

Problem Set 3: solutions

1. A spherical volume of radius a is filled with charge of uniform density ρ . We want to know the potential energy U of this sphere of charge, that is, the work done in assembling it. Calculate it by building up the sphere up layer by layer, making use of the fact that the field outside a spherical distribution of charge is the same as if all the charge were at the center. Express the result in terms of the total charge Q of the sphere.

Solution: There are in my opinion three reasonably straightforward ways to go about this problem, each with its own merits. Below I will outline all three, you only needed to have one valid solution for full credit.

Method 1: use $dU = V dq$

We know that we can find the potential energy for any charge distribution by calculating the work required to bring in each bit of charge in the distribution in from an infinite distance away. If we bring a bit of charge dq in from infinitely far away (where the electric potential is zero) to a point $P(\vec{r})$, at which the electric potential is V , work is required. The potential V results from all of the other bits of charge already present; we bring in one dq at a time, and as we build up the charge distribution, we calculate the new potential, and that new potential determines the work required to bring in the next dq .

In the present case, we will build our sphere out of a collection of spherical shells of infinitesimal thickness dr . We'll start with a shell of radius 0, and work our way up to the last one of radius R . A given shell of radius r will have a thickness dr , which gives it a surface area of $4\pi r^2$ and a volume of (thickness)(surface area) = $4\pi r^2 dr$.

Once we've built the sphere up to a radius r , Gauss' law tells us that the potential at the surface is just that of a point charge of radius r :

$$V(r) = \frac{k_e q(r)}{r} \tag{1}$$

Where $q(r)$ is the charge built up so far, contained in a radius r . Bringing in the next spherical shell of radius $r + dr$ and charge dq will then require work to be done, in the amount $V(r) dq$, since

we are bringing a charge dq from a potential of 0 at an infinite distance to a potential $V(r)$ at a distance r . Thus, the little bit of work required to bring in the next shell is

$$-dW = dU = V(r) dq = \frac{k_e q(r)}{r} dq \quad (2)$$

Now, first we'll want to know what $q(r)$ is. If we have built the sphere to a radius r , then the charge contained so far is just the charge density times the volume of a sphere of radius r :

$$q(r) = \frac{4}{3}\pi r^3 \rho \quad (3)$$

Next, we need to know what dq is, the charge contained in the next shell of charge we want to bring in. In this case the charge is just the volume of the shell times the charge density:

$$dq = 4\pi r^2 dr \rho \quad (4)$$

Putting that all together:

$$dU = \frac{k_e \frac{4}{3}\pi r^3 \rho}{r} 4\pi r^2 dr \rho = \frac{16\pi^2 k_e \rho^2}{3} r^4 dr \quad (5)$$

All we need to do now is integrate up all the small bits of work required from the starting radius $r=0$ to the final radius $r=R$:

$$U = \int_0^R dU = \int_0^R \frac{16\pi^2 k_e \rho^2}{3} r^4 dr = \frac{16\pi^2 k_e \rho^2}{3} \left[\frac{r^5}{5} \right]_0^R \quad (6)$$

$$\implies U = \frac{16\pi^2 k_e \rho^2}{15} R^5 \quad (7)$$

If we remember that the total charge on the sphere is $Q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho$, we can rewrite a bit more simply:

$$U = \frac{3k_e Q_{\text{tot}}^2}{5R} \quad (8)$$

Method 2: use $dU = \frac{1}{2}\epsilon_0 E^2 dV_{\text{ol}}$

We have also learned that the potential energy of a charge distribution can be found by integrating

its electric field (squared) over all space:¹

$$U = \frac{1}{2} \int_{\text{everywhere}} \epsilon_0 E^2 dV_{\text{ol}} = \frac{1}{8\pi k_e} \int_{\text{everywhere}} E^2 dV_{\text{ol}} \quad (9)$$

If we have a sphere of charge of radius R , we can break all of space up into two regions: the space outside of the sphere, where the field is $\vec{E}_{\text{out}}(r)$ at a distance r from the center of the sphere, and the space inside the sphere, where the field is $\vec{E}_{\text{in}}(r)$. Therefore, we can break the integral above over all space into two separate integrals over each of these regions, representing the energy contained in the field outside the sphere, and that contained inside the sphere itself:

$$U = U_{\text{outside}} + U_{\text{inside}} = \frac{1}{8\pi k_e} \int_{\text{outside}} E_{\text{out}}^2 dV_{\text{ol}} + \frac{1}{8\pi k_e} \int_{\text{inside}} E_{\text{in}}^2 dV_{\text{ol}} \quad (10)$$

