

## Problem Set 2 Solutions

1. A charge of  $1\ \mu\text{C}$  is at the origin. A charge of  $-2\ \mu\text{C}$  is at  $x=1$  on the  $x$  axis.

(a) Find the point on the  $x$  axis where the electric field is zero.

(b) Locate, at least approximately, a point on the  $y$  axis here the electric field is parallel to the  $x$  axis [a calculator should help].

**Solution:** Consider a charge  $q_1$  at the origin and a charge  $q_2$  at position  $x_o > 0$  on the  $x$  axis. We will assume  $|q_2| > |q_1|$ ,  $q_1 > 0$  and  $q_2 < 0$ .

First, we need to do the physics before we get to any math. Where *could* the field be zero along the  $x$  axis? In the region between the two charges, the positive charge  $q_1$  gives a field in the  $\hat{x}$  direction, and the negative charge  $q_2$  *also* gives a field in the  $\hat{x}$  direction. Since both fields act in the same direction, they cannot possibly cancel each other.

How about for  $x > x_o$  (i.e., to the right of the negative charge)? Well, in this region we are farther from  $q_1$  than we are  $q_2$ , so the field from  $q_1$  is smaller than that of  $q_2$ . Further, if  $|q_2| > |q_1|$ , the field of  $q_1$  is smaller to start with, even at the same distance. Even though the fields are in opposite directions, since they decay as  $1/x^2$  and  $q_1$  is smaller and farther away, the two fields cannot possibly cancel each other.

We are left with  $x < 0$ . This makes sense: the two fields are in opposite directions, and we are closer to the smaller charge. The smaller field at a given distance can be compensated by simply getting closer to the smaller charge. Let us consider a position  $-x$  along the  $x$  axis, which means we are a distance  $x$  from  $q_1$  and  $x + x_o$  from  $q_2$ . The total field is then:

$$E = E_1 + E_2 = \frac{kq_1}{x^2} + \frac{kq_2}{(x + x_o)^2} \quad (1)$$

Keep in mind  $q_2$  is negative. We desire  $E=0$ . Thus,

$$\frac{kq_1}{x^2} = -\frac{kq_2}{(x + x_o)^2} \quad (2)$$

Since we already know our point of interest along the negative  $x$  axis, we can just take the square root of both sides and solve this thing quickly – we don't need to worry about the  $\pm$  since we already figured that part out. Thus,

$$\begin{aligned}
\frac{x}{\sqrt{q_1}} &= \frac{x + x_o}{\sqrt{-q_2}} \\
\frac{x_o}{\sqrt{-q_2}} &= x \left( \frac{1}{\sqrt{q_1}} - \frac{1}{\sqrt{-q_2}} \right) \\
x &= \frac{x_o}{\frac{\sqrt{-q_2}}{\sqrt{q_1}} - 1} = \frac{x_o \sqrt{q_1}}{\sqrt{-q_2} - \sqrt{-q_1}}
\end{aligned} \tag{3}$$

With the numbers given,

$$x = \frac{1}{\sqrt{2} - 1} = -\sqrt{2} - 1 \approx -2.4 \text{ m} \tag{4}$$

For the next part, we want a point along the  $y$  axis where the field is purely in the horizontal direction. If that is to be true, then we must find a point where the  $y$  components of the field from each charge exactly balance. From the charge  $q_1$  this is easy: if we are a distance  $y$  along the  $y$  axis, then

$$E_{1y} = \frac{kq_1}{y^2} \tag{5}$$

The charge  $q_2$  is at a distance  $\sqrt{x_o^2 + y^2}$  from the same point on the  $y$  axis. Its field in the  $y$  direction is then found similarly, accounting for the geometric factor  $\sin \theta$  to pick out the  $y$  component.

$$E_{2y} = \frac{kq_2}{x_o^2 + y^2} \sin \theta = \frac{kq_2}{x_o^2 + y^2} \frac{y}{\sqrt{x_o^2 + y^2}} = \frac{kq_2 y}{(x_o^2 + y^2)^{3/2}} \tag{6}$$

