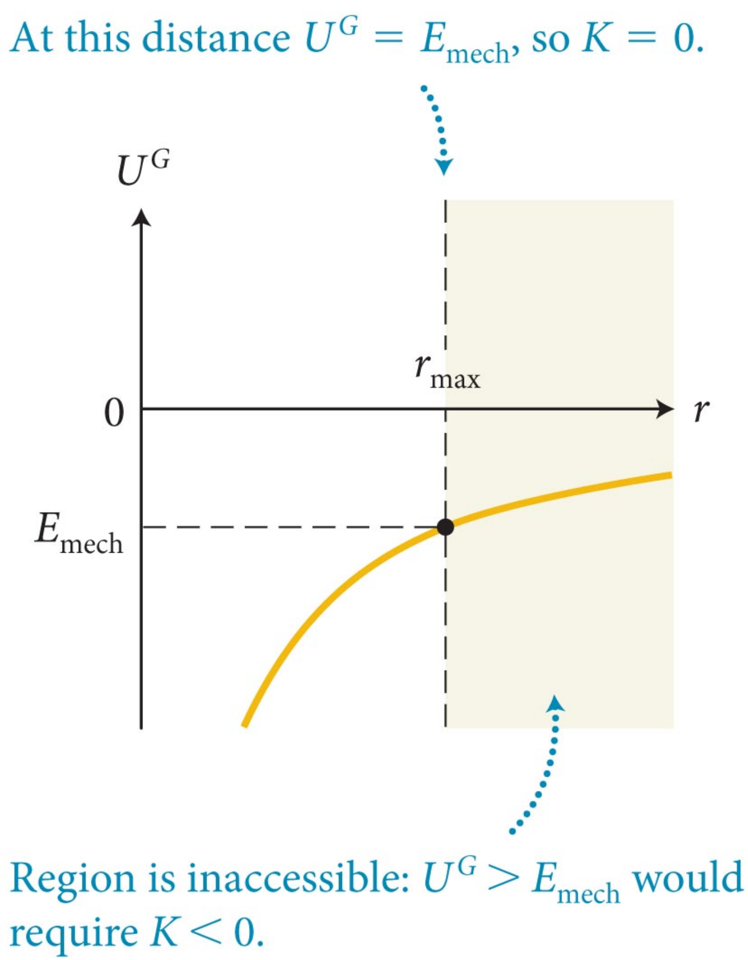
Section 13.7: Celestial mechanics

- $E_{mech} = U^G + K$ of a star-satellite system can be negative.
- Since $K > 0$, for systems where $E_{mech} < 0$, we must have $r < r_{max}$. r_{max} is the maximum separation distance, given by

$$-\frac{GMm}{r_{max}} = E_{mech} \qquad \text{(all PE)}$$

$$r_{max} = \frac{GMm}{-E_{mech}}$$

- In other words, if $E_{mech} < 0$, the satellite is *bound* to the star within r_{max}
- If $E_{mech} \geq 0$, r can take any value, and the satellite is *unbound*.



Conic Sections

Second-degree equations in two variables of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