Obviously, the region outside the sphere is defined as $r > R$, and the region inside the sphere is $r \leq R$. Just like in the last example, we can perform the volume integral over spherical shells of radius r and width dr , letting the radius run from $0 \rightarrow R$ inside the sphere, and $R \rightarrow \infty$ outside. Our volume elements are thus $dV_{\text{ol}} = 4\pi r^2 dr$, just as in the previous method.

What about the fields? Outside the sphere, the electric field is just that of a point charge (thanks to Gauss' law). Noting again that the total charge on the sphere is $Q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho$,

$$\vec{E}_{\text{out}}(r) = \frac{k_e Q_{\text{tot}}}{r^2} \hat{r} = \frac{4\pi k_e R^3 \rho}{3r^2} \hat{r} \quad (r > R) \quad (11)$$

Inside the sphere at a radius r , the electric field depends only on the amount of charge contained within a sphere of radius r , $q(r) = \frac{4}{3}\pi r^3 \rho$ as noted above, which we derived previously from Gauss' law:

$$\vec{E}_{\text{in}}(r) = \frac{k_e q(r)}{r^2} \hat{r} = \frac{4\pi k_e r \rho}{3} \hat{r} \quad (r \leq R) \quad (12)$$

Now all we have to do is perform the integrals . . .

¹The odd notation for volume dV_{ol} is to avoid confusion with electric potential without having to introduce more random Greek symbols. Some people will use $d\tau$ to represent volume to this end.

$$U = U_{\text{outside}} + U_{\text{inside}} = \frac{1}{8\pi k_e} \int_{\text{outside}} E_{\text{out}}^2 dV_{\text{ol}} + \frac{1}{8\pi k_e} \int_{\text{inside}} E_{\text{in}}^2 dV_{\text{ol}} \quad (13)$$

$$= \frac{1}{8\pi k_e} \int_R^\infty \left(\frac{4\pi k_e R^3 \rho}{3r^2} \right)^2 4\pi r^2 dr + \frac{1}{8\pi k_e} \int_0^R \left(\frac{4\pi k_e r \rho}{3} \right)^2 4\pi r^2 dr \quad (14)$$

$$= \frac{8\pi^2 k_e R^6 \rho^2}{9} \int_R^\infty \frac{1}{r^2} dr + \frac{8\pi^2 k_e \rho^2}{9} \int_0^R r^4 dr \quad (15)$$

$$= \frac{8\pi^2 k_e R^6 \rho^2}{9} \left[-\frac{1}{r} \right]_R^\infty + \frac{8\pi^2 k_e \rho^2}{9} \left[\frac{r^5}{5} \right]_0^R = \frac{8\pi^2 k_e \rho^2 R^5}{9} \left(\frac{1}{5} + 1 \right) \quad (16)$$

$$= \frac{16\pi^2 k_e \rho^2}{15} R^5 = \frac{3}{5} \frac{kQ_{\text{tot}}^2}{R} \quad (17)$$

In the last line, we used $\rho = Q / (\frac{4}{3}\pi R^3)$. Somehow, it is reassuring that these two different methods give us the same result ... or at least it should be!

Method 3: use $dU = \frac{1}{2} \rho V dV_{\text{ol}}$

As in the first method, we know that we can find the potential energy for any charge distribution by calculating the work required to bring in each bit of charge in the distribution in from an infinite distance away. For a collection of discrete point charges, we found:

$$U = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{k_e q_i q_j}{r_{ij}} = \frac{1}{2} \sum_i \sum_{j \neq i} q_i V_j = \frac{1}{2} \sum_i \sum_{j \neq i} q_j V_i \quad (18)$$

Recall that the factor 1/2 is so that we don't count all pairs of charges twice. If we have a *continuous* charge distribution - like a sphere - our sum becomes an integral:

$$U = \frac{1}{2} \int_{\text{everywhere}} \rho V dV_{\text{ol}} \quad (19)$$

We still need the factor 1/2 to avoid double counting. Like the last example, we break this integral over all space up into two separate ones: one over the volume inside the sphere, and one outside. Outside the sphere at a distance $r > R$, the charge density ρ is zero, so that integral is zero. All we have to do is integrate the potential inside the sphere times the constant charge density over the volume of the sphere! But wait ... what is the potential inside the sphere?

