PH585: Magnetic dipoles and so forth

	SI units	**CGS units**
Energy	1 Joule	10^7 erg
Force	1 Newton	10^5 dyne
Electric Charge	1 Coulomb	"3" x 10^9 esu
Electric Current	1 Ampere	"3" x 10^9 esu/sec
Electric Potential	"3" x 10^2 Volts	1 statvolt (erg/esu)
Electric Field	"3" x 10^4 Volts/m	1 statvolt/cm (dyne/esu)
Magnetic Field ***B***	1 Tesla	10^4 gauss (10^4 dynes/esu)
Magnetization ***M***	1 Ampere/m	4π x 10^{-3} Oersted
Magnetization ***M***	1 Ampere/m	10^{-3} emu/cm^3
Magnetic Field ***H***	1 Ampere/m	4π x 10^{-3} Oersted
Capacitance	1 farad	"9" x 10^{11} cm
Resistance	1 ohm	1/("9" x 10^{11}) sec/cm
Inductance	1 henry	1/("9" x 10^{11}) sec^2/cm

"3" = 2.9979 "9" = "3" x "3"

1 Magnetic Moments

Magnetic moments $\vec{\mu}$ are analogous to dipole moments $\vec{\mathbf{p}}$ in electrostatics. There are two sorts of magnetic dipoles we will consider: a dipole consisting of two magnetic charges p separated by a distance d (a true dipole), and a current loop of area A (an approximate dipole).

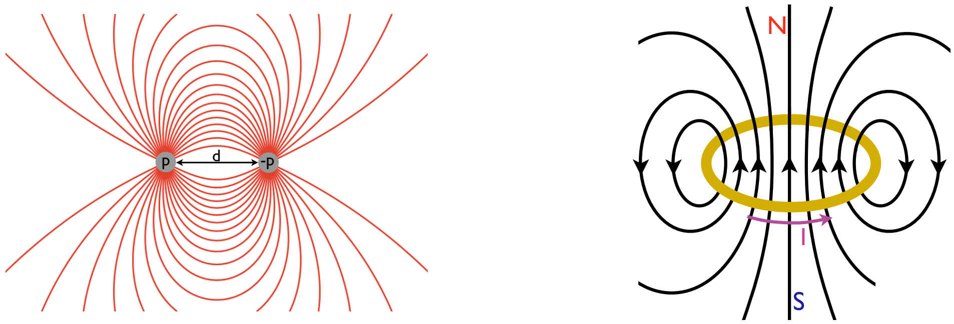

Figure 1: (left) *A magnetic dipole, consisting of two magnetic charges p separated by a distance d. The dipole moment is $|\vec{\mu}| = pd$.* **(right)** *An approximate magnetic dipole, consisting of a loop with current I and area A. The dipole moment is $|\vec{\mu}| = IA$.*

In the case of two separated magnetic charges, the dipole moment is trivially calculated by comparison with an electric dipole – substitute $\vec{\mu}$ for $\vec{\mathbf{p}}$ and p for q.[i] In the case of the current loop, a bit more work is required to find the moment. We will come to this shortly, for now we just quote the result $\vec{\mu} = IA\,\hat{n}$, where $\hat{n}$ is a unit vector normal to the surface of the loop.

2 An Electrostatics Refresher

In order to appreciate the magnetic dipole, we should remind ourselves first how one arrives at the field for an electric dipole. Recall Maxwell's first equation (in the absence of polarization density):

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_r \epsilon_0} \tag{1}$$

If we assume that the fields are static, we can also write:

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{2}$$

This means, as you discovered in your last homework, that $\vec{\mathbf{E}}$ can be written as the gradient of a scalar function φ, the electric potential:

$$\vec{\mathbf{E}} = -\vec{\nabla}\varphi \tag{3}$$

This lets us rewrite Eq. 1 in terms of the Laplacian of the potential – in other words, the electric potential satisfies Poisson's equation:

[i] It is unfortunate that the electric dipole strength and magnetic charge both use the same letter p. We will try to make which is which clear by context.

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = -\nabla \cdot \nabla\varphi = -\nabla^2\varphi = \frac{\rho}{\epsilon_0} \tag{4}$$

The general solution for this potential can be readily found:

$$\varphi(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \, d^3r' \tag{5}$$

The solution is based on the fact that

$$\nabla^2 \left(\frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} \right) = -4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right) \tag{6}$$

which makes $|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^{-1}$ an elementary example of a Green's function. You can read all about this in, for example, *Classical Electrodynamics* by Jackson.

We may need to augment the solution above with, *e.g.,* a function Ψ to satisfy the boundary conditions, and we can do this so long as $\nabla^2\Psi = 0$. This requirement essentially states that we can't add anything to our potential that would imply additional charge density. A sensible requirement, right?

Now we can solve Poisson's equation for a given charge distribution, find the (scalar) electric potential everywhere, and use $\vec{\mathbf{E}} = -\nabla\varphi$ to find the electric field everywhere. For the task at hand – finding the field due to an electric dipole – this is overkill. We will just calculate the potential directly in this case.

