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Central forces and gravitation

1 Motion along a curved path

Some time ago, we derived the components of acceleration required to move along a generic path in two dimensions. Given the equation of a path a particle follows, described by $\vec{\mathbf{r}}(t)$, we could use some calculus and Newton's laws to constrain the force balances parallel and perpendicular to that path. Our result was something like this:

$$\vec{\mathbf{a}}(t) = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\,\hat{\mathbf{N}} = \frac{d^2s}{dt^2}\,\hat{\mathbf{T}} + \frac{|\vec{\mathbf{v}}|^2}{R}\,\hat{\mathbf{N}} \equiv a_N\,\hat{\mathbf{T}} + a_T\,\hat{\mathbf{N}} \tag{1}$$

Where $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ were local unit vectors tangential (parallel) to and normal (perpendicular) to the path at any instant:

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} = \frac{dx}{ds}\,\hat{\boldsymbol{\imath}} + \frac{dy}{ds}\,\hat{\boldsymbol{\jmath}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \tag{2}$$

$$\hat{\mathbf{N}} = -\sin\varphi\,\hat{\boldsymbol{\imath}} + \cos\varphi\,\hat{\boldsymbol{\jmath}} = \frac{\dfrac{d\hat{\mathbf{T}}}{ds}}{\left|\dfrac{d\hat{\mathbf{T}}}{ds}\right|} \tag{3}$$

At any instant, φ is the angle that a tangent line makes with the x axis. What this told us was that if we want to stay on a curved path, we need to have a certain acceleration parallel to the path, dictated by how fast we are traveling along the path, and a certain acceleration perpendicular to the path, dictated by how fast we are turning (the local radius of curvature R). These are not real *forces*, but merely constraints – given our condition that a particle follows $\vec{\mathbf{r}}(t)$, we have automatically placed a condition on the acceleration components that must be present. These are the acceleration components that external forces must supply, in net, to produce the motion described by $\vec{\mathbf{r}}(t)$.

1.1 Fictitious forces

It is confusing sometimes to realize what is a real, live force, and what is not. For instance, circular motion. You draw a free-body diagram, and sum up the forces. This is one side of the equation. The other side is your constraint on the motion, namely, that for a circular path a net force of mv^2/r is required. Sometimes, we call this "centripetal force," and that is highly misleading. It is not a real force, but a fictitious force. This basically means that if you impose the requirement that the motion is circular, an specific constraint is placed on your force balance.

When you think about it, your force balance always has a constraint, namely, that the net force has to result in mass times the acceleration required to produce the observed path. For a straight line path, zero acceleration, the force balance is zero: equilibrium. For a circular path, the required acceleration is v^2/r, so if circular motion is observed then the net force divided by mass must give v^2/r - if not, then you don't have circular motion. For generic paths, it is more complex: an observed path constrains the force balance, but it depends on the speed along the path and the local radius of curvature.

The confusion is, in my opinion, largely an unfortunate artifact of history and terminology. There is no centripetal force, it is just a boundary condition on your force balance that enforces circular motion.

2 Motion along a curved path in polar coordinates

Back to our arbitrary path. In order to study interactions between objects, such as gravitational or electric forces, we will have to deal with forces that vary as the distance between two masses. As it turns out, both gravitational and electrical forces are *central forces*, which depend only on the relative separation between point-like objects. More specifically, we will want to deal with orbiting systems like planets, for example. Both of these cases can be much more simply described in polar coordinates. For this reason, we would like to derive a version of Eq. 1 in polar coordinates.

In fact, we can derive quite a bit about the general form of the gravitational force between objects based on a description of motion in polar coordinates combined with a few astronomical observations and symmetry considerations. First things first: how to describe motion along a general path in polar coordinates?

In fact, our most general expression for $\vec{\mathbf{a}}$ is coordinate-free – it depends only on $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and R (and the path itself), which can be defined without reference to any coordinate system. For this reason, the job is not as hard as it might seem at first.

2.1 Unit Vectors

In order to tie ourselves to a polar (r,θ) coordinate system, we need to have unit vectors in this system. Positions in polar coordinates can be described by a vector $\vec{\mathbf{r}}$, or equivalently, a distance r from the origin, and an angle θ. This means that the two unit vectors we want are (1) one pointing in the radial direction toward the point (r,θ), and (2) one pointing in the direction of increasing θ. By convention, we call these unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, respectively.