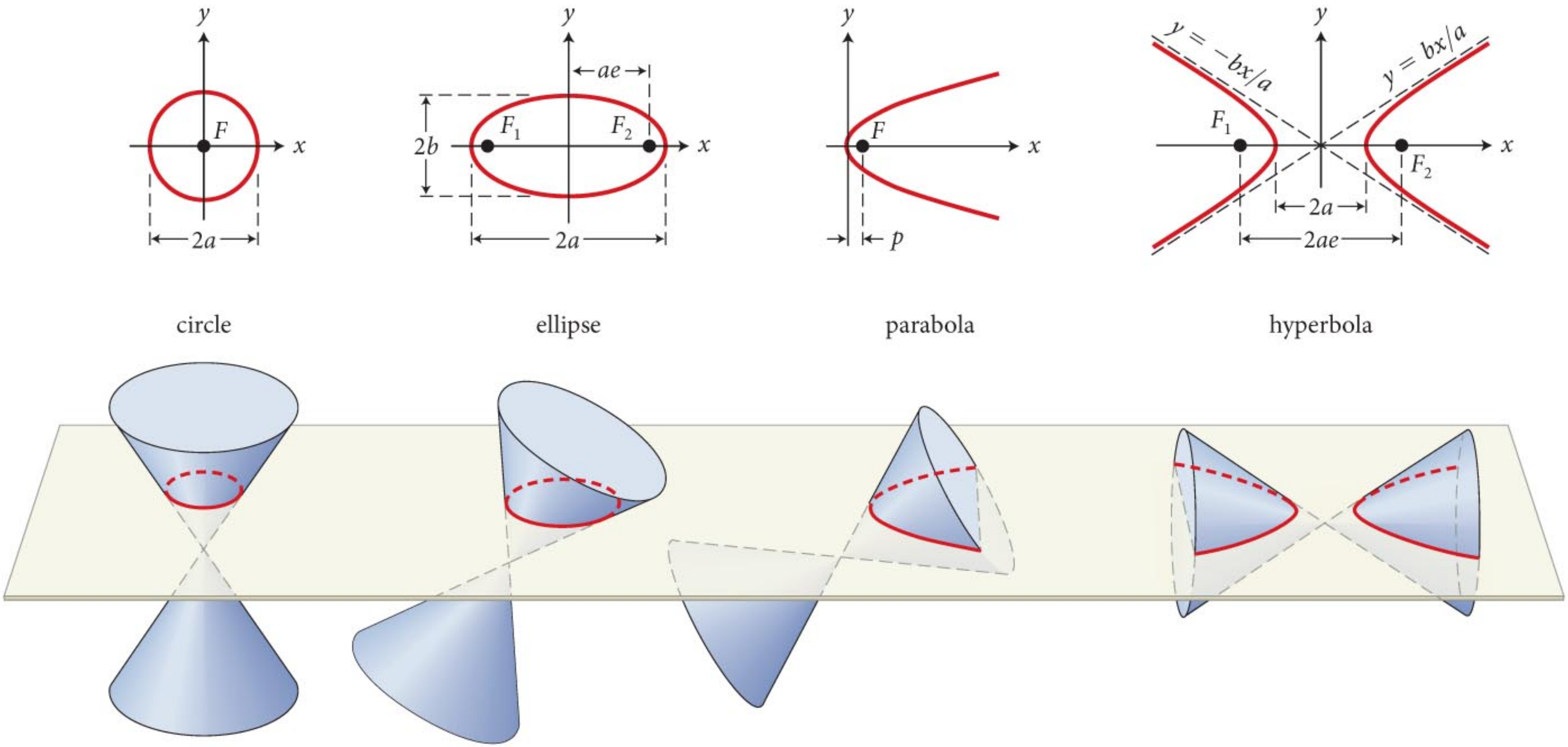
are called **conic sections** because the curves they represent can be obtained by intersecting a plane and a circular cone (Figure 13.35). These curves can also be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

where a is the semimajor axis (see Figure 13.7) of the section and e is the eccentricity.

Section 13.7: Celestial mechanics

Conic Sections



Conic Sections

Ellipse: For $0 \le e < 1$, we obtain an ellipse with a semimajor axis a and a semiminor axis b with $b^2 = a^2(1 - e^2)$, where e is the eccentricity of the ellipse.

For each point on the ellipse, the sum of the distances to the two foci at $(\pm ae, 0)$ is equal to $2a$. A special type of ellipse is obtained for $e = 0$: The two foci coincide, and the ellipse becomes a **circle** of radius a.

Conic Sections

Hyperbola: When $e > 1$, the term that contains y^2 becomes negative, and Eq. 1 yields the two branches of a hyperbola with foci at $(\pm ae, 0)$ and oblique asymptotes $y = \pm(b/a)x$, with $b^2 = a^2(e^2 - 1)$. For each point on the hyperbola, the difference of the distances to the foci is equal to $2a$.

The limiting case between an ellipse and a hyperbola, which occurs when $e = 1$, yields a **parabola** of the form $y^2 = 4px$. The parabola can be thought of as an elongated ellipse with one focus at $(p, 0)$ and the other at $(\infty, 0)$.

