University of Alabama
Department of Physics and Astronomy

PH 126 LeClair — Fall 2011

Problem Set 1: solutions

1. Water is poured into a container that has a leak. The mass m of the water is as a function of time t is

$$m = 5.00t^{0.8} - 3.00t + 20.00$$

with $t \geqslant 0$, m in grams, and t in seconds. At what time is the water mass greatest?

Solution: **Given:** Water mass versus time $m(t)$.

Find: The time t at which the water mass m is greatest. This can be accomplished by finding the time derivative of $m(t)$ and setting it equal to zero, followed by checking the second derivative to be sure we have found a maximum.

Sketch: It is useful to plot the function $m(t)$ and graphically estimate about where the maximum should be, roughly.[i]

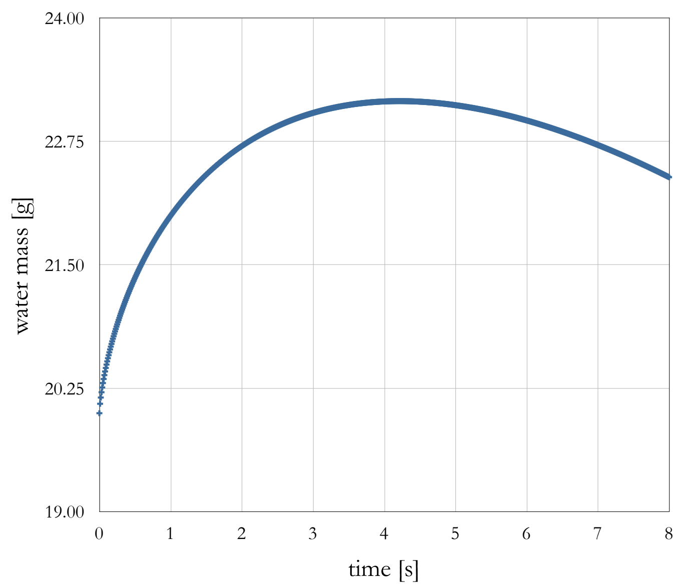

Figure 1: *Water mass versus time, problem 1. Note the rather expanded vertical axis, with offset origin.*

It is clear that there is indeed a maximum water mass, and it occurs just after $t = 4\,\text{s}$.

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[i] It is relatively easy to do this on a graphing calculator, which can be found online these days: `http://www.coolmath.com/graphit/`.

Relevant equations: We need to find the maximum of $m(t)$. Therefore, we need to set the first derivative equal to zero. We must also check that the second derivative is negative to ensure that we have found a maximum, not a minimum. Therefore, only two equations are needed:

$$\frac{dm}{dt} = \frac{d}{dt}\left[m(t)\right] = 0 \qquad \text{and} \qquad \frac{d^2m}{dt^2} = \frac{d^2}{dt^2}\left[m(t)\right] < 0 \quad \Longrightarrow \quad \text{maximum in } m(t)$$

Symbolic solution:

$$\begin{aligned}
\frac{dm}{dt} &= \frac{d}{dt}\left[5t^{0.8} - 3t + 20\right] = 0.8\left(5t^{0.8-1}\right) - 3 = 4t^{-0.2} - 3 = 0 \\
4t^{-0.2} - 3 &= 0 \\
t^{-0.2} &= \frac{3}{4} \\
\Longrightarrow \quad t &= \left(\frac{3}{4}\right)^{-5} = \left(\frac{4}{3}\right)^{5}
\end{aligned}$$

Thus, $m(t)$ takes on an extreme value at $t{=}(4/3)^5$. We did not prove whether it is a maximum or a minimum however! This is important ...so we should apply the *second derivative* test.

Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a}{=}\, 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a}{<}\, 0$, you have a maximum, if $d^2f/dx^2|_{x=a}{>}\, 0$ you have a minimum, and if $d^2f/dx^2|_{x=a}{=}0$, the test basically wasted your time. Anyway:

$$\begin{aligned}
\frac{d^2m}{dt^2} &= \frac{d}{dt}\left[\frac{dm}{dt}\right] = \frac{d}{dt}\left[4t^{-0.2} - 3\right] = -0.2\left(4t^{-0.2-1}\right) = -0.8t^{-1.2} \\
\frac{d^2m}{dt^2} &< 0 \quad \forall \quad t > 0
\end{aligned}$$

Since $t^{-1.2}$ is always positive for $t{>}0$, $\frac{d^2m}{dt^2}$ is always less than zero[ii], which means we have indeed found a maximum.

Numeric solution: Evaluating our answer numerically, remembering that t has units of seconds (s):

$$t = \left(\frac{4}{3}\right)^5 \approx 4.21399 \xrightarrow[\text{digits}]{\text{sign.}} 4.21\,\text{s}$$

The problem as stated has only three significant digits, so we round the final answer appropriately.

[ii] You can read the symbol $\forall$ above as “for all." Thus, $\forall\ t{>}0$ is read as “for all t greater than zero.”

Double check: From the plot above, we can already graphically estimate that the maximum is somewhere around $4\frac{1}{4}$ s, which is consistent with our numerical solution to 2 significant figures. The dimensions of our answer are given in the problem, so we know that t is in seconds. Since we solved $dm/dt(t)$ for t, the units must be the same as those given, with t still in seconds – our units are correct.

2. Find the angle between the *body* diagonals of a cube. Use one of the vector products.

Solution: Put one corner of the cube at the origin, and let it extend in the region where x, y, z are positive, such that it has vertices at (000), (100), (110), (010), (101), (001), (011), and (111). We could represent two body diagonals by the vectors

$$\vec{\mathbf{A}} = \hat{\imath} + \hat{\jmath} - \hat{\mathbf{k}}$$
$$\vec{\mathbf{B}} = \hat{\imath} + \hat{\jmath} + \hat{\mathbf{k}}$$

Note that for $\vec{\mathbf{A}}$ one should translate the whole vector by 1 unit along $\hat{\mathbf{k}}$ for both diagonals to be within the cube. You should make a sketch to be sure you understand the geometry here. We can use the scalar ("dot") vector product to find the angle θ between the diagonals:

$$\cos\theta = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|} = \frac{1+1-1}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \quad \Longrightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5^\circ$$

3. If $\vec{\mathbf{a}} = \hat{\imath} - \hat{\jmath} + \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = 2\hat{\imath} - \hat{\jmath}$, and $\vec{\mathbf{c}} = 3\hat{\imath} + 5\hat{\jmath} - 7\hat{\mathbf{k}}$, verify the identity

$$\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\,\vec{\mathbf{b}} - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)\vec{\mathbf{c}}$$

Solution: We just need to grind through it. For the left-hand side:

$$\vec{\mathbf{b}} \times \vec{\mathbf{c}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 2 & -1 & 0 \\ 3 & 5 & -7 \end{vmatrix} = 7\hat{\imath} + 14\hat{\jmath} + 13\hat{\mathbf{k}}$$

$$\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 7 & 14 & 13 \end{vmatrix} = -27\hat{\imath} + -6\hat{\jmath} + 21\hat{\mathbf{k}}$$

For the right-hand side:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{c}} = \left(\hat{\imath}-\hat{\jmath}+\hat{\mathbf{k}}\right)\cdot\left(3\hat{\imath}+5\hat{\jmath}-7\hat{\mathbf{k}}\right) = \begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}3 & 5 & -7\end{bmatrix} = 3-5-7 = -9$$

$$\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\vec{\mathbf{b}} = -9\vec{\mathbf{b}} = -18\hat{\imath}+9\hat{\jmath}$$

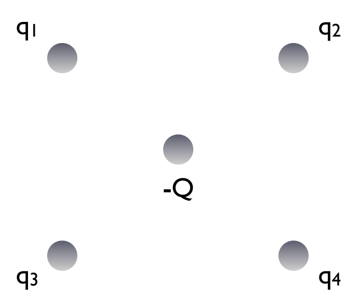
$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}2 & -1 & 0\end{bmatrix} = 2+1 = 3$$

$$\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\vec{\mathbf{c}} = 3\vec{\mathbf{c}} = 9\hat{\imath}+15\hat{\jmath}-21\hat{\imath}$$

$$\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\vec{\mathbf{b}}-\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\vec{\mathbf{c}} = -27\hat{\imath}-6\hat{\jmath}+21\hat{\mathbf{k}} = \vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right)$$