The electric *field* inside such a sphere at a radius r we have already calculated, and it depended only on the amount of charge contained in a sphere of radius r , $q(r)$. That is because electric field from

the spherical shell of charge more distant than r canceled out. This is *not* true of the potential, because the potential is a *scalar*, not a vector like the electric field, and there is no directionality that will cause the potential to cancel from the charge more distant than r .

Thus, the potential inside the sphere will actually have two components at a distance r inside the sphere: a term which looks like the potential of a point charge, due to the charge at a distance less than r from the center, and one that looks like the potential due to the remaining shell of charge from r to R . How do we calculate this mess? We can use the definition of the potential and the known electric field inside the sphere.

$$\vec{\mathbf{E}}_{\text{in}}(r) = \frac{k_e Q_{\text{tot}} r}{R^3} \hat{\mathbf{r}} \quad (20)$$

$$V_B - V_A = - \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} \quad (21)$$

In this case, we want to find the potential difference between a point at which we already know what it is - such as on the surface of the sphere - and a point r inside the sphere. Then we can use the known potential on the surface to find the unknown potential at r . We can most simply follow a radial path, $d\vec{\mathbf{l}} = \hat{\mathbf{r}} dr$. Since the electric field is conservative, the path itself doesn't matter, just the endpoints - so we always choose a very convenient path.

$$V(r) - V(R) = - \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = - \int_R^r \frac{k_e Q_{\text{tot}} r}{R^3} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr \quad (22)$$

$$= - \int_R^r \frac{k_e Q_{\text{tot}} r}{R^3} dr = - \frac{k_e Q_{\text{tot}}}{R^3} \left[\frac{r^2}{2} \right]_R^r \quad (23)$$

$$= \frac{k_e Q_{\text{tot}}}{2R^3} (R^2 - r^2) \quad (24)$$

We know what the potential at the surface $V(R)$ is, since outside the sphere the situation is identical to that of a point charge. Thus,

$$V(r) = V(R) + \frac{k_e Q_{\text{tot}}}{2R^3} (R^2 - r^2) = \frac{k_e Q_{\text{tot}}}{R} + \frac{k_e Q_{\text{tot}}}{2R^3} (R^2 - r^2) = \frac{k_e Q_{\text{tot}}}{2R} \left(3 - \frac{r^2}{R^2} \right) \quad (25)$$

Now we are ready! Once again, we integrate over our spherical volume by taking spherical shells of radius r and width dr , giving a volume element $dV_{\text{el}} = 4\pi r^2 dr$. Noting that ρ is zero outside the sphere,

$$U = \frac{1}{2} \int_{\text{everywhere}} \rho V dV_{\text{ol}} = U = \frac{1}{2} \int_0^R \rho V(r) 4\pi r^2 dr \quad (26)$$

$$= \frac{1}{2} \int_0^R \frac{k_e Q_{\text{tot}} \rho}{2R} \left(3 - \frac{r^2}{R^2}\right) 4\pi r^2 dr = \frac{\pi k_e \rho Q_{\text{tot}}}{R} \int_0^R 3r^2 - \frac{r^4}{R^2} dr \quad (27)$$

$$= \frac{\pi k_e Q_{\text{tot}} \rho}{R} \left[r^3 - \frac{r^5}{5R^2} \right]_0^R = \frac{\pi k_e \rho Q_{\text{tot}}}{R} \left[R^3 - \frac{R^3}{5} \right] = \frac{4\pi k_e Q_{\text{tot}} \rho}{5} R^2 = \frac{3}{5} \frac{k Q_{\text{tot}}^2}{R} \quad (28)$$

One problem, three methods, and the same answer every time. Just how it ought to be. Which method should you use? It is a matter of taste, and the particular problem at hand. I tried to present the methods in order of what I thought was increasing difficulty, your opinion may differ.