Section 13.7: Celestial mechanics

- The value of E_{mech} determines the shape of the orbits of the satellite:
  - $E_{mech} < 0$: elliptical orbit (bound)
    - Magnitude of gravitational PE larger than max KE
  - $E_{mech} = 0$: parabolic orbit (unbound)
    - Magnitude of gravitational PE equal to max KE
  - $E_{mech} > 0$: hyperbolic orbit (unbound)
    - KE larger than max magnitude of gravitational PE

Checkpoint 13.21

13.21 Consider a planet of mass m moving at constant speed v in a circular orbit of radius R under the influence of the gravitational attraction of a star of mass M.

(a) What is the planet's kinetic energy, in terms of m, M, G, and R?

(b) What is the energy of the star-planet system?

Checkpoint 13.21

- If the trajectory is circular with radius *R* and constant speed *v*, the gravitational force must provide centripetal force:

$$\frac{GMm}{R^2} = m\frac{v^2}{R}$$

- To get KE, multiply both sides by ½*R*

$$\frac{1}{2}\frac{GMm}{R} = \frac{1}{2}mv^2$$

Checkpoint 13.21

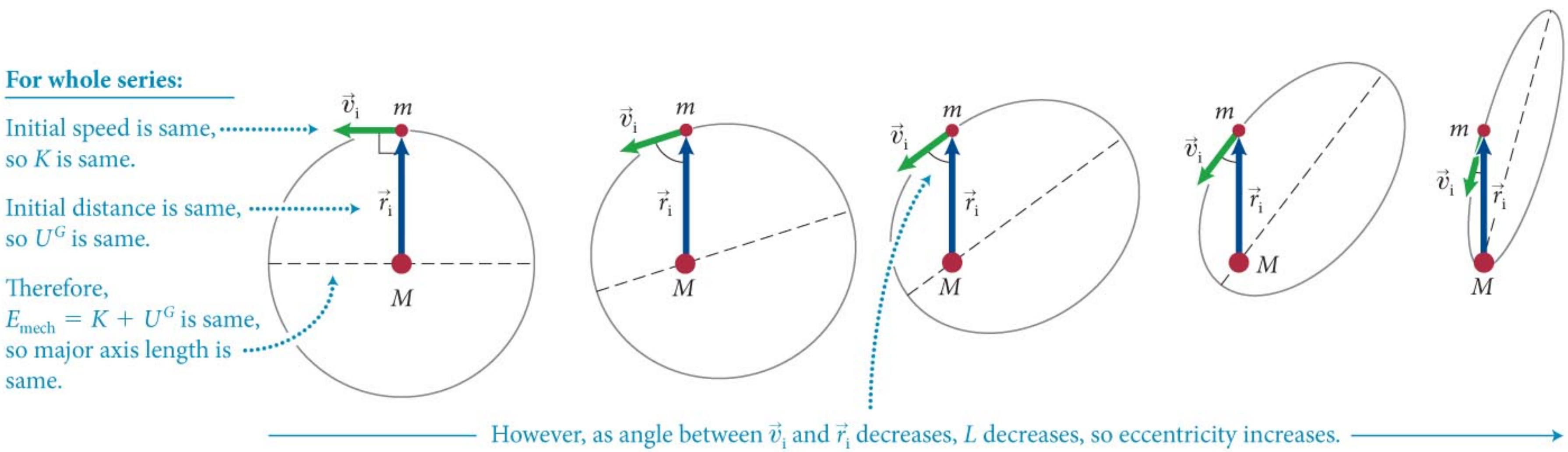
- The mechanical energy is the sum of kinetic and potential

$$E = K + U = \frac{1}{2}\frac{GMm}{R} - \frac{GMm}{R} = -\frac{1}{2}\frac{GMm}{R}$$

- Note this is the negative of KE! Total energy is half the magnitude of the potential energy for a circular orbit.
- Energy is negative, indicating this is a bound state

