University of Alabama
Department of Physics and Astronomy

PH 126 LeClair Fall 2011

Problem Set 4

1. A battery has an ideal voltage ΔV and an internal resistance r. A variable load resistance R is connected to the battery. Determine the value of R such that the power delivered to the resistor is maximum.

Solution: The circuit we are considering is just a series combination of the (ideal) internal voltage source ΔV, the internal resistance R_i, and the external resistance R. Since all the elements are in series, the current is the same in each, which we will call I. Applying conservation of energy,

$$\Delta V - IR - IR_i = 0 \qquad \Longrightarrow \qquad I = \frac{\Delta V}{R + R_i}$$

The power delivered *to the external resistor* $\mathscr{P}_R$ is just I^2R:

$$\mathscr{P}_R = I^2R = \left(\frac{\Delta V}{R + R_i}\right) R = (\Delta V)^2 \frac{R}{(R + R_i)^2}$$

We can maximize the power delivered to the resistor R by differentiating the power with respect to R and setting the result equal to zero:

$$\begin{aligned}
\frac{d\mathscr{P}_R}{dR} &= \frac{d}{dR}\left[(\Delta V)^2 \frac{R}{(R + R_i)^2}\right] = (\Delta V)^2 \left[\frac{1}{(R + R_i)^2} + \frac{-2R}{(R + R_i)^3}\right] = 0 \\
\Longrightarrow \qquad \frac{1}{(R + R_i)^2} &= \frac{2R}{(R + R_i)^3} \\
1 &= \frac{2R}{R + R_i} \\
R + R_i &= 2R \\
\Longrightarrow \qquad R_i &= R
\end{aligned}$$

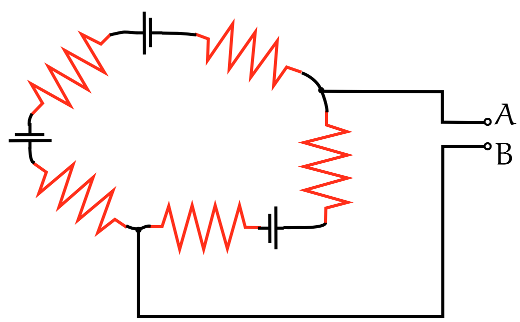
The power is indeed extremal when the external resistor matches the internal resistance of the battery. We should apply the second derivative test to see whether this is a maximum or a minimum. Recall briefly that after finding the extreme point of a function $f(x)$ via $df/dx|_{x=a} = 0$, one should calculate $d^2f/dx^2|_{x=a}$: if $d^2f/dx^2|_{x=a} < 0$, you have a maximum, if $d^2f/dx^2|_{x=a} > 0$ you have a minimum, and if $d^2f/dx^2|_{x=a} = 0$, the test basically wasted your time. Anyway, let's find the second derivative, and simplify it as much as possible.

$$
\begin{aligned}
\frac{d^2\mathscr{P}_R}{dR^2} &= \frac{d}{dR}\left[(\Delta V)^2\left(\frac{1}{(R+R_i)^2} - \frac{2R}{(R+R_i)^3}\right)\right] \\
&= (\Delta V)^2\left[\frac{-2}{(R+R_i)^3} - \frac{2}{(R+R_i)^3} + \frac{6R}{(R+R_i)^4}\right] \\
&= (\Delta V)^2\left[\frac{-4}{(R+R_i)^3} + \frac{6R}{(R+R_i)^4}\right] \\
&= \frac{(\Delta V)^2}{(R+R_i)^3}\left[\frac{6R}{R+R_i} - 4\right]
\end{aligned}
$$

We are concerned with the value of the second derivative at the point $R=R_i$, the extreme point:

$$
\left.\frac{d^2\mathscr{P}_R}{dR^2}\right|_{R=R_i} = \frac{(\Delta V)^2}{(R_i+R_i)^3}\left[\frac{6R_i}{R_i+R_i} - 4\right] = \frac{(\Delta V)^2}{8R_i^3}\left[3-4\right] = -\frac{(\Delta V)^2}{8R_i^3} < 0
$$

The second derivative is always negative, so we have found a maximum. Thus, the power delivered to an external resistor is maximum when $R=R_i$.



2. In the circuit above, all five resistors have the same value, $100\,\Omega$, and each battery has a rated voltage of $1.5\,\mathrm{V}$ and no internal resistance. Find the open-circuit voltage and the short-circuit current for the terminals A, B. Then find the Thèvenin equivalent circuit (i.e., the ideal battery and resistor that could replace this circuit between terminals A, B.)