2. At the beginning of the 20th century the idea that the rest mass of the electron might have a purely electrical origin was very attractive, especially when the equivalence of energy and mass was revealed by special relativity. Imagine the electron as a ball of charge, of constant volume density ρ out to some maximum radius r_o . Using the result of the previous problem, set the potential energy of the system equal to mc^2 and see what you get for r_o . One defect of this model is rather obvious: nothing is provided to hold the charge together!

Solution: Using the result of the previous problem, and using $a=r_o$,

$$U = \frac{3}{5} \frac{k Q_{\text{tot}}^2}{R} = mc^2 \quad (29)$$

$$r_o = \frac{3k Q^2}{5mc^2} \approx 1.7 \times 10^{-15} \text{ m} \quad (30)$$

Here we used $Q = \frac{4}{3}\pi r_o^3 \rho$ to make things simpler. In terms of the charge density ρ , we have

$$r_o = \sqrt[5]{\frac{15mc^2}{16\pi^2 k \rho^2}} \quad (31)$$

3. A spherical conductor A contains two spherical cavities. The total charge on the conductor itself is zero. However, there is a point charge q_b at the center of one cavity and q_c at the center of the other. A considerable distance r away, outside the conductor, is a point charge q_d . What force acts on each of the four objects, A, q_b , q_c , and q_d ? Which answers, if any, are only approximate, and depend on r being relatively large?

Solution: The force on the point charges q_b and q_c are zero. The field inside the spherical cavity is independent of anything outside. Since we require the field to be zero inside a conductor, the field

just outside the cavities but inside the conductor must be zero. For this to be true, charge equal and opposite to the point charges must be induced on the inside surface of the cavities, $-q_b$ on cavity B, and $-q_c$ on cavity C. In order for the spherical conductor itself to be overall electrically neutral, its surface must then have a charge $q_b + q_c$ spread out uniformly on it to cancel out the induced charge on the cavity surfaces.

If there were no charge q_d , the field outside would look like that of a point charge of magnitude $q_b + q_c$, the total charge enclosed by the conductor: $E = |q_b + q_c|/r^2$. The presence of q_d will slightly alter the distribution of charge on the conductor's surface (if q_d is positive, it would push some of the previously uniform positive surface charge to the far side), but not the total amount of charge. Since the distribution would no longer be spherically symmetric, it would no longer be exactly like a point charge. If q_d is far enough from the conductor, the effect is small, and the force on q_d would be approximately

$$\vec{\mathbf{F}}_d \approx \frac{k_e q_d (q_b + q_c)}{r^2} \hat{\mathbf{r}} \quad (32)$$

Recall that by Newton's third law that the force on the cavity must be equal and opposite, $\vec{\mathbf{F}}_A = -\vec{\mathbf{F}}_d$. Only these two forces are approximate and depend on r being relatively large; the net force on q_b and q_c is zero regardless.

4. We want to design a spherical vacuum capacitor with a given radius a for the outer sphere, which will be able to store the greatest amount of electrical energy subject to the constraint that the electric field strength on the surface of the inner sphere may not exceed E_0 . What radius b should be chosen for the inner spherical conductor, and how much energy can be stored?

Solution: First we need to know the capacitance of the two sphere system. Let r be the distance measured from their common centers. For that, we need only find the potential between the two spheres, $b < r < a$. Inside the outer sphere of radius a , we know that the field due to outer sphere is zero by Gauss' law. That means that the *potential* due to the outer sphere is constant for $r \leq a$, since $\vec{\mathbf{E}} = -\vec{\nabla}V$. Thus, the potential *difference* between the two spheres depends only on the field due to the inner sphere of radius b . This we can find easily enough by integrating $\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ over a radial path from b to a , since the field of the inner sphere will be that of a point charge for $r > b$. Let the outer sphere have a charge $-Q$, meaning the inner sphere has a charge Q .

$$\Delta V = - \int_b^a \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}} = - \int_b^a \frac{k_e Q}{r^2} dr = k_e Q \left(\frac{1}{b} - \frac{1}{a} \right) = k_e Q \frac{a - b}{ab} \quad (33)$$