We need  $E_{1y} = -E_{2y}$  for the net vertical field to be zero (resulting in a purely horizontal field):

$$E_{1y} = -E_{2y} \tag{7}$$

$$\frac{kq_1}{y^2} = \frac{-kq_2 y}{(x_o^2 + y^2)^{3/2}} \tag{8}$$

$$q_1 (x_o^2 + y^2)^{3/2} = -q_2 y^3 \tag{9}$$

$$q_1^{2/3} (x_o^2 + y^2) = (-q_2)^{2/3} y^2 \tag{10}$$

$$y^2 (q_1^{2/3} - (-q_2)^{2/3}) = q_1^{2/3} x_o^2 \tag{11}$$

$$y = \frac{\pm q_1^{1/3} x_o}{\sqrt{(-q_2)^{2/3} - q_1^{2/3}}} = \frac{\pm 1}{\sqrt{2^{2/3} - 1}} \approx \pm 1.30 \text{ m} \tag{12}$$

Clearly, by symmetry there are two equivalent points along the  $y$  axis, so our  $\pm$  makes sense here.

**2.** Two long, thin parallel rods, a distance  $2b$  apart, are joined by a semicircular piece of radius  $b$ , as shown below. Charge of uniform linear density  $\lambda$  is deposited along the whole filament. Show that the field  $\vec{E}$  of this charge distribution vanishes at point C. One way to do this is by comparing the contribution of the element at A to that of the element at B which is defined by the same values of  $\theta$  and  $d\theta$ .

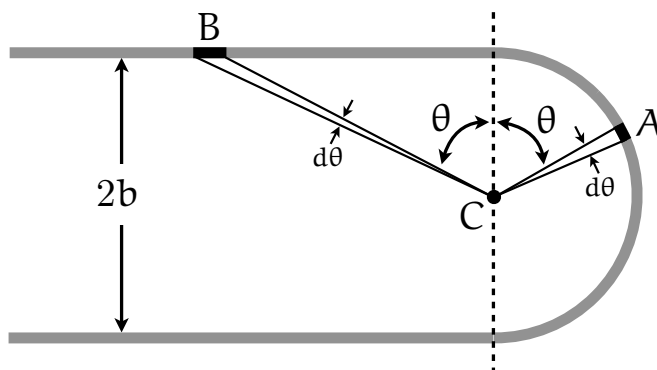


Figure 1: Problem 2.

**Solution:** A tiny segment of the circular part  $ds$  will have charge  $dq$ , giving rise to a field at C which points to the right at an angle  $\theta$ . If we define vertical to be angle equal to zero, building up the entire semicircle means running the angle from 0 to  $\pi$ . Let's redraw the figure with more detail:

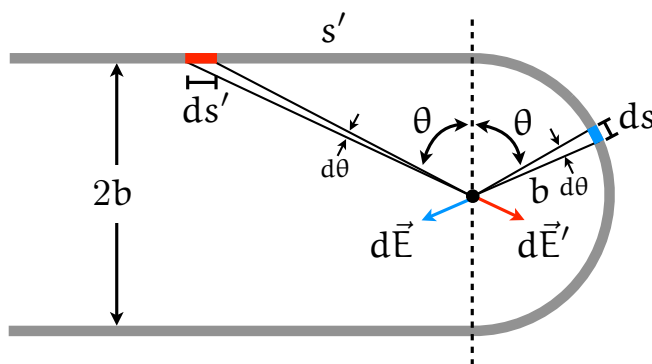


Figure 2: Problem 2, more so.

Now consider first the contributions from the two line charges. A tiny segment  $ds'$  of the upper line will have charge  $dq'$ , giving rise to a field  $dE'$  at C which points to the *left* at angle  $\theta$ . Here, the angle  $\theta$  is defined relative to the vertical, so building the entire upper line means running  $\theta$

from 0 to  $\pi/2$ . Building the *lower* line is the same construction, except now we run  $\theta$  from  $\pi/2$  to  $\pi$ . Thus, building *both* line charges runs over the same angular range as building the semicircle, so we may consider the lines together. Now we need to write down the field contributions  $dE$  and  $dE'$  from the semicircle and line charges.

By symmetry, the vertical component of the field clearly cancels - for both wires and semicircle, any segment giving a vertical component to the field will have a partner giving an equal and opposite vertical component. Thus, we need only write down the horizontal component of the field for line and semicircle; let us call that the  $x$  direction, with positive  $x$  toward the right. For the semicircle, a segment of length  $ds$  has charge  $dq = \lambda ds$ . For a circle, arclength is angle times radius, so  $ds = r d\theta$  and  $dq = \lambda r d\theta$ . The horizontal component of the field from a segment of the semicircle is then readily found:

$$dE_x = -\frac{k dq}{b^2} \sin \theta = -\frac{k\lambda d\theta}{b} \sin \theta \quad (13)$$