Solution: Let the batteries have voltage V and the resistors have resistance R, and presume a current I exists in both branches (when A and B are open, the current must be the same everywhere since there is just one loop). We can get the current by walking all the way around the loop, which must give us a net zero change in potential:

$$0 = 2V - 3IR - 2IR + V \tag{1}$$

$$3V = 5IR \tag{2}$$

$$I = \frac{3V}{5R} \tag{3}$$

The open-circuit voltage is now readily found by "walking" from A to B and summing the changes in potential. We can do this via either the upper or lower branch, and the two results must be equal. We'll do both just to be sure: first walk the lower branch, then the upper, taking care to keep the signs straight.

$$\Delta V_{AB} = -3IR + 2V = 2IR - V \tag{4}$$
$$= -\frac{9}{5}V + 2V = \frac{6}{5}V - V = \frac{1}{5}V \tag{5}$$

Once we short A and B, we will have different currents in the two branches, and the current from A to B will be their difference. To make that point more clear, connect the two sides of the loop that go to points A and B directly across the center of the loop. Then there must be a counterclockwise current in one branch of the loop and a clockwise current in the other, and through the center short-cut from A to B, the two currents must meet head on. To get the currents, one can walk around each of the two smaller loops we've now created by shorting across the center. In the loop with three resistors and two batteries, we have

$$0 = 3I_1R - 2V \qquad \Longrightarrow I_1 = \frac{2V}{3R} \tag{6}$$

in the other loop, we have

$$0 = 2I_2R - V \qquad \Longrightarrow I_2 = \frac{V}{2R} \tag{7}$$

The short-circuit current in the branch shorting A to B is the difference between these two currents,

$$I_s = I_1 - I_2 = \frac{V}{6R} \tag{8}$$

The Thévenin equivalent resistance is then the open-circuit voltage divided by the short-circuit current:

$$R_{th} = \frac{\Delta V_{AB}}{I_s} = \frac{6}{5}R \tag{9}$$

Thus, the Thévenin equivalent of this circuit is a single battery of value $V/5$ in series with a resistor of value $6R/5$. The Norton equivalent circuit would be a current source of value $V/6R$ in parallel with a resistor of value $6R/5$. Numerically,

$$V_{th} = \Delta V_{AB} = 0.3\,\text{V} \tag{10}$$
$$R_{th} = 120\,\Omega \tag{11}$$
$$I_N = I_s = 2.5\,\text{mA} \tag{12}$$

3. Two resistors are connected in parallel, with values R_1 and R_2. A total current I_o divides somehow between them. Show that the condition $I_1 + I_2 = I_o$, together with the requirement of minimum power dissipation, leads to the same current values that we would calculate with normal circuit formulas. This illustrates a general variational principle that holds for direct current networks: the distribution of currents within the networks, for a given input current I_o, is always that which gives the least total power dissipation.

Solution: First, let's figure out the current in each resistance using the normal circuit formulas. Since the two resistors are in parallel, they will have the same potential difference across them, but in general different currents (unless $R_1 = R_2$, in which case the currents are the same). Let I_1 and I_2 be the currents in resistors R_1 and R_2, respectively, with the total current then given by conservation of charge, $I_o = I_1 + I_2$. Given the current through each resistor, we can readily calculate the voltage drop on each, which again must be the same for both resistors:

$$\begin{aligned}
\Delta V_1 &= I_1 R_1 \\
\Delta V_2 &= I_2 R_2 \\
\Delta V_1 &= \Delta V_2 \qquad \Longrightarrow \qquad I_1 R_1 = I_2 R_2
\end{aligned}$$

We can find the current in each resistor from the known total current I_o by noting that $I_o = I_1 + I_2$, and thus $I_2 = I_o - I_1$

$$\begin{aligned}
I_1 R_1 &= I_2 R_2 \\
I_1 R_1 &= (I_o - I_1)\, R_2 \\
I_1 R_1 + I_1 R_2 &= I_o R_2 \\
\Longrightarrow \qquad I_1 &= \frac{R_2}{R_1 + R_2} I_o = \left[\frac{1}{1 + \frac{R_1}{R_2}}\right] I_o
\end{aligned}$$

Thus, the fraction of the total current in first resistor depends on the ratio of the two resistors. The larger resistor 2 is, the more current that will flow through the first resistor – not shocking! Given the above expression for I_1, we can easily find I_2 from $I_2 = I_o - I_1$, which yields

$$I_2 = \frac{R_1}{R_1 + R_2} I_o = \left[\frac{1}{1 + \frac{R_2}{R_1}}\right] I_o$$

Our derivation of the currents in each resistor has so far only relied on conservation of energy (components in parallel have the same voltage) and conservation of charge ($I_o = I_1 + I_2$), we have not invoked any special "laws" about combining parallel resistors. In fact, that is what we have just derived!

Now for the requirement of minimum power dissipation. We want to find the distribution of currents that results in minimum power dissipation in the most general way, specifically *not* using the results of the previous portion of this problem. We will only assume that resistors R_1 and R_2 carry currents I_1 and I_2, respectively, and that these two currents add up to the total current, $I_o = I_1 + I_2$. In other words, we only assume conservation of charge to start with.