The capacitance is charge divided by potential difference:

$$C = \frac{Q}{\Delta V} = \frac{1}{k_e} \frac{ab}{a-b} = 4\pi\epsilon_0 \frac{ab}{a-b} \quad (34)$$

The stored energy can be found in terms of the capacitance and voltage:

$$U = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2k_e} \frac{ab}{a-b} k_e^2 Q^2 \left(\frac{a-b}{ab} \right)^2 = \frac{1}{2} k_e Q^2 \left(\frac{a-b}{ab} \right) \quad (35)$$

Now we know that at the inner surface $r=b$ the field can be at most E_o , and we know that the field is just that of a point charge of magnitude Q . This lets us recast the energy in terms of the maximum allowed field, since one is proportional to the other:

$$E_o = \frac{k_e Q}{b^2} \implies Q = \frac{b^2 E_o}{k_e} \quad (36)$$

$$U = k_e \frac{b^4 E_o^2}{k_e^2} \left(\frac{a-b}{ab} \right) = \frac{E_o^2}{2k_e} \left(b^3 - \frac{b^4}{a} \right) \quad (37)$$

Maximizing the energy means $\partial U / \partial b = 0$ (and $\partial^2 U / \partial b^2 < 0$), which gives us a relationship between b and a :

$$\frac{\partial U}{\partial b} = \frac{E_o^2}{2k_e} \left(3b^2 - 4 \frac{b^3}{a} \right) = 0 \implies b = \frac{3}{4} a \quad (38)$$

Of course, this could be a minimum; we must apply the second derivative test (or at least graph the function):

$$\left. \frac{\partial^2 U}{\partial b^2} \right|_{b=\frac{3}{4}a} = 6b - 12 \frac{b^2}{a} \Big|_{b=\frac{3}{4}a} = \left(\frac{9}{2} - \frac{27}{4} \right) a < 0 \quad (39)$$

Indeed, $b = \frac{3}{4} a$ is a maximum, giving the maximum stored energy if the field strength on the inner conductor is the constraint. The actual energy stored is then found by plugging this value back into our expression for U :

$$U_{\max} = \frac{27}{512k_e} E_o^2 a^3 \quad (40)$$

5. We have two point charges connected by a rigid rod, forming a dipole. It is placed in an external electric field $\vec{E}(\vec{r})$.

(a) Suppose that the electric field is uniform: $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0$ where $\vec{\mathbf{E}}_0$ is a constant vector. What will be the total force on the dipole?

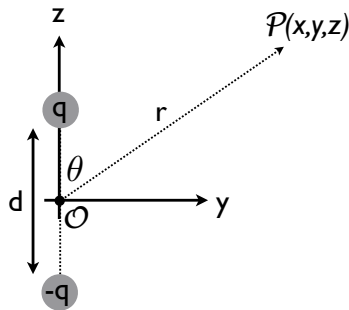
(b) Now suppose the field is not uniform, but that it only changes by a small amount over the distance d . Show that the z -component of the total force on the dipole is approximately

$$F_z = p_z \frac{\partial E_z}{\partial z} \quad (41)$$

where p_z is the z -component of the dipole moment $\vec{\mathbf{p}}$.ⁱⁱ

(c) Why is a charged rubber rod able to attract bits of paper without touching them?

Solution: Reference the figure below. We will imagine that there is a rigid rod holding the two charges together.



(a) The total force on the rigid dipole is the sum of the forces on each charge:

$$\vec{\mathbf{F}}_{\text{net}} = q\vec{\mathbf{E}}_0 - q\vec{\mathbf{E}}_0 = 0 \quad (42)$$

(b) For a non-uniform field, the force is in general not zero, but depends on the difference between the field at the position of positive charge and the negative charge. If we worry only about a single component of the total force, say the z component, the force depends only on that component of the field, and we have

$$F_z = q\vec{\mathbf{E}}_z\left(\frac{d}{2}\hat{\mathbf{z}}\right) - q\vec{\mathbf{E}}_z\left(-\frac{d}{2}\hat{\mathbf{z}}\right) = q\left[\vec{\mathbf{E}}_z\left(\frac{d}{2}\hat{\mathbf{z}}\right) - \vec{\mathbf{E}}_z\left(-\frac{d}{2}\hat{\mathbf{z}}\right)\right] \quad (43)$$

ⁱⁱFor an arbitrary dipole orientation, this generalizes to $\vec{\mathbf{F}} = (\vec{\mathbf{p}} \cdot \vec{\nabla}) \vec{\mathbf{E}}$.

