University of Alabama
Department of Physics and Astronomy

PH 126 LeClair Fall 2011

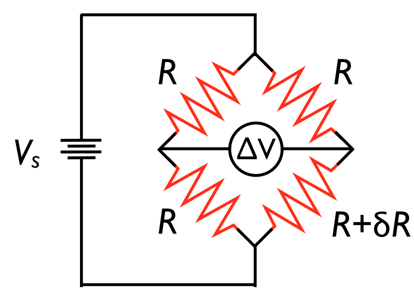
PH126: Exam 1

Instructions:

1. Answer four of the five questions below. All problems have equal weight.
2. You must show your work for full credit.

□ **1.** The circuit below is known as a *Wheatstone Bridge*, and it is a useful circuit for measuring small changes in resistance. Perhaps you can figure out why. Three of the four branches on our bridge have identical resistance R, but the fourth has a slightly different resistance, by an amount δR such that its total resistance is $R+\delta R$.

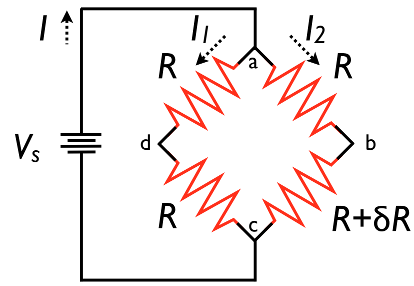
In terms of the source voltage V_s, base resistance R and change in resistance δR, what is the reading on the voltmeter, ΔV? You may assume the voltmeter and voltage source are perfect (drawing no current and having no internal resistance, respectively).



Solution: Label the nodes on the bridge a-d, as shown in the figure below, and let a current I_1 flow from point d through a to c, and a current I_2 flow from d through b to c.

Looking more carefully at the bridge, we notice that it is nothing more than two sets of series resistors, connected in parallel with each other. This immediately means that the voltage drop across the left side of the bridge, following nodes d→a→c, must be the same as the voltage drop across the right side of the bridge, following nodes d→b→c. Both are ΔV_{dc}, and both must be the same as the source voltage: $\Delta V_{dc}=V_s$. If we can find the current in each resistor, then with the known source potential difference we will know the voltage at any point in the circuit we like, and finding ΔV_{ab} is no problem.

Let the current from the source V_s be I. This current I leaving the source will at node a split into the separate currents I_1 and I_2; conservation of charge requires $I=I_1+I_2$. At node c, the currents



Labeling notes and currents in the Wheatstone Bridge

recombine into I. On the leftmost branch of the bridge, the current I_1 creates a voltage drop I_1R across each resistor. Similarly, on the rightmost branch of the bridge, the resistor R has a voltage drop I_2R and the lower resistor has a voltage drop $I_2\left(R+\delta R\right)$. Equating the total voltage drop on each branch of the bridge:

$$
\begin{aligned}
& V_s = I_1R + I_1R = I_2R + I_2\left(R+\delta R\right) \\
\implies \quad & I_1 = \frac{V_s}{2R} \\
& I_2 = \frac{V_s}{2R+\delta R}
\end{aligned}
$$

Now that we know the currents in terms of known quantities, we can find ΔV_{ab} by "walking" from point a to point b and summing the changes in potential difference. Starting at node a, we move toward node d *against* the current I_1, which means we *gain* a potential difference I_1R. Moving from node d to node b, we move *with* the current I_2, which means we *lose* a potential difference I_2R. Thus, the total potential difference between points a and b must be

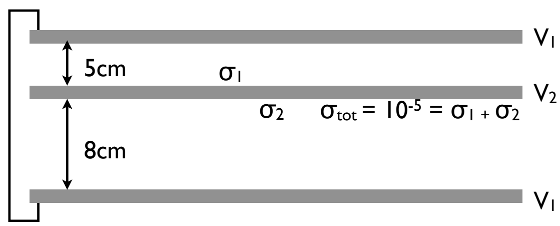
$$
\begin{aligned}
\Delta V_{ab} &= I_1R - I_2R = R\left(I_1+I_2\right) = R\left(\frac{V_s}{2R} - \frac{V_s}{2R+\delta R}\right) \\
\Delta V_{ab} &= V_s\left(\frac{1}{2} - \frac{R}{R+\delta R}\right) = V_s\left(\frac{\delta R}{4R+2\delta R}\right)
\end{aligned}
$$