The total power dissipated is just the sum of the individual power dissipations in the two resistors:

$$\mathscr{P}_{\text{tot}} = \mathscr{P}_1 + \mathscr{P}_2 = I_1^2 R_1 + I_2^2 R_2 = I_1^2 R_1 + (I_o - I_1)^2 R_2 = I_1^2 (R_1 + R_2) + I_o^2 R_2 - 2IR_2 I_1$$

For the last part, we invoked our conservation of charge condition ($I_o = I_1 + I_2$). What to do next? We have now the total power $\mathscr{P}_{\text{tot}}$ in both resistors as a function of the current in R_1. If we minimize the total power with respect to I_1, we will have found the value of I_1 which leads to the minimum power dissipation. Since I_2 is then fixed by the total current I once we know I_1, $I_2 = I_o - I_1$, this is sufficient to establish the values of *both* I_1 and I_2 that lead to minimum power dissipation. Of course, to find the minimum of $\mathscr{P}_{\text{tot}}$ for any value of I_1, we need to take a derivative[i] ...

$$\frac{d\mathscr{P}_{\text{tot}}}{dI_1} = 2I_1 (R_1 + R_2) - 2I_o R_2 = 0$$
$$\Longrightarrow I_1 = \frac{R_2}{R_1 + R_2} I_o$$

Lo and behold, the minimum power dissipation occurs when the currents are distributed exactly as we expect for parallel resistors. At this point, you can easily find I_2 as well, given $I_2 = I_o - I_1$. The general rule is that current in a dc circuit distributes itself such that the total power dissipation is minimum, which we will not prove here.

Of course ... by finding $\frac{d\mathscr{P}_{\text{tot}}}{dI_1}$ and setting it to zero, we have certainly found an extreme value for $\mathscr{P}_{\text{tot}}$. We did not prove whether it is a maximum or a minimum. Once again, the second derivative test is necessary.

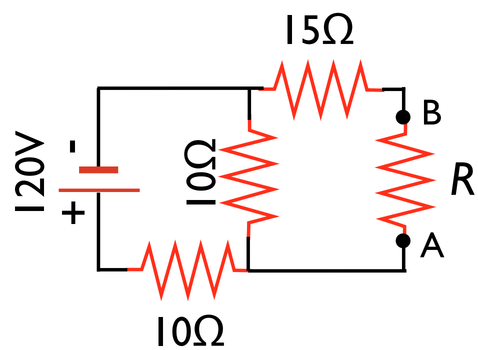
$$\frac{d^2\mathscr{P}_{\text{tot}}}{dI_1^2} = 2(R_1 + R_2) > 0$$

Since resistances are always positive, we have in fact found a minimum of $\mathscr{P}_{\text{tot}}$. Crisis averted.

[i] Keep in mind that the total current I_o is fixed, so $dI_o/dI_1 = 0$. And, yes we should technically be using partial derivatives here (differentiating with respect to I_1 while holding everything else constant), but since only I_1 varies that would be a bit pedantic. Plus, the ∂ symbols seem to scare people.

Don't let that lull you into complacency, however: you always need to apply the second derivative test to see what you've really found. At the very least, you should invoke the symmetry of the function to justify having found a minimum or maximum, and not just take derivatives and set them to zero all willy-nilly.

4. A resistor R is to be connected across the terminals A, B of the circuit below. **(a)** For what value of R will the power dissipated in the resistor be the greatest? To answer this, construct the Thévenin equivalent circuit and then invoke the result of the first problem. **(b)** How much power will be dissipated in R?



Solution: For the moment, imagine that R has been removed. We can find the Thévenin equivalent of the rest of the circuit, just a single battery and resistor. Plugging the resistor R into the Thévenin equivalent would just make two resistors in series with a battery, simple. Further, based on the result of the first problem, we know the power will be maximum when R is equal to the Thévenin equivalent resistance between points A and B. Thus, once we've found the Thévenin equivalent, we are basically done.

When R is removed and A and B are left unconnected, the entire circuit is just a 120 V battery in series with two 10 Ω resistors. The 15 Ω resistor does nothing, since one end is unconnected. This circuit is just a voltage divider, the 120 V of the battery splits up evenly between the two 10 Ω resistors, so the voltage between A and B is simply $V_{th} = 60\,\text{V}$.

When R is (still) removed and A and B are shorted, we have the vertical 10 Ω resistor in parallel with 15 Ω, and that combination in series with 10 Ω. The equivalent resistance is then

$$R_{eq} = 10\,\Omega + \frac{(10\,\Omega)\,(15\,\Omega)}{10\,\Omega + 15\,\Omega} = 16\,\Omega \tag{13}$$