In the limit of small d and slowly-varying $\vec{E}(\mathbf{r})$, we can approximate the difference term as

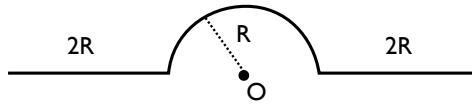
$$\vec{E}_z\left(\frac{d}{2}\hat{z}\right) - \vec{E}_z\left(-\frac{d}{2}\hat{z}\right) \approx d\frac{\partial E_z}{\partial z} \quad (44)$$

Noting that qd is the dipole moment \mathbf{p} ,

$$F_z \approx qd\frac{\partial E_z}{\partial z} \quad (45)$$

(c) When a charged rubber rod is brought near bits of paper, dipole moments are induced in the bits of paper. The non-uniform electric field emanating from the end of the rubber rod then interacts with the induced dipole moments to attract the bits of paper.

6. A wire having uniform linear charge density λ is bent into the shape shown below. Find the electric potential at O .



Solution: The two straight bits of wire will give the same contribution to the potential at O . Take the straight segment on the right side and break it up into infinitesimal segments of length dx , each of which will have charge $dq = \lambda dx$. The potential from each dq is that of a point charge, we can find the potential at O by integrating over the line segment from R to $3R$ all such contributions:

$$V_{\text{line}} = \int_{\text{line}} \frac{k_e dq}{r} = k_e \int_R^{3R} \frac{\lambda dx}{x} = k \ln 3 \quad (46)$$

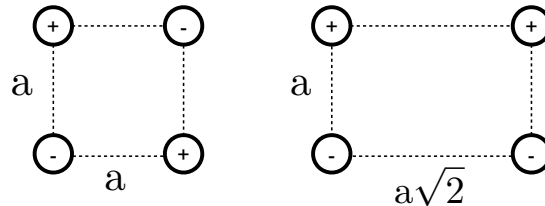
For the semicircle, each infinitesimal bit of arclength $ds = R d\theta$ has charge λds , and also gives a contribution to the potential $k dq/R$. Integrating over the semicircle means running the angle θ from $-\pi/2$ to $\pi/2$:

$$V_{\text{semicircle}} = \int_{\text{semicircle}} \frac{k_e dq}{r} = k_e \int_{-\pi/2}^{\pi/2} \frac{\lambda R d\theta}{R} = k_e \pi \lambda \quad (47)$$

The total potential at O is due to two line segments and one semicircle:

$$V_O = 2V_{\text{line}} + V_{\text{semicircle}} = 2k\lambda \ln 3 + k\lambda\pi = k\lambda(2 \ln 3 + \pi) \quad (48)$$

7. The two figures below show small sections of two different possible surfaces of a NaCl surface. In the left arrangement, the NaCl(100) surface, charges of $+e$ and $-e$ are arranged on a square lattice as shown. In the right arrangement, the NaCl(110) surface, the same charges are arranged in a rectangular lattice. What is the potential energy of each arrangement (symbolic answer)? Which is more stable?



Solution: We need only add up the potential energies of all possible pairs of charges. In each case we have four charges, so there must be $\binom{4}{2} = 6$ combinations. Let the upper left charge be q_1 , and number the charges in a clockwise fashion. The combinations are thus

$$q_1 q_2, q_1 q_3, q_1 q_4 \quad (49)$$

$$q_2 q_3, q_2 q_4 \quad (50)$$

$$q_3 q_4 \quad (51)$$

For either arrangement, the energy is then

$$U = \frac{k_e q_1 q_2}{r_{12}} + \frac{k_e q_1 q_3}{r_{13}} + \frac{k_e q_1 q_4}{r_{14}} + \frac{k_e q_2 q_3}{r_{23}} + \frac{k_e q_2 q_4}{r_{24}} + \frac{k_e q_3 q_4}{r_{34}} \quad (52)$$