If the change in resistance δR is small compared to R ($\delta R \ll R$), the term in the denominator can be approximated $4R+\delta R \approx 4R$, and we have

$$
\Delta V_{ab} = \frac{1}{4}V_s\left(\frac{\delta R}{R}\right) \qquad (\delta R \ll R)
$$

Thus, for small changes in resistance, the voltage measured across the bridge is directly proportional to the change in resistance, which is the basic utility of this circuit: it allows one to measure small changes on top of a large 'base' resistance. Fundamentally, it is a *difference* measurement, meaning that one directly measures *changes* in the quantity of interest, rather than measuring the whole thing and trying to uncover subtle changes. This behavior is very useful for, e.g., strain gauges, temperature sensors, and many other devices.

□ **2.** Three conducting plates are placed parallel to one another as shown below. The outer plates are connected by a wire. The inner plate is isolated and carries a charge amounting to 10^{-5} C per square meter of plate. In what proportion must this charge divide itself into a surface charge σ_1 on one face of the inner plate and a surface charge σ_2 on the other side of the same plate?



Solution: Let's tackle the specific case first, and then work out the more general case. If we connect the two outer plates, they are effectively part of the same conductor, and thus they will be at the same potential. Call that V_1. Let the inner plate be at potential V_2. Conservation of charge dictates that the total surface charge on plate two is $\sigma = \sigma_1 + \sigma_2 = 10^{-5}$ C/m.

In the region between the upper plate and the middle plate, which we will call region I, the total electric field must be the sum of the field from the upper plate and that of the middle plate. The field from the middle plate in this region is that of an infinite plate with surface charge σ_1, $E = \sigma_1/2\epsilon_o$. The upper plate, under the influence of the top side of the middle plate, will have an induced charge $-\sigma_1$. It will contribute the same electric field in the region between the upper and middle plates,[i] and in total we have

$$E_I = \sigma_1/2\epsilon_o + \sigma_1/2\epsilon_o = \sigma_1/\epsilon_o \tag{1}$$

The electric potential between the upper and middle plates must be $V_2 - V_1$, and it must be equivalent to integrating the electric field across the gap between the plates. Since the electric field is independent of distance, this is easy:

$$V_2 - V_1 = -\int_I \vec{\mathbf{E}}_I \cdot d\vec{l} = El = E\,(5\,\text{cm}) = (5\,\text{cm})\,\sigma_1/\epsilon_o \tag{2}$$

[i]The induced charge on the upper plate is negative, but the field is in the opposite direction.

Proceeding similarly in region II between the lower and middle plates, we find

$$V_2 - V_1 = (8\,\text{cm})\,\sigma_2/\epsilon_o \tag{3}$$

Dividing the last two equations, we find

$$(5\,\text{cm})\,\sigma_1/\epsilon_o = (8\,\text{cm})\,\sigma_2/\epsilon_o \qquad \Longrightarrow \qquad \frac{\sigma_1}{\sigma_2} = \frac{8}{5} \tag{4}$$

Noting $\sigma = \sigma_1 + \sigma_2$, we find

$$\sigma_1 = \frac{8}{13}\sigma \qquad \sigma_2 = \frac{5}{13}\sigma \tag{5}$$

What about the more general case? Let the spacing between the upper and middle plates be d, and the spacing between the upper and lower plates be D (and thus the spacing between the lower and middle plates is $D-d$). Proceeding as above, we still have $E_I = \sigma_1/\epsilon_o$, and

$$V_2 - V_1 = -\int_I \vec{\mathbf{E}}_I \cdot d\vec{\mathbf{l}} = El = Ed = d\sigma_1/\epsilon_o \tag{6}$$

In the region between the lower and middle plates, we have $E_{II} = \sigma_2/\epsilon_o$, and

$$V_2 - V_1 = -\int_{II} \vec{\mathbf{E}}_{II} \cdot d\vec{\mathbf{l}} = El = E\,(D-d) = (D-d)\,\sigma_2/\epsilon_o \tag{7}$$