Here the minus sign reminds us that the field points to the left. For an arbitrary line segment  $ds'$ , the infinitesimal length can be found by considering the distance  $s'$  from  $C$  to the segment:

$$s' = b \tan \theta \quad (14)$$

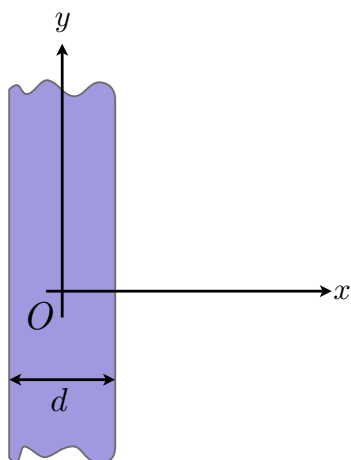
$$\frac{ds'}{d\theta} = b \sec^2 \theta \quad (15)$$

$$ds' = b \sec^2 \theta d\theta \quad (16)$$

The charge  $dq'$  is then just  $\lambda ds'$ . This bit of charge is a distance  $r' = b / \cos \theta = b \sec \theta$  away from  $C$ , so the field is easy to find:

$$dE'_x = \frac{k dq'}{(r')^2} \sin \theta = \frac{k\lambda b \sec^2 \theta d\theta}{b^2 \sec^2 \theta} \sin \theta = \frac{k\lambda d\theta}{b} \sin \theta \quad (17)$$

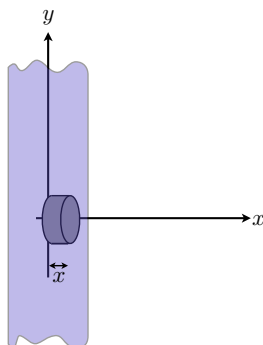
Thus,  $dE'_x = -dE_x$ . Since there is a 1:1 mapping between segments of the two lines and segments of the semicircle, the field vanishes. That is, the horizontal component of the field from a line segment and a semicircle segment cancel each other, and there are the same number of each, so everything cancels, the field vanishes.



**3.** A slab of insulating material, infinite in two of its three dimensions, has a uniform positive charge density  $\rho$ , shown at left. **(a)** Show that the magnitude of the electric field a distance  $x$  from the center and inside the slab is  $E = \rho x / \epsilon_0$ . **(b)** Suppose an electron of charge  $-e$  and mass  $m_e$  can move freely within the slab. It is released from rest at a distance  $x$  from the center. Show that the electron exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{\rho e}{m_e \epsilon_0}}$$

**Solution:** Consider a cylindrical shaped Gaussian surface perpendicular to the  $yz$  plane, as shown below. One circular face is in the  $yz$  plane, and the other is a distance  $x$  in the positive  $\hat{x}$  direction.



We know that for a thin slab of charge, the electric field must be perpendicular to the surface of the slab, or along the  $\hat{x}$  direction in this case. Further, by symmetry the field must be zero exactly at the center of the slab, where we have placed the left-most end cap. If we apply Gauss' law to this surface, the net flux is non-zero only through the rightmost end cap of the cylinder. Thus, if we call the area of the cylinder's end cap  $A_{\text{end cap}}$ ,

$$\oint \vec{E} \cdot \hat{n} dA = EA_{\text{end cap}}$$

The amount of charge contained in the Gaussian cylinder is just the charge density of the slab times the volume of the cylinder,  $x A_{\text{end cap}}$ , so

$$E A_{\text{end cap}} = \frac{\rho A_{\text{end cap}} x}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\rho x}{\epsilon_0}$$

Fair enough. What is the force on an electron placed at a distance  $x$ ? We know it is the electron's charge  $-e$  times the electric field at  $x$ , and if this is the only force acting, it must equal the electron's mass  $m_e$  times the resulting acceleration.

$$\vec{F} = -e\vec{E} = \frac{-e\rho x}{\epsilon_0} \hat{x} = m_e \vec{a}$$

Thus, the acceleration must be in the  $\hat{x}$  direction, and it is proportional to the electron's position  $x$ . How do we show that simple harmonic motion results? We just did! Recall from mechanics that simple harmonic motion occurs whenever we can show that a system's acceleration are related by some constant  $-\omega^2$ , viz.,  $\mathbf{a} = -\omega^2 \mathbf{x}$ .<sup>i</sup> If this is true, then  $\omega$  is the angular frequency of oscillation, and  $\omega = 2\pi f$ . Rewriting the equation above, dropping the now redundant vector notation,

$$\begin{aligned} \mathbf{a} &= \frac{d^2x}{dt^2} = -\frac{e\rho}{m_e \epsilon_0} x \\ \Rightarrow \quad \omega &= \sqrt{\frac{e\rho}{m_e \epsilon_0}} \quad \text{and} \quad f = \frac{1}{2\pi} \sqrt{\frac{e\rho}{m_e \epsilon_0}} \end{aligned}$$