For the first arrangement, NaCl(100), we need only plug in the distances and charges:

$$U_{100} = \frac{-k_e e^2}{a} + \frac{k_e e^2}{a\sqrt{2}} + \frac{-k_e e^2}{a} + \frac{-k_e e^2}{a} + \frac{k_e e^2}{a\sqrt{2}} + \frac{-k_e e^2}{a} = \frac{k_e e^2}{a} \left(-4 + \sqrt{2} \right) \approx -2.58 \frac{k_e e^2}{a} \quad (53)$$

For the second arrangement, NaCl(110), we have:

$$U_{110} = \frac{k_e e^2}{a\sqrt{2}} + \frac{-k_e e^2}{a\sqrt{3}} + \frac{-k_e e^2}{a} + \frac{-k_e e^2}{a} + \frac{-k_e e^2}{a\sqrt{3}} + \frac{k_e e^2}{a\sqrt{2}} = \frac{k_e e^2}{a} \left(-2 + \sqrt{2} - \frac{2}{\sqrt{3}} \right) \approx -1.74 \frac{k_e e^2}{a} \quad (54)$$

Since $U_{100} < U_{110}$, the (100) surface is more stable, in agreement with experiments.

8. A charge Q is located h meters above a conducting plane. How much work is required to bring this charge out to an infinite distance above the plane? *Hint: Consider the method of images.*

Solution: There are two ways to approach this one. First, the presence of the conducting plane a distance h from the positive charge means that this problem is equivalent to a dipole of spacing $2h$ (see figure below).ⁱⁱⁱ Thus, we need to find the work required to move a charge q from a distance $2h$ from a second charge $-q$ out to infinity. Let the origin be halfway between the real charge q and its image charge $-q$. The work required to move the positive charge away is:

$$W = - \int \vec{F} \cdot d\vec{l} = - \int_h^\infty q\vec{E} \cdot \hat{r} dr = - \int_h^\infty \frac{kq^2}{(2r)^2} dr = \frac{kq^2}{4} \left[\frac{1}{z} \right]_h^\infty = \frac{-kq^2}{4h} \quad (55)$$

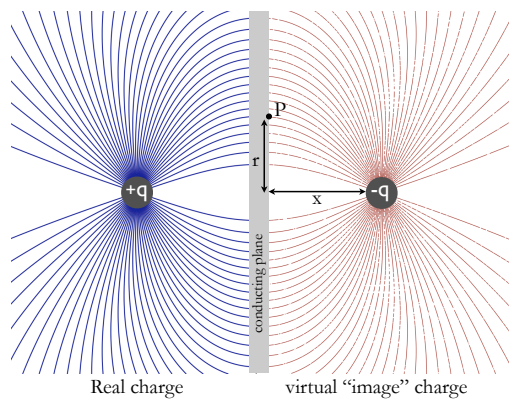


Figure 1: The field of a charge near a conducting plane, found by the method of images.

A much sneakier way is to realize that the energy in the electric field must just be half of that due to a real dipole. We learned that the energy of a charge configuration can be found by integrating the electric field over all space, $U \sim \int E^2 dV$. The electric field due to our point charge above the conducting plane is identical to that of a dipole, but *only for the region of space above the plane*. Below the plane, half of all volume in space, the field is zero. We can immediately conclude that the point charge and infinite plane have half as much energy, since there is no field below the conducting plane.

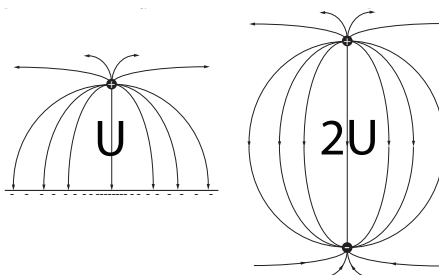


Figure 2: The field energy of our single charge with a conducting plate is half that of a dipole.

ⁱⁱⁱSee http://faculty.mint.ua.edu/~pleclair/ph106/Exercises/EX3_SOLN.pdf, problem 4.

The energy of a dipole we found already when we considered point charges. If the dipole spacing is $2h$,

$$U_{\text{dip}} = \frac{kq^2}{2h} \quad \Rightarrow \quad U = \frac{1}{2}U_{\text{dip}} = \frac{kq^2}{4h} \quad (56)$$