Thus,

$$\frac{\sigma_1}{\sigma_2} = \frac{D-d}{d} \tag{8}$$

Again noting $\sigma = \sigma_1 + \sigma_2$,

$$\sigma_1 = \left(\frac{D-d}{D}\right)\sigma \qquad \sigma_2 = \left(\frac{d}{D}\right)\sigma \tag{9}$$

We can find the energy stored by integrating the electric field squared over all space. Outside all of the plates, $\vec{\mathbf{E}} = 0$. We can break up the integral over the region between the two plates into an integral over region I and an integral over region II. Since the electric field is constant in each region, the integrals simply reduce to the volume contained in the region between the two plates. Assume each plate as an area A.

$$U = \frac{1}{2}\epsilon_o \int E^2\,d\tau = \frac{1}{2}\epsilon_o \int_I E_I^2\,d\tau + \frac{1}{2}\epsilon_o \int_{II} E_{II}^2\,d\tau = \frac{1}{2}\epsilon_o\left(\sigma_1/\epsilon_o\right)^2 \int_I d\tau + \frac{1}{2}\epsilon_o\left(\sigma_2/\epsilon_o\right)^2 \int_{II} d\tau$$

$$U = \frac{\sigma_1^2}{2\epsilon_o}Ad + \frac{\sigma_2^2}{2\epsilon_o}A\left(D-d\right) \tag{10}$$

Now we may substitute $\sigma_2 = \frac{d}{D}\sigma$ and $\sigma_1 = \left(\frac{D-d}{d}\right)\sigma$:

$$\begin{aligned}
U &= \frac{\sigma_1^2}{2\epsilon_o}Ad + \frac{\sigma_2^2}{2\epsilon_o}A\left(D-d\right) = \frac{A}{2\epsilon_0}\left[d\left(\frac{D-d}{D}\sigma\right)^2 + \left(\frac{d}{D}\sigma\right)^2\left(D-d\right)\right] \\
&= \frac{A\sigma^2}{2D^2\epsilon_0}\left[d\left(D-d\right)^2 + d^2\left(D-d\right)\right] = \frac{A\sigma^2}{2D^2\epsilon_0}\left(dD^2 - 2Dd^2 + d^3 + d^2D - d^3\right) \\
\therefore \quad U &= \frac{A\sigma^2}{2D^2\epsilon_0}\left(dD^2 - d^2D\right) = \frac{A\sigma^2}{2D\epsilon_0}\left(dD - d^2\right)
\end{aligned} \tag{11}$$

We wish to optimize U with respect to d:

$$\frac{dU}{dd} = \frac{A\sigma^2}{2D\epsilon_0}\left(D-2d\right) = 0 \qquad \Longrightarrow \qquad d = \frac{D}{2} \tag{12}$$

We can verify this is a *maximum* potential energy by noting $d^2U/dd^2 < 0$ for all d. A nice result: maximal energy is stored for a symmetric placement of the middle plate, just what we might have expected. Another way to approach this problem would be to notice that this is really just two capacitors connected in series, which leads you to the same result.

□ **3.** Two graphite rods are of equal length. One is a cylinder of radius a. The other is conical, tapering linearly from a radius a at one end to radius b at the other. Show that the end-to-end electrical resistance of the conical rod is a/b times that of the cylindrical rod. *Hint: consider the rod to be made up of thin, disk-like slices, all in series.*

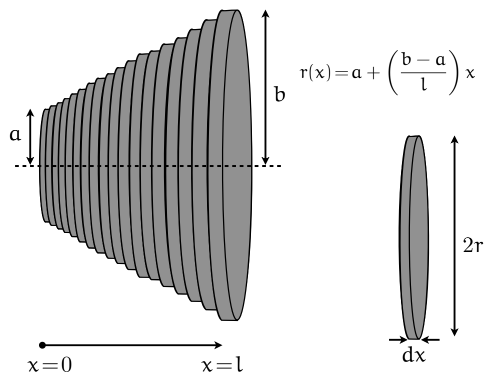
Solution: The cylindrical conductor is trivial: if it is of radius a and length l, and has resistivity ρ, then

$$R_{\text{cyl}} = \frac{\rho l}{\pi a^2} \tag{13}$$

Of course, we don't know the length l or resistivity ρ, but they will not matter in the end.