**4.** Imagine a sphere of radius  $a$  filled with negative charge of uniform density, the total charge being equivalent to that of two electrons. Imbed in this jelly of negative charge two protons and assume that in spite of their presence the negative charge distribution remains uniform. Where must the protons be located so that the total force on each of them is zero? (This is a surprisingly realistic model of a hydrogen atom; the magic that keeps the electron cloud in the molecule from collapsing around the protons is explained by quantum mechanics!)

**Solution:** The protons should be placed at a distance  $a/2$  from the center of the sphere of negative charge, symmetric about the sphere's midpoint.

The forces on the protons from each other will be equal and opposite. Therefore, the forces on them from the negative charge distribution must be equal and opposite also. This requires that they lie on a line through the center and are equidistant from the center. The force on each proton at radius  $r$  from the negative charge will be proportional to the amount of negative charge lying inside a sphere of radius  $r$ . For purposes of finding the electric field, we may treat all of this charge

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<sup>i</sup>Somewhat more precisely, we want to show that the differential equation  $d^2x/dt^2 = -\omega^2 x$  is obeyed, which has the simplest physical solution  $x(t) = A \cos(\omega t + \delta)$ .

as if it were a point charge sitting in the center. We ignore all negative charge outside the radius of the proton positions. The negative charge inside the radius  $r$  is:

$$\begin{aligned} q(r) &= -2e \left( \frac{\text{volume enclosed by sphere of radius } r}{\text{total volume}} \right) \\ &= -2e \left( \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a_0^3} \right) \\ &= -\frac{2er^3}{a_0^3} \end{aligned}$$

This charge  $q(r)$  will give an electric field at the position of each proton. Since the charge  $q(r)$  is spherically symmetric, it will be the same as the field from a point charge  $q(r)$  at a distance  $r$ :

$$E(r) = \frac{k_e q(r)}{r^2} = -\frac{2ek_e r^3}{a_0^3 r^2} = -\frac{2k_e e r}{a_0^3}$$

The force on each proton must be zero, the sum of the attractive force due to the charge  $q(r)$  and the repulsive force from the other proton. Since a proton has a charge  $e$ , the attractive force is  $qE(r)$ . The repulsive force between the protons is easily calculated noting their charge  $e$  and separation  $2r$ :

$$\begin{aligned} \sum F &= eE(r) + \frac{k_e e^2}{(2r)^2} = -\frac{2k_e e^2 r}{a_0^3} + \frac{k_e e^2}{4r^2} = 0 \\ \implies \frac{2k_e e^2 r}{a_0^3} &= \frac{k_e e^2}{4r^2} \\ 8k_e e^2 r^3 &= e^2 k_e a_0^3 \\ r &= \frac{a_0}{2} \end{aligned}$$

**5. (a)** Twelve equal charges  $q$  are situated at the corners of a regular 12-sided polygon (for instance, one on each numeral of a clock face). What is the net force on a test charge  $Q$  at the center?

**(b)** Suppose *one* of the 12  $q$ 's is removed (say, the one at 6 o'clock). What is the force on  $Q$ ? Use superposition ... and explain your reasoning carefully.

**Solution:** When there are charges on every position of the clock, all the forces cancel by symmetry – for each charge, there is another directly opposite the center of the clock that gives an equal and opposite contribution to the net force on  $Q$ . When one is missing, the charge *opposite* the missing one is no longer canceled, meaning there is a net force due to a single  $q$  on the center charge  $Q$ , pointing *toward* the missing charge:  $F = kqQ/r^2$ . Alternatively, you could imagine that we “re-

moved” the 6 o’clock charge by putting a  $-q$  charge right on top of it. All the positive charges still cancel, but we’d have a extra  $-q$  pulling the center charge  $Q$  toward it.

Now a question: would it still work out if there were 13 charges on a 13-sided polygon, where the symmetry isn’t quite as obvious?