The radial unit vector $\hat{\mathbf{r}}$ is easily determined from the position vector $\vec{\mathbf{r}}$. It is instructive to see what form $\hat{\mathbf{r}}$ takes when expressed in normal cartesian coordinates. If the distance to a point is r, and the position vector makes an angle θ with the x axis, we can write

$$\vec{\mathbf{r}} = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}} \tag{4}$$

The position vector, by definition, points in the radial direction. We can make a unit vector pointing in the same direction by simply dividing $\vec{\mathbf{r}}$ by its magnitude:

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \tag{5}$$

What about a $\hat{\boldsymbol{\theta}}$? It is easiest to construct by thinking about what the conditions on it must be. First, we must have $|\hat{\boldsymbol{\theta}}|=1$ if it is to be a unit vector. Second, we must have $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ perpendicular, which really says $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}}=0$. Finally, we must be consistent with the right-hand rule. In this case means that $\hat{\boldsymbol{\theta}}$ must point toward increasing θ, which means the direction of counter-clockwise rotation about the origin. Equivalently, we could say that $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}=\hat{\mathbf{z}}$. These three requirements together fix a form for $\hat{\boldsymbol{\theta}}$:

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} \tag{6}$$

For a circular path, the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are the same as the $\hat{\mathbf{N}}$ and $\hat{\mathbf{T}}$ vectors in our general equations, which you should be able to quickly verify.

2.2 Variation of unit vectors

When we defined $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ unit vectors, it was with the understanding that they were not *global* unit vectors, but *local* ones. They change with time and position or angle as a particle moves along its path. Things are no different now. Using the cartesian forms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ above, we can easily calculate their variation with angle and time.

As the particle moves along its path, the angle θ changes, and the unit vectors change correspondingly. This variation is

$$\begin{aligned}
\frac{d\hat{\mathbf{r}}}{d\theta} &= \hat{\boldsymbol{\theta}} \\
\frac{d\hat{\boldsymbol{\theta}}}{d\theta} &= -\hat{\mathbf{r}}
\end{aligned} \tag{7}$$

Of course, moving along the path means that we are moving in time as well. The variation of the unit vectors in time is a bit sneakier, but still simple if we use the chain rule:

$$\begin{aligned}
\frac{d\hat{\mathbf{r}}}{dt} &= \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \equiv \omega\hat{\boldsymbol{\theta}} \\
\frac{d\hat{\boldsymbol{\theta}}}{dt} &= \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} \equiv -\omega\hat{\mathbf{r}}
\end{aligned} \tag{8}$$

2.3 Velocity in polar coordinates

Armed with the variation of the unit vectors, we can now find the velocity in terms of radial ($\hat{\mathbf{r}}$) and angular ($\hat{\boldsymbol{\theta}}$) components. First, we can now trivially rewrite the equation for our path:

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}} \tag{9}$$

This is not profound, but a definition – the radial position vector is its magnitude times a unit vector pointing along the radial direction …We can differentiate this with respect to time to get the velocity, taking care to apply the chain rule (since both $\hat{\mathbf{r}}$ and r vary in time).

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\,\hat{\mathbf{r}} + r\,\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\,\hat{\mathbf{r}} + r\frac{d\theta}{dt}\,\hat{\boldsymbol{\theta}} \equiv v\hat{\mathbf{r}} + r\omega\,\hat{\boldsymbol{\theta}} \tag{10}$$

Nicely, this agrees perfectly with our previous studies: the velocity has a component related to the rate at which radial distance changes, and one which represents the rate at which the angle changes. If we restrict ourselves to a circular path, where $dr/dt = 0$, we recover $\vec{\mathbf{v}}(t) = r\omega\,\hat{\boldsymbol{\theta}}$, precisely the result we expect. The velocity for circular motion is only along the (circular) path, and must have a magnitude given by the angular acceleration times the radial distance.