This means the total current is the battery voltage divided by this resistance

$$I = \frac{120\,\text{V}}{16\,\Omega} = 7.5\,\text{A} \tag{14}$$

The current through the (shorted) A to B path is the same as the current in the $15\,\Omega$ resistor. To find that, we can first note that the voltage across the $15\,\Omega$ resistor would be 120 V minus the drop across the horizontal $10\,\Omega$ resistor:

$$\Delta V_{15} = 120 - (10\,\Omega)\,(7.5\,\text{A}) = 45\,\text{V} \tag{15}$$

The current through the $15\,\Omega$ resistor, the short circuit current, is then just

$$I_s = \frac{45\,\text{V}}{15\,\Omega} = 3\,\text{A} \tag{16}$$

Finally, this gives us the Thévenin equivalent resistance:

$$R_{th} = \frac{V_{th}}{I_s} = 20\,\Omega \tag{17}$$

The power delivered will thus be maximal when $R = 20\,\Omega$. To find the power dissipated in R, we need the current *in the circuit with* R *connected.* In that case, our equivalent circuit is a 60 V battery in series with $R = 20\,\Omega$ and the $20\,\Omega$ Thévenin equivalent, or 60 V in series with $40\,\Omega$. The current would then be $I = 60\,\text{V}/40\,\Omega = 1.5\,\text{A}$. We could have noted that directly be realizing that connecting R doubles the resistance compared to the short-circuit case, which means half the short-circuit current results. The power in R is then

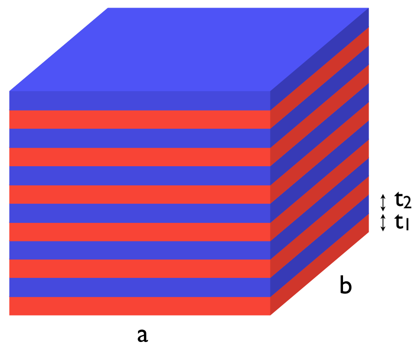
$$P_R = I^2R = (1.5\,\text{A})^2\,(20\,\Omega) = 45\,\text{W} \tag{18}$$

The total power dissipation would be twice as much, since the $20\,\Omega$ Thévenin equivalent has the same current and thus dissipates the same power.

5. A laminated conductor was made by depositing, alternately, layers of silver 10 nm thick and layers of tin 20 nm thick. The composite material, considered on a larger scale, may be considered a homogeneous but anisotropic material with electrical conductivity $\sigma_\perp$ for currents perpendicular to the planes of the layers, and a different conductivity $\sigma_{||}$ for currents parallel to that plane. Given that the conductivity of silver is 7.2 times that of tin, find the ratio $\sigma_\perp/\sigma_{||}$.

Solution: First, let us sketch out the situation given:
Now, let's solve the problem in general way, and only use the given numbers once we've found a symbolic solution.

We are not told how many layers of each type we have, and it will not matter in the end. For now, however, assume we have n_1 layers of tin of conductivity σ_1 and n_2 layers of silver of conductivity σ_2. Instead of conductivity, we can equivalently use resistivity ρ when it is more convenient, with



$\rho = 1/\sigma$. We will also say the tin layers have thickness t_1, and the silver layers thickness t_2. The total thickness of our entire multi-layer stack is then $t_{tot} = n_1 t_1 + n_2 t_2$.

First, consider the perpendicular conductivity, the case where we pass current upward through the stack, perpendicular to the planes of the layers. When a current is flowing, electrons pass through each layer in sequence, and we can consider the stack of layers to be resistors in series. If the layers have an area of $A = ab$ (see the Figure above) and a thickness t_1 or t_2, we can readily calculate the resistance presented by a single tin or silver layer with current perpendicular to the layers:

$$\begin{aligned} R_{1,\perp} &= \frac{\rho_1 t_1}{A} = \frac{t_1}{\sigma_1 A} \\ R_{2,\perp} &= \frac{\rho_2 t_2}{A} = \frac{t_2}{\sigma_2 A} \end{aligned}$$

For reasons that should become apparent below, it will be convenient in this case to work with the resistivity rather than the conductivity, and invert the result later. The total resistance of the stack is then just a series combination of n_1 resistors of value R_1 and n_2 resistors of value R_2:

$$R_{tot,\perp} = n_1 R_{1,\perp} + n_2 R_{2,\perp} = \frac{1}{A} \left(\rho_1 t_1 n_1 + \rho_2 t_2 n_2 \right)$$

If we measure the whole stack and find this resistance, we can define an effective resistivity or conductivity for the whole stack in terms of the total resistance and total thickness of the multilayer. If the resistivity of the whole stack for perpendicular currents is $\rho_\perp = 1/\sigma_\perp$, then:

$$R_{tot,\perp} = \frac{\rho_\perp t_{tot}}{A} \qquad \Longrightarrow \qquad \rho_\perp = \frac{A R_{tot,\perp}}{t_{tot}}$$

Now we just need to plug in what we know and simplify . . .

$$\rho_\perp = \frac{AR_{\text{tot},\perp}}{t_{\text{tot}}} = \frac{A}{t_{\text{tot}}}\left[\frac{1}{A}\left(\rho_1 t_1 n_1 + \rho_2 t_2 n_2\right)\right]$$
$$\rho_\perp = \frac{\rho_1 t_1 n_1 + \rho_2 t_2 n_2}{t_{\text{tot}}} = \frac{\rho_1 t_1 n_1 + \rho_2 t_2 n_2}{n_1 t_1 + n_2 t_2}$$