**6. (a)** Find the electric field a distance  $z$  above the center of a flat circular disk of radius  $R$  which carries a uniform surface charge  $\sigma$ .

**(b)** What happens in the limits  $R \rightarrow \infty$  and  $z \gg R$ ? Comment on the mathematical and physical nature of those limits.

**Solution:** We can break the disk up into tiny little annular slices, as in the figure below.<sup>ii</sup> Each little patch of area  $dA$  is defined by an angular spread  $d\theta$  and a radial width  $dr$ . If the little patches are infinitesimally small, then the outer and inner curved surfaces have essentially the same length, the arclength  $ds = r d\theta$ . This makes the area

$$dA = r dr d\theta \tag{18}$$

If the charge per unit area is  $\sigma$ , then each patch has charge  $dq = \sigma dA$ . What is the field contribution from each  $dq$ ? By symmetry, the lateral components of the field ( $x$  and  $y$ ) will vanish, leaving only a vertical ( $z$ ) component. Each  $dq$  is a distance  $\sqrt{r^2 + z^2}$  from the point of interest. Picking out the vertical component means multiplying by  $\cos \varphi = z/\sqrt{r^2 + z^2}$ , so each  $dq$  contributes a vertical field

$$dE_z = \frac{k dq}{r^2 + z^2} \cos \varphi = \frac{k\sigma r dr d\theta}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}} = \frac{k\sigma z r dr d\theta}{(r^2 + z^2)^{3/2}} \tag{19}$$

Building up the whole disk means running the angle  $\theta$  from 0 to  $2\pi$ , and the radius from 0 to  $R$ , the total radius of the disk. Thus, we integrate  $dE_z$  with respect to  $\theta$  and  $r$  over those limits to get the total vertical field  $E_z$ . The  $\theta$  integral is trivial, since there is no  $\theta$  dependence.<sup>iii</sup>

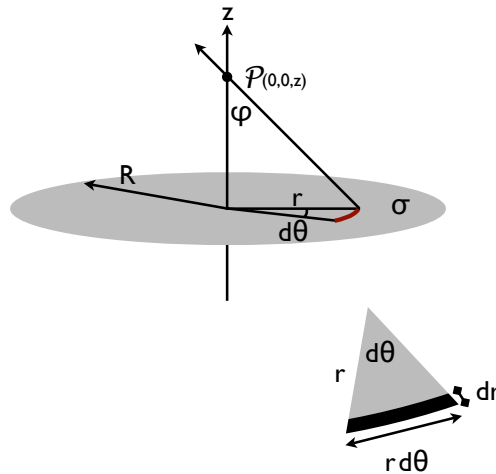
$$E_z = \int_0^{2\pi} d\theta \int_0^R dr \frac{k\sigma z r}{(r^2 + z^2)^{3/2}} = 2\pi \int_0^R dr \frac{k\sigma z r}{(r^2 + z^2)^{3/2}} \tag{20}$$

$$= 2\pi k\sigma z \frac{1}{\sqrt{r^2 + z^2}} \Big|_0^R = 2\pi k\sigma \left( 1 - \frac{|z|}{\sqrt{R^2 + z^2}} \right) = \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{|z|}{\sqrt{R^2 + z^2}} \right) \tag{21}$$

<sup>ii</sup>We could also build the disk out of little rings, since the field of a ring is easy to calculate. We chose the more general approach here just for the sake of completeness. As we note below, the general approach can be thought of as first building a ring, then building the disk out of rings.

<sup>iii</sup>Integrating over  $\theta$  first, we build an annulus (ring) out of little segments first, integrating over  $r$  then builds the disk out of rings.





In the limit  $R \rightarrow \infty$ , we recover  $E_z = \sigma/2\epsilon_0$ , the field of an infinite sheet of charge. In the limit  $z \gg R$ , when we are far from the disc, it should reduce to a point charge. We can use the binomial expansion  $(1+x)^n \approx 1 + nx$  for  $x \ll 1$ :

$$(R^2 + z^2)^{-1/2} = z^{-1} \left(1 + \frac{R^2}{z^2}\right)^{-1/2} \approx \frac{1}{z} \left(1 - \frac{R^2}{2z^2}\right) \quad (22)$$

This gives

$$E_z = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{|z|}{\sqrt{R^2 + z^2}}\right) \approx \frac{\sigma}{2\epsilon_0} \left(1 - |z| \frac{1}{z} \left(1 - \frac{R^2}{2z^2}\right)\right) = \frac{\sigma}{2\epsilon_0} \frac{R^2}{2z^2} \quad (23)$$

Noting that  $Q = \pi R^2 \sigma$  is the total charge, or  $\sigma = Q/\pi R^2$ ,

$$E_z \approx \frac{\sigma}{2\epsilon_0} \frac{R^2}{2z^2} = \frac{Q}{2\pi\epsilon_0 R^2} \frac{R^2}{2z^2} = \frac{Q}{4\pi\epsilon_0 z^2} \quad (24)$$

Exactly the expression for a point charge  $Q$  at a distance  $z$ .