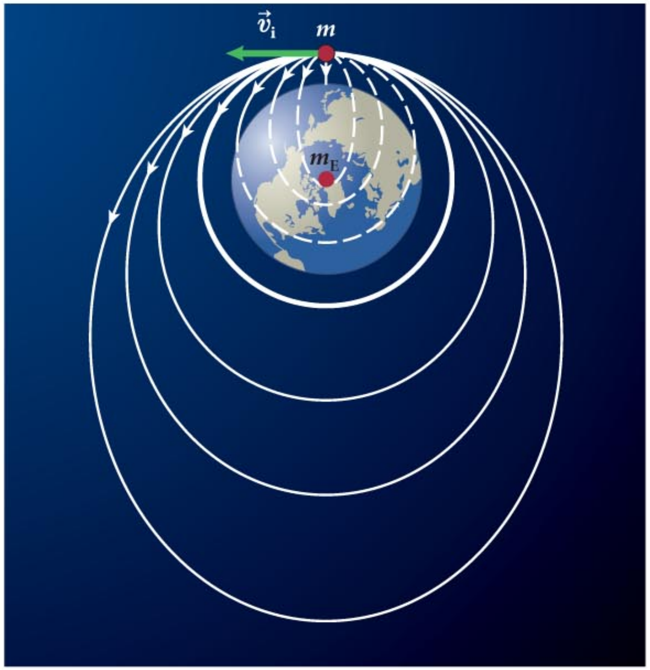
Section 13.7: Celestial mechanics

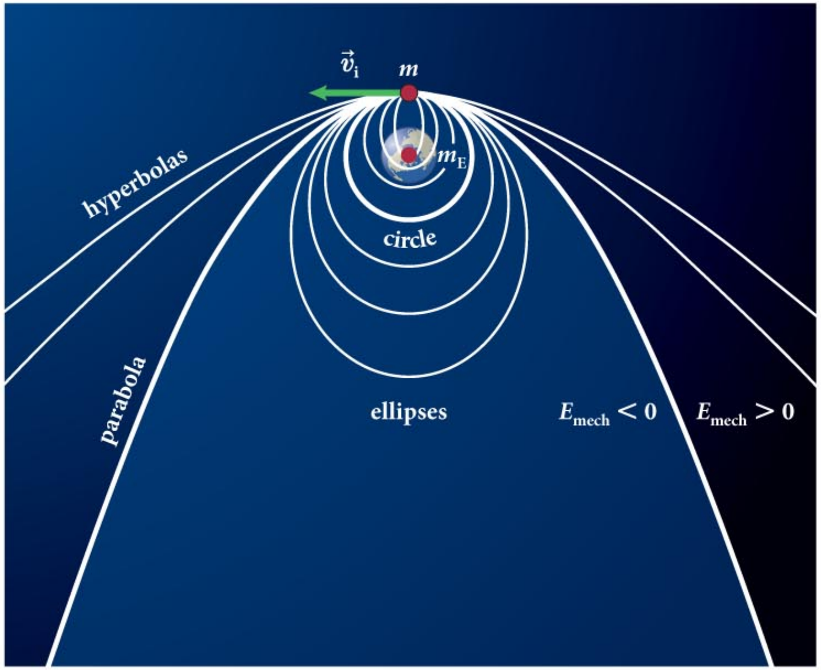
- The figure shows five orbits with the same fixed energy E_{mech} but different values of L.



Section 13.7: Celestial mechanics

- The figure shows the orbit of an object launched multiple times from a fixed location that is a distance r_i from Earth's center.





- If v_i exceeds v_{esc}, such that $E_{mech} > 0$, then the satellite is unbound. v_{esc} is the object's *escape velocity* given by

$$E_{mech} = \tfrac{1}{2} m v_{esc}^2 - \frac{G m m_E}{r_i} = 0$$

Checkpoint 13.22

13.22 (*a*) Determine an expression for the escape speed at Earth's surface. (*b*) What is the value of this escape speed? (*c*) Does it matter in which direction an object is fired at the escape speed?

At earth's surface, $r = R_E$. Kinetic energy needed? Equal to gravitational potential energy at surface.

$$\frac{1}{2} m v_{esc}^2 = \frac{G m m_E}{R_E} \qquad \text{or} \qquad v_{esc} = \sqrt{2 G m_E / R_E}$$

Using known values, around 1.12×10^4 m/s

Direction doesn't matter, as long as you're not pointing at the ground. If you have enough energy ($E = 0$) you always escape.