2.4 Acceleration in polar coordinates

We can differentiate once more, and we have the acceleration. It is merely a bit messier.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2 r}{dt^2}\,\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + r\frac{d^2\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} \tag{11}$$

Using equations 7 and 8 and collecting terms,

$$\vec{\mathbf{a}}(t) = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left[r\frac{d^2\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{\boldsymbol{\theta}} \tag{12}$$

The first terms in brackets are the radial acceleration, the second the circumferential acceleration. Let's look at the terms separately, beginning with the $\hat{\mathbf{r}}$ parts.

The first term for the radial part is easy. If you move along a straight-line path, where θ is constant, this is the only surviving term, and we recover our 1-D motion result. The second term is also familiar. For a circular path, $d\theta/dt$ is just ω, so this term is $r\omega^2$, or v^2/r – the centripetal acceleration. In this coordinate system, we pick up a minus sign to represent the fact that the centripetal acceleration points toward the center of the circle.

The first term for the circumferential part can also be understood by considering circular motion. In that case, $d^2\theta/dt^2$ is just α, so we have $r\alpha$, just as we expect. If we have *uniform* circular motion, at constant velocity, then this term vanishes. The second term in the circular case is a bit trickier. The first part dr/dt is the velocity in the radial direction, or the velocity *normal* to the circle's surface. If we follow a circular path, r is constant, and this term vanishes. If we are on a straight-line path, $d\theta/dt$ vanishes. For a more general path, this term says that we have an acceleration in the angular direction which depends on the outward velocity from the origin and the rate of rotation. We pick up an extra acceleration when we both increase the radial *and* angular velocities. Writing down the special cases,

$$\begin{aligned} \text{general circular motion:} &\qquad \vec{\mathbf{a}}(t) = -r\omega^2\,\hat{\mathbf{r}} + r\alpha\,\hat{\boldsymbol{\theta}} \\ \text{uniform circular motion:} &\qquad \vec{\mathbf{a}}(t) = -r\omega^2\,\hat{\mathbf{r}} \end{aligned} \tag{13}$$

3 Why gravity should be a central force

4 Motion under the influence of a central force

If our force has only a radial dependence (i.e., no angular variation), we may write it as

$$\vec{\mathbf{F}} = -f(r)\,\hat{\mathbf{r}} \tag{14}$$

where $f(r)$ is a function of r only, independent of θ, and the minus sign indicates that it is an attractive force. If this is the only force acting, then this radial force must yield mass times the radial acceleration, and the angular acceleration must be zero. This is just applying Newton's laws and writing down a force balance for the radial and circumferential directions, which are constrained by our path $\vec{\mathbf{r}}(t)$ as shown above.

$$\sum F_r = -f(r) = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \tag{15}$$

$$\sum F_\theta = 0 = m\left[r\frac{d^2\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] \tag{16}$$

These two constraint equations, as it turns out, will be enough to derive some important properties of central forces in general, and gravitation in particular. Keep in mind: so far we have *only* assumed that the force has no θ dependence, and that our masses are point-like.[i]

Let us look at the second equation, Eq. 16 first and see what it implies. If we multiply both sides of Eq. 16 by r, our equation is a perfect differential:

$$mr\left[r\frac{d^2\theta}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] = \left[mr^2\frac{d^2\theta}{dt} + 2mr\frac{dr}{dt}\frac{d\theta}{dt}\right] = \frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = 0 \tag{17}$$

This implies that $mr^2d\theta/dt$ is a constant of the motion (since its time derivative vanishes). What is it? In fact, it is the angular momentum of the system:

$$\begin{aligned} \vec{\mathbf{L}} &= \vec{\mathbf{r}} \times \vec{\mathbf{p}} = mr^2\vec{\boldsymbol{\omega}} \\ |\vec{\mathbf{L}}| &= L = mr^2\omega = mr^2\frac{d\theta}{dt} = \text{const.} \end{aligned} \tag{18}$$

Basically, we have derived that central forces conserve angular momentum. Of course we knew this already, but it is quite reassuring! This result is not useless however. We can rearrange it a bit first:

$$\begin{aligned} r^2\frac{d\theta}{dt} &= \frac{L}{m} \\ r^2\,d\theta &= \frac{L}{m}\,dt \end{aligned} \tag{19}$$

Why is this useful? The quantity $r^2d\theta$ is twice the unit of differential area in polar coordinates.

[i]We do not even have to assume point-like masses, it is enough if they are at least spherically symmetric. We will justify this point later.