We can simplify this somewhat if we realize that we have the *same* number of silver and tin layers - we are told that the layers are deposited alternatingly. If we let $n_1 = n_2 \equiv n_{bi}$, meaning we count the number of bilayers instead, then $t_{\text{tot}} = n_{bi}\left(t_1 + t_2\right)$, and

$$\rho_\perp = \frac{n_{bi}\rho_1 t_1 + n_{bi}\rho_2 t_2}{n_{bi} t_1 + n_{bi} t_2} = \frac{\rho_1 t_1 + \rho_2 t_2}{t_1 + t_2}$$

This is a nice, simple result: for current perpendicular to the planes, the effective *resistivity* is just a thickness-weighted average of the resistivities of the individual layers. Given the resistivity in the perpendicular case, we can now find the conductivity $\sigma_\perp$

$$\sigma_\perp = \frac{1}{\rho_\perp} = \frac{t_1 + t_2}{\rho_1 t_1 + \rho_2 t_2} = \frac{t_1 + t_2}{\frac{t_1}{\sigma_1} + \frac{t_2}{\sigma_2}} = \frac{\sigma_1 \sigma_2 \left(t_1 + t_2\right)}{\sigma_2 t_1 + \sigma_1 t_2}$$

As a consistency check, we can take a couple of limiting cases. First, let $\sigma_1 = \sigma_2 \equiv \sigma$. This corresponds to a homogeneous lump of a single material, and we find $\sigma_\perp = \sigma$, as expected. Next, we can check for $\sigma_1 = 0$. In this case, one layer is not conducting at all, and since the layers are in series, this means no current flows through the stack at all, and $\sigma_\perp = 0$ as expected. Finally, we notice that the number of bilayers is irrelevant. Since the layers do not affect each other in our simple model of conduction, there is no reason to expect otherwise. So far so good. What other limiting cases can you check?

Next, let us consider current flowing parallel to the plane of the layers, from (for example) left to right in the figure above. Now the stack looks like many parallel resistors. A single tin layer of thickness t_1 and in-plane dimensions a and b now presents a resistance

$$R_{1,||} = \frac{\rho_1 a}{t_1 b} = \frac{a}{t_1 b \sigma_1}$$

Similarly, each silver layer presents a resistance

$$R_{2,||} = \frac{\rho_2 a}{t_2 b} = \frac{a}{t_2 b \sigma_2}$$

One bilayer of silver and tin means a *parallel* combination of these two resistances:

$$\frac{1}{R_{bi,||}} = \frac{1}{R_{1,||}} + \frac{1}{R_{2,||}} = \frac{b}{a}\left(t_1\sigma_1 + t_2\sigma_2\right)$$

If we have n_{bi} bilayers, then the total equivalent resistance is easily found:

$$\frac{1}{R_{tot,||}} = n_{bi}\frac{1}{R_{bi,||}} = n_{bi}\frac{b}{a}\left(t_1\sigma_1 + t_2\sigma_2\right)$$

Given the total resistance, we can now calculate the conductivity directly (in this case, first finding the resistivity does not save us any algebra), noting that the length of the whole stack along the direction of the current is just a, and the cross-sectional area is $bt_{tot} = bn_{bi}\left(t_1 + t_2\right)$:

$$\sigma_{||} = \frac{1}{\rho_{||}} = \frac{a}{n_{bi}\left(t_1 + t_2\right) bR_{tot,||}} = \frac{a}{n_{bi}\left(t_1 + t_2\right) b}\left(\frac{b}{a}\right) n_{bi}\left(t_1\sigma_1 + t_2\sigma_2\right) = \frac{t_1\sigma_1 + t_2\sigma_2}{t_1 + t_2}$$

Again, a sensible result: the effective *conductivity* for current parallel to the planes is just a thickness-weighted average of the conductivities of the individual layers. Again, you can convince yourself with a couple of limiting cases that this result makes some sense.

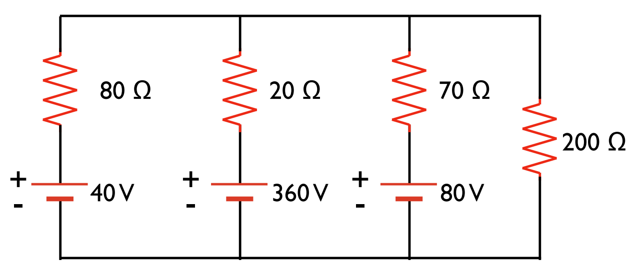
Now that we have both parallel and perpendicular conductivities, we can easily find the anisotropy $\sigma_{\perp}/\sigma_{||}$.

$$\sigma_{\perp}/\sigma_{||} = \frac{\rho_{||}}{\rho_{\perp}} = \frac{\sigma_1\sigma_2\left(t_1 + t_2\right)}{\sigma_2 t_1 + \sigma_1 t_2}\frac{t_1 + t_2}{\sigma_1 t_1 + \sigma_2 t_2} = \frac{\sigma_1\sigma_2\left(t_1 + t_2\right)^2}{\left(t_1\sigma_1 + t_2\sigma_2\right)\left(t_1\sigma_2 + t_2\sigma_1\right)}$$