**7.** Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin,  $\rho = cr$ , for some constant  $c$ . Note: the charge density is not uniform, and you must integrate to get the enclosed charge.

**Solution:** Just for fun, we'll find the field everywhere. The charge distribution is spherically symmetric, which means according to Gauss' law we can treat it as a point charge. That means that if the total charge is  $q$ , the field outside the sphere is  $kq/r^2$  pointing radially outward, where  $r$  is the distance from the center of the sphere. To get  $q$ , we have to integrate  $\rho dV$ . A thin spherical

shell of thickness  $dr$  has volume  $4\pi r^2 dr$  (thickness times surface area), so

$$q = \int \rho dV = \int_0^R 4\pi r^2 \rho dr = \int_0^R 4\pi r^3 \rho dr = \frac{4\pi c R^4}{4} = \pi c R^4 \quad (25)$$

Thus,

$$E = \frac{kq}{r^2} = \frac{\pi k c R^4}{r^2} \quad r \geq R \quad (26)$$

How about *inside* the sphere at some distance  $r$  from the center? Gauss' law says that the field is still that of a point charge, but one whose magnitude is determined by how much charge is enclosed by a sphere of radius  $r$ . That is, how much charge is still "beneath" the point of interest. That enclosed charge is found by just changing the upper limit of integration for  $q$  from  $R$  to  $r$ :  $q_{\text{encl}} = \pi c r^4$ , so

$$E = \frac{kq_{\text{encl}}}{r^2} = \frac{\pi k c r^4}{r^2} = \pi k c r^2 \quad r \leq R \quad (27)$$

One can readily verify that the field is zero at the origin, and that the two formulae agree for  $r=R$ .

**8.** Consider an infinite number of identical charges (each of charge  $q$ ) placed along the  $x$  axis at distances  $a, 2a, 3a, 4a, \dots$  from the origin. What is the electric field at the origin due to this distribution?

**Solution:** We have identical charges  $q$  placed at distances which are integer multiples of  $a$  from the origin. The total electric field is the sum of each individual electric field from each individual charge. Since all charges are along the  $x$  axis, their fields all point in the  $-\hat{x}$  direction, and we can simply add the magnitudes.

If we number these charges sequentially, starting from the closest one and continuing until the  $n^{\text{th}}$ , the magnitude of the electric fields from the first few charges can be calculated easily, and we begin to notice a pattern:

$$\begin{aligned}
E_1 &= -\frac{k_e q}{a^2} \\
E_2 &= -\frac{k_e q}{(2a)^2} = -\frac{k_e q}{4a^2} \\
E_3 &= -\frac{k_e q}{(3a)^2} = -\frac{k_e q}{9a^2} \\
&\vdots \\
E_n &= -\frac{k_e q}{(na)^2} = -\frac{k_e q}{n^2 a^2}
\end{aligned}$$

The minus sign is to indicate that the field is in the  $-\hat{x}$  direction. Having noticed a pattern, we can more easily represent the total electric field as an infinite sum, noting that the  $n^{\text{th}}$  charge from the origin is at a distance  $na$ , and all charges are identical:

$$E_{\text{tot}} = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \frac{-k_e q_n}{r_n^2} = \sum_{n=1}^{\infty} \frac{-k_e q}{(na)^2} = \frac{-k_e q}{a^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

This infinite sum is convergent, and has the remarkably simple value of  $\pi^2/6$ .<sup>iv</sup> The total electric field is then

$$E_{\text{tot}} = \frac{-k_e q}{a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{-k_e q}{a^2} \left( \frac{\pi^2}{6} \right) = \frac{-\pi^2 k_e q}{6a^2}$$

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<sup>iv</sup>There are some very clever proofs of this, my favorite perhaps being the one using Fourier series and Parseval's identity. The first proof was due to Euler: [http://en.wikipedia.org/wiki/Basel\\_problem](http://en.wikipedia.org/wiki/Basel_problem)