What about the cone? Break the cone up into many disks of thickness dx. Stacking these disks up with increasing radius can build us a cone:

If we start out with a radius a at one end of the cone, and the other end has a radius b, then the radius as a function of position along the cone is easily determined. Let our origin $(x=0)$ be



the end of the cone with radius a, and assume the cone has a total length l, same as the cylinder. Again, we will not need this length in the end, but it is convenient now. The radius at any position along the cone is then

$$r(x) = a + \left(\frac{b-a}{l}\right) x \tag{14}$$

If the current is in the x direction, then each infinitesimally thick disk is basically just a tiny segment of wire with thickness dx and cross-sectional area $\pi\left[r(x)\right]^2$. If we assume the same resistivity ρ, the resistance of each disk must be

$$dR_{\text{cone}} = \frac{\rho dx}{\pi\left[r(x)\right]^2} = \frac{\rho dx}{\pi\left(a + \left(\frac{b-a}{l}\right) x\right)^2} \tag{15}$$

The total resistance of the cone is found by integrating over all such disks, from $x=0$ to the end of the cone at $x=l$. For convenience, let $c = (b-a)/l$.

$$R_{\text{cone}} = \int dR_{\text{cyl}} = \int_0^l \frac{\rho dx}{\pi\left(a+cx\right)^2} = \frac{\rho}{\pi}\left[\frac{-1}{c\left(a+cx\right)}\right]_0^l = \frac{\rho}{\pi}\left[\frac{-1}{cb} - \frac{-1}{ca}\right] = \frac{\rho}{\pi c}\left[\frac{b-a}{ab}\right] = \frac{\rho l}{\pi ab} \tag{16}$$

Here we have a nice result: the resistance of a cone is the same as a resistance of a cylinder whose radius is the geometric mean cone's radii. That is, if we substitute $a^2 \to ab$ in our usual formula for the resistance of a cylinder, we have the result for a cone. Anyway: the desired result now follows readily,

$$\frac{R_{\text{cone}}}{R_{\text{cyl}}} = \frac{a}{b} \tag{17}$$

□ **4.** Three protons and three electrons are to be placed at the vertices of a regular octahedron of edge length a. We want to find the potential energy of the system, or the work required to assemble it starting with the particles infinitely far apart. There are essentially two different arrangements possible. What is the energy of each? Symbolic answer, please.

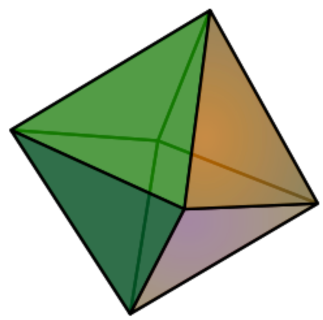


Figure 1: *An octahedron. It has eight faces and six vertices.*

Solution: Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just:

$$\mathsf{PE}_{\text{pair}} = \frac{k_e q^2}{d}$$

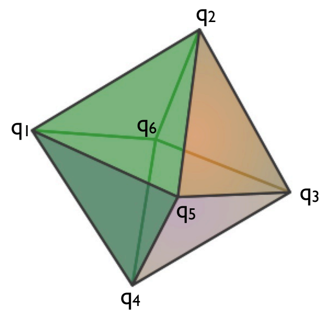
First, we need figure out how many pairs there are for charges arranged on the vertices of an octahedron, and for each pair, how far apart the charges are. Once we've done that, we need to figure out the two different arrangements of charges and run the numbers.