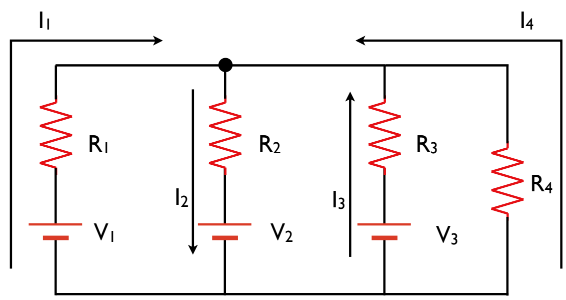
Finally, we are given that the conductivity of silver is 7.2 times that of tin, and the tin layers' thickness is twice that of the silver. Thus, $t_1 = 2t_2$ and $\sigma_2 = 7.2\sigma_1$. The actual values and units do not matter, as this is a dimensionless ratio (you should verify this fact ...), and you should find $\sigma_{\perp}/\sigma_{||} \approx 0.457$.

And, once again, you can check that for $\sigma_1 = \sigma_2$, we have $\sigma_{\perp}/\sigma_{||} = 1$, as it must if both materials are the same.

6. In the circuit below, determine the current in each resistor and the voltage across the $200\,\Omega$ resistor.



Solution: First we must label the currents in each branch and pick their directions. We will pick the directions arbitrarily - in fact, we will intentionally choose directions that are extremely unlikely to be correct just to drive home the point that it doesn't matter at all. If we chose the current direction in a particular branch incorrectly, the current comes out negative, so it really doesn't matter. What does matter is that we solve the problem in a general way, purely symbolically. Get rid of the numbers and reformulate the problem in an abstract way:



Now we just apply conservation of charge (current) and energy (voltage). We have 4 loops which will give us three conservation of energy equations, and 4 unknown currents. That means we need a single node equation based on conservation of current, which we can apply at the point indicated by the dot in the figure above:

$$I_1 + I_2 + I_4 = I_2 \tag{19}$$

Now we'll walk the three smaller loops clockwise, from left to right.

$$V_1 - I_1R_1 - I_2R_2 - V_2 = 0 \tag{20}$$

$$V_2 - I_2R_2 - I_3R_3 - V_2 = 0 \tag{21}$$

$$V_3 - I_3R_3 + I_4R_4 = 0 \tag{22}$$

At this point, we have four equations and four unknowns. We could just plug the numbers into any number of linear algebra programs or online equations solvers, but this lacks a certain elegance. A symbolic solution is required if at all possible, just on general principles. To facilitate this, let's write our four equations in matrix form.

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ R_1 & R_2 & 0 & 0 \\ 0 & R_2 & R_3 & 0 \\ 0 & 0 & R_3 & -R_4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 - V_2 \\ V_3 - V_2 \\ V_3 \end{bmatrix} \tag{23}$$

One alternative is to use Gaussian elimination. It is reasonably efficient, and it is simple (Cramer's rule, for instance, will be messier). The entire method basically consists of adding the equations together in various fashions until you're left with one that has just a single variable in it. Let's start by adding R_4 times the first equation to the last equation:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ R_1 & R_2 & 0 & 0 \\ 0 & R_2 & R_3 & 0 \\ R_4 & -R_4 & R_3 + R_4 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 - V_2 \\ V_3 - V_2 \\ V_3 \end{bmatrix} \tag{24}$$

Now add $(-R_4/R_1)$ times the second equation to the last equation:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ R_1 & R_2 & 0 & 0 \\ 0 & R_2 & R_3 & 0 \\ 0 & -R_4 - R_4 R_2/R_1 & R_3 + R_4 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 - V_2 \\ V_3 - V_2 \\ V_3 + (V_2 - V_1)R_4/R_1 \end{bmatrix} \tag{25}$$

Now add $-(R_4 + R_3)/R_3$ times the third equation to the last equation:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ R_1 & R_2 & 0 & 0 \\ 0 & R_2 & R_3 & 0 \\ 0 & -R_4 - R_4 R_2/R_1 - R_2(R_3 + R_4)/R_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ V_1 - V_2 \\ V_3 - V_2 \\ V_3 + (V_2 - V_1)R_4/R_1 + (V_2 - V_3)(R_3 + R_4)/R_3 \end{bmatrix} \tag{26}$$

The last line now gives

$$\left[-R_4 - R_4 R_2/R_1 - R_2(R_3 + R_4)/R_3\right] I_2 = (V_3 + (V_2 - V_1)R_4/R_1 + (V_2 - V_3)(R_3 + R_4)/R_3 \tag{27}$$

$$I_2 = \frac{(V_1 - V_2)R_4/R_1 + (V_3 - V_2)(R_3 + R_4)/R_3 - V_3}{R_4 + R_4 R_2/R_1 + R_2(R_3 + R_4)/R_3} \approx -5.99\,\text{A} \tag{28}$$