4. At each corner of a square is a particle with charge q. Fixed at the center of the square is a point charge with opposite sign, of magnitude Q. What value must Q have to make the total force on each of the four particles zero? With Q set at that value, the system, in the absence of other forces, is in equilibrium. Do you think the equilibrium is stable?

Solution: The configuration of interest is thus:



Clearly, the four charges on the corners all have the same force, so we need only worry about the force on a single charge. Let the length of the square's side be a. Let the $\hat{\jmath}$ direction be upward, and the $\hat{\imath}$ direction be to the right. On charge 1, charges 2, 3, and 4 all give a repulsive force, while the charge $-Q$ gives an attractive force. Charges 2 and 3 are a distance a away, charge 4 is $a\sqrt{2}$ away, and the $-Q$ charge is $a\sqrt{2}/2$ away. The forces from charges 2 and 3 are purely along the $\hat{\imath}$ and $\hat{\jmath}$ directions, respectively:

$$\vec{\mathbf{F}}_3 = \frac{k_e q_1 q_3}{a^2}\hat{\boldsymbol{\jmath}} \tag{1}$$

$$\vec{\mathbf{F}}_2 = -\frac{k_e q_1 q_2}{a^2}\hat{\boldsymbol{\imath}} \tag{2}$$

Charge 4 exerts a force on charge 1 along the square diagonal:

$$\vec{\mathbf{F}}_4 = -\frac{k_e q_1 q_4}{2a^2}\cos 45\,\hat{\boldsymbol{\imath}} + \frac{k_e q_1 q_4}{2a^2}\sin 45\,\hat{\boldsymbol{\jmath}} = \frac{\sqrt{2}k_e q_1 q_4}{4a^2}\left(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}\right) \tag{3}$$

The $-Q$ charge works out the same way as q_4, but at half the distance:

$$\vec{\mathbf{F}}_{-Q} = \frac{\sqrt{2}k_e q_1 Q}{a^2}\left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}\right) \tag{4}$$

Now we have all the relevant forces. We need both the x and y components to vanish, but the symmetry of the problem makes the x and y directions equivalent. Thus, we can balance either one of them and be certain the other is as well – the equations are the same for the x and y directions. Picking the x direction arbitrarily,

$$\sum F_x = F_{2x} + F_{4x} + F_{-Qx} = -\frac{k_e q_1 q_2}{a^2} - \frac{k_3 q_1 q_4}{a^2}\frac{\sqrt{2}}{4} + \frac{k_e q_1 Q}{a^2}\sqrt{2} = 0 \tag{5}$$

$$0 = \frac{k_e}{a^2}\left[-q_1 q_2 - q_1 q_4 \frac{\sqrt{2}}{4} + q_1 Q_2 \sqrt{2}\right] \tag{6}$$

Since the four corner charges are all equivalent, we can just call them q:

$$0 = -q^2 - \frac{\sqrt{2}}{4}q^2 + \sqrt{2}qQ \tag{7}$$

$$Q = \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)q \approx 0.957q \tag{8}$$

Is this a stable equilibrium? If so, the system should be stable against small displacements of any given charge. Strictly, we should find the potential energy with the center charge displaced by some amount δ, and show that the potential energy is larger. However, qualitatively we can see that the equilibrium is unstable. Consider a tiny displacement of the central charge: any movement will bring it slightly closer to one of the corner charges, which will make it even more attracted to that corner. The central charge will rapidly be pulled toward one corner with even an infinitesimal displacement away from the exact center. A similar argument holds for displacing one of the corner charges. The system as drawn is indeed in an equilibrium state, but since the tiniest infinitesimal perturbation will destroy the given configuration, the equilibrium is unstable.