How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force – which we will do anyway to calculate the energy – but we can also calculate how many there are. In both distinct configurations, we have 6 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair $(2,1)$ is the same as $(1,2)$. Pairings like this are known as combinations, as opposed to *permutations* where $(1,2)$ and $(2,1)$ are *not* the same. Calculating the number of possible combinations is done like this:[ii]

$$\text{ways of choosing pairs from six charges} = \binom{6}{2} = {}^{6}C_2 = \frac{6!}{2!\,(6-2)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot4\cdot3\cdot2\cdot1} = 15$$

[ii] A nice discussion of combinations and permutations is here: `http://www.themathpage.com/aPreCalc/permutations-combinations.htm`

We can verify this by simply enumerating all the possible pairings. Label the charges at each vertex in some fashion, such as this:



We have six charges at six vertices, and thus ${}_6C_2 = \frac{6!}{2!4!} = 15$ unique pairings of charges. Namely,

q_1q_2, $\boldsymbol{q_1q_3}$, q_1q_4, q_1q_5, q_1q_6

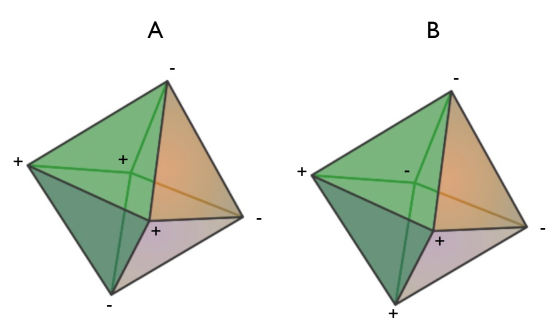
q_2q_3, $\boldsymbol{q_2q_4}$, q_2q_5, q_2q_6

q_3q_4, q_3q_5, q_3q_6

q_4q_5, q_4q_6

$\boldsymbol{q_5q_6}$

Here all the q_i have the same magnitude, the labels are just to keep things straight. At a given vertex, all four nearest-neighbor vertices are at distance a, while the single "next-nearest neighbor" is at a distance $a\sqrt{2}$. This means that there are three *pairs* charges which are separated by a distance $a\sqrt{2}$, and the other twelve pairings are at a distance a. We have highlighted the $a\sqrt{2}$ pairings above. How can we find two different arrangements? Since there are an odd number of next-nearest neighbor pairings, the first suspicion is that the difference between the two arrangements will be in next-nearest neighbor pairings. If you experiment for a while, the two different arrangements are these:



Now we need only add up the potential energies of all possible pairs of charges. All the nearest-neighbor pairs will have the same energy, viz.,

$$|U_{nn}| = \frac{kq^2}{a} \tag{18}$$

All the next-nearest neighbor pairs will have

$$|U_{nnn}| = \frac{kq^2}{a\sqrt{2}} \tag{19}$$

For the first arrangement we have 12 nearest-neighbor pairs: eight of them are $+-$ pairings, and four of them are $++$ or $--$ pairs. We have three next-nearest neighbor pairs, two $++$ or $--$, and one $+-$. Thus, the total energy must be

$$U_A = 8\left[\frac{-kq^2}{a}\right]+4\left[\frac{kq^2}{a}\right]+2\left[\frac{kq^2}{a\sqrt{2}}\right]+1\left[\frac{-kq^2}{a\sqrt{2}}\right] = \frac{kq^2}{a}\left[\frac{1}{\sqrt{2}}-4\right] = \left[\frac{1}{\sqrt{2}}-4\right]|U_{nn}| \approx -3.29|U_{nn}| \tag{20}$$

For the second arrangement, of the 12 nearest-neighbor pairs we have six $+-$ pairs and six $++$ or $--$ pairs, and thus the total energy of nearest-neighbor pairs will be zero. We are left with only the next-nearest neighbor terms, and for this arrangement, all three are $+-$ pairs. Thus,

$$U_B = -3\frac{kq^2}{a\sqrt{2}} = \frac{3}{\sqrt{2}}|U_{nn}| \approx -2.12|U_{nn}| \tag{21}$$

Thus, $U_A < U_B$, and the first lattice is more stable, owing to its lower nearest-neighbor energy. Though the second lattice has a smaller next-nearest neighbor energy, there are fewer next-nearest neighbor pairs, and their energy is smaller than the nearest neighbor pairs. Usually, minimizing the nearest-neighbor energy gives the most stable crystal, simply because the potential is decreasing with distance.

□ **5.** Show that the expression $Q^2/2C$ is the energy stored in a spherical capacitor (two concentric hollow metal spheres) by integrating the energy density $u = \frac{1}{2}\epsilon_o E^2$ over the region between the spheres. Use the volume between two spheres or radius r and $r+dr$ as a volume element.