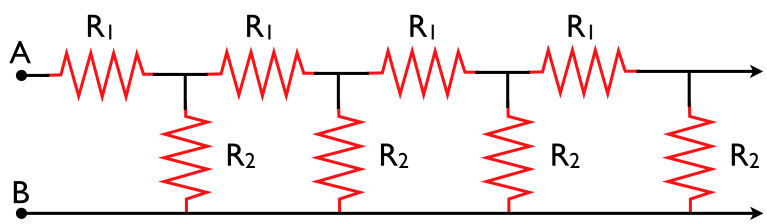
Substitution back into the original equations then readily gives the other currents:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = - \begin{bmatrix} 2.50 \\ 5.99 \\ 2.29 \\ 1.20 \end{bmatrix} \text{A} \tag{29}$$

Thus, the directions are exactly the opposite of our original choice, but this isn't a problem at all. The current through the $200\,\Omega$ resistor is thus $1.20\,\text{A}$ downward, and the voltage across it is

$I_4R_4 \approx 240\,\mathrm{V}$.

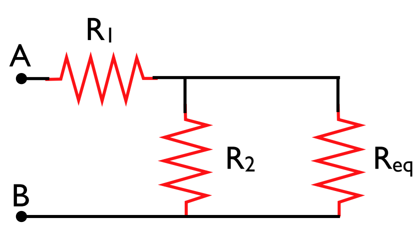
7. Find the input resistance (between terminals A and B) of the following infinite series of resistors.



Show that, if voltage V_o is applied at the input to such a chain, the voltage at successive nodes decreases in a geometric series. What ratio is required for the resistors to make the ladder an attenuator that halves the voltage at every step? Can you suggest a way to terminate the ladder after a few sections without introducing any error in its attenuation? *Hint: If we put another "link" on the left of this infinite chain, we get exactly the same configuration.*

Solution: The infinite ladder of resistors must have a finite overall resistance, since adding more "rungs" in the chain will only reduce the overall resistance. To find out what the equivalent *is*, we can exploit the fact that the network is infinite, so adding or subtracting a few nodes makes no difference.

Let's say we we wished to terminate the chain after some N resistor pairs. We could add a last resistor R_{eq} on the very end which has a resistance equivalent to all those that would come further in the chain. But, how many are left? Take N out of infinity, and you still have infinity left! It doesn't matter if you terminate the chain after N pairs or just 1, the resistance equivalent to the rest of the chain is always R_{eq}. Moreover, since the rest of the chain we're leaving off is infinite just like the original one, *the resistance* R_{eq} *must also be the equivalent resistance of the entire ladder.* In short, taking a finite number of resistors off of an infinite chain makes no difference. If that is the case, why not just terminate it after one pair and be done with it? That means we can replace the whole infinite ladder with this:



where R_{eq} is the equivalent resistance of the remainder of the ladder. Of course, the equivalent resistance of *this* circuit must also be R_{eq}, it is the same infinite ladder. Finding the equivalent of what we've drawn above is easy enough,

$$R_{eq} = R_1 + R_{eq}||R_2 = R_1 + \frac{R_1 R_{eq}}{R_1 + R_{eq}} \tag{30}$$

Solving for R_{eq},

$$R_{eq} = \frac{R_1 \pm \sqrt{R_1^2 + 4R_1R_2}}{2} = \frac{1}{2}R_1\left[1 + \sqrt{1 + 4\left(\frac{R_2}{R_1}\right)}\right] \tag{31}$$

Only the positive resistance has physical meaning. This solves our issue of termination - after as many nodes as we need, just end the ladder by shorting the far end with a resistor of value R_{eq} and it is the same as if the ladder continued on. Also note that if $R_1 = R_2$, we have $R_{eq} = R_1\left(1 + \sqrt{5}\right)/2 = \phi R_1$ where $\phi = \left(1 + \sqrt{5}\right)/2$ is the golden ratio. A useless fact, but sort of cute.

How about the voltage at successive nodes? Say we apply V_o at the input to the ladder. The net current is then $I = V_o/R_{eq}$. After the first resistor R_1, we have a voltage drop of IR_1, so the first node after the input must have a voltage V_1 of

$$V_1 = V_o - IR_1 = V_o - V_o\frac{R_1}{R_{eq}} = V_o\left(1 - \frac{R_1}{R_{eq}}\right) \tag{32}$$

This argument will work for *any* two adjacent nodes. The current at node n is always $I_n = V_n/R_{eq}$, which means node $(n+1)$ has a voltage which is lower than the n^{th} by I_nR_1:

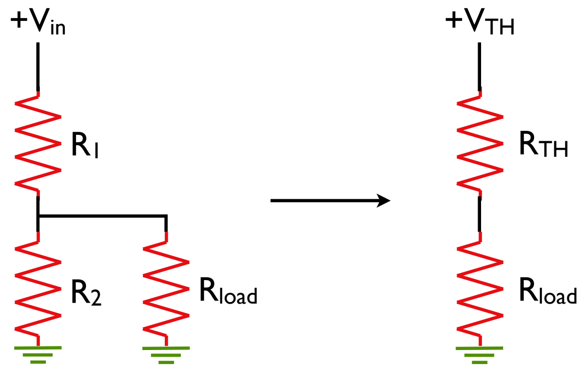
$$V_{n+1} = V_n - I_nR_1 = V_n - V_n\frac{R_{eq}}{R_1} = V_n\left(1 - \frac{R_1}{R_{eq}}\right) \tag{33}$$

The ratio of the voltage at successive nodes is thus the definition of a geometric series:

$$\frac{V_{n+1}}{V_n} = 1 - \frac{R_1}{R_{eq}} = 1 - \frac{R_1}{\frac{1}{2}R_1\left[1 + \sqrt{1 + 4\left(\frac{R_2}{R_1}\right)}\right]} \tag{34}$$

For an attenuation of $\frac{1}{2}$ at every node, it is easy to see that we need $R_{eq} = 2R_1$. Using our formula for R_{eq}, this means $R_1 = \frac{1}{2}R_2$

8. For the circuit shown below, with $V_{in} = 30\,\text{V}$ and $R_1 = R_2 = 10\,\text{k}\Omega$, find **(a)** the output voltage (between the R_1 and R_2) with no load attached; **(b)** the output voltage with a $10\,\text{k}\Omega$ load resistance; **(c)** the Thèvenin equivalent circuit (rightmost circuit in the figure below); **(d)** the power in each resistor with and without a load present.



Solution: With no load attached, the output voltage is just the voltage across R_2. Since we have only two series resistors, the current in both is the same, and it must be $V_{in}/(R_1+R_2)$. The output voltage is then

$$V_o = IR_2 = V_{in}\frac{R_2}{R_1+R_2} = 15\,\text{V} \qquad \text{no load} \tag{35}$$

With a load R_L attached, we now have R_2 and R_L in parallel. This simply changes R_2 in the previous case to the equivalent of $R_2||R_L$, meaning we can just make that substitution in our previous formula.

$$V_o = \frac{V_{in}R_2}{R_1+R_2||R_L} = \frac{V_{in}R_LR_2}{(R_L+R_2)\left(\frac{R_LR_2}{R_L+R_2}\right)} = \frac{V_{in}R_2R_L}{R_1R_2+R_1\left(R_L+R_2\right)} = 10\,\text{V} \quad \text{with load} \tag{36}$$

This is the basic problem with a simple voltage divider - as soon as you plug in a load to it, the output changes. One can make the divider "stiff" by ensuring that R_L is large compared to R_2 (say, 10 times larger for $\sim$10% accuracy).

Without a load present, the output voltage *is* the open-circuit voltage, and thus the Thèvenin equivalent V_{th}. A short circuit between output and ground would take R_2 out of the circuit (V_{in} goes straight through R_1 to ground), so the current would be $I_s=V_{in}/R_1$. The Thèvenin equivalent resistance is then $R_{th}=V_{th}/I_s$:

$$V_{th} = \frac{V_{in}R_2}{R_1+R_2} = 15\,\text{V} \qquad \text{no load} \tag{37}$$

$$R_{th} = \frac{V_{th}}{I_s} = \frac{R_1R_2}{R_1+R_2} = 5\,\text{k}\Omega \qquad \text{no load} \tag{38}$$

The Thèvenin resistance is just that of R_1 parallel to R_2, which is exactly what the output "sees" looking back to the source. With a load present, the open-circuit voltage is now the modified output voltage we found above. The short-circuit current is exactly the same as in the no load situation, since both R_2 and the load will be bypassed by the short circuit on the output. Thus,

$$V_{th} = \frac{V_{in} R_2 R_L}{R_1 R_2 + R_1 \left(R_L + R_2\right)} = 10\,\text{V} \qquad \text{with load} \tag{39}$$

$$R_{th} = \frac{V_{th}}{I_s} = \frac{R_1 R_2 R_L}{R_1 R_2 + R_1 \left(R_L + R_2\right)} = 3.3\overline{3}\,\text{k}\Omega \qquad \text{with load} \tag{40}$$

The power without a load present is I^2R for each resistor. Since the resistors are the same, the power in each is

$$P = I^2 R_1 = \frac{V_{in}^2 R_1}{\left(R_1 + R_2\right)} = 22.5\,\text{mW} \qquad \text{no load} \tag{41}$$

With a load present, the output voltage is V_o, which is the voltage across both R_2 and R_L. Since $R_2 = R_L$, they have the same power. That means that the voltage across R_1 is $V_{in} - V_o$. The power in each must then be

$$P_1 = \frac{\left(V_{in} - V_o\right)^2}{R_1} = 40\,\text{mW} \qquad \text{with load} \tag{42}$$

$$P_2 = P_L = \frac{V_o^2}{R_2} = 10\,\text{mW} \qquad \text{with load} \tag{43}$$