2.1 Electric Dipoles

Figure 2 shows the dipole we wish to study, two charges q separated by d. We will choose the origin $\mathcal{O}$ to be precisely between the two charges, along the line connecting the charges, and we wish to calculate the field at an arbitrary point $\mathcal{P}(x, y, z) = \mathscr{P}(r)$ far from the dipole $(r \gg d)$.[ii].

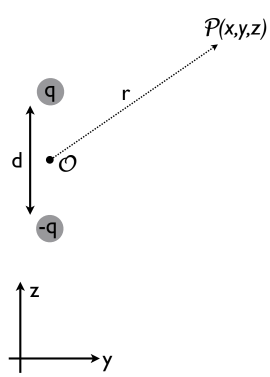

Figure 2: *An electric dipole consisting of two charges q separated by d. The origin $\mathcal{O}$ is chosen between the two charges, and we wish to calculate the field at an arbitrary point $\mathcal{P}(x, y, z) = \mathcal{P}(r)$ far from the dipole $(r \gg d)$.*

We can readily write down the potential at the point $\mathcal{P}$ – it is just a superposition of the potential due to each of the charges alone:

[ii]It is not much harder to solve without assuming $r \gg d$, but we don't really *need* the solution close to the dipole.

$$\varphi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{\left(z-\frac{d}{2}\right)^2+x^2+y^2}} + \frac{-q}{\sqrt{\left(z+\frac{d}{2}\right)^2+x^2+y^2}}\right] \tag{7}$$

Since we are assuming $r \gg d$, we can simplify this a bit by noting that $(z-d/2)^2 \approx z^2 - zd$ using a binomial expansion. This simplifies the denominator quite a bit:

$$\frac{1}{\sqrt{\left(z-\frac{d}{2}\right)^2+x^2+y^2}} \approx \frac{1}{\sqrt{z^2-zd+x^2+y^2}} = \frac{1}{\sqrt{r^2-zd}} = \frac{1}{r\sqrt{1-\frac{zd}{r^2}}} \approx \frac{1}{r}\left(1+\frac{zd}{2r^2}\right) \tag{8}$$

Here we used the binomial expansion once again. We can do this for both terms in the potential, and arrive at a fairly simple form for φ:

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{z}{r^3}qd \tag{9}$$

Now we define the electric dipole as $\vec{\mathbf{p}} = qd\,\hat{z} = q\vec{\mathbf{d}}$, where $\vec{\mathbf{d}}$ is just a vector going from one charge to the other. We also note that z/r is nothing more than $\cos\theta$, where θ is the angle between the vector $\vec{\mathbf{r}}$ pointing to $\mathcal{P}$ and the y axis. With this in hand,

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{|\vec{\mathbf{p}}|\cos\theta}{r^2} \tag{10}$$

Finally, $|\vec{\mathbf{p}}|\cos\theta$ is nothing more than a dot product of $\vec{\mathbf{p}}$ and the radial unit vector $\hat{\mathbf{r}}$ ...

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}}{r^3} \tag{11}$$

This is the usual expression for the potential relatively far from an electric dipole. How do we find the field? We just use $\vec{\mathbf{E}} = -\nabla\varphi$ and grind through the math ...

$$E_x = \frac{|\vec{\mathbf{p}}|}{4\pi\epsilon_0}\frac{3zx}{r^5}\,\hat{\mathbf{x}} \tag{12}$$

$$E_y = \frac{|\vec{\mathbf{p}}|}{4\pi\epsilon_0}\frac{3zy}{r^5}\,\hat{\mathbf{y}} \tag{13}$$

$$E_z = \frac{|\vec{\mathbf{p}}|}{4\pi\epsilon_0}\left(\frac{1}{r^3}-\frac{3z^2}{r^5}\right)\hat{\mathbf{z}} \tag{14}$$

With a bit more tedium, we would discover an ostensibly more elegant formula for $\vec{\mathbf{E}}$:

$$\vec{\mathbf{E}} = \frac{3\left(\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\vec{\mathbf{p}}}{4\pi\epsilon_0 r^3} \tag{15}$$

You can find form this by directly taking the gradient of Eq. 11, if you remember a few arcane vector identities. What we will discover shortly is that the solution for a magnetic dipole is *exactly the same*, provided we make the substitutions

$$\vec{\mathbf{p}} \rightarrow \frac{\vec{\boldsymbol{\mu}}}{c^2} \tag{16}$$

We know already that we pick up a factor c^2 scaling the magnetic dipole moment because the magnetic field is just the electric field viewed from another reference frame. Remembering that $\mu_0\epsilon_0 = c^{-2}$, you have the magnetic dipole formula:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{3\left(\vec{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \vec{\boldsymbol{\mu}}}{r^3} \tag{17}$$

We will derive this shortly to make sure our substitution is correct.

Just as an aside, instead of resolving the components of $\vec{\mathbf{E}}$ along the x and y axes, we could just as easily resolve them along the direction parallel and perpendicular to the dipole axis:

$$E_{||} = \frac{|\vec{\mathbf{p}}|}{4\pi\epsilon_0}\frac{3z}{r^5}\sqrt{x^2+y^2}\,\hat{\mathbf{z}} \tag{18}$$

$$E_{\perp} = \frac{|\vec{\mathbf{p}}|}{4\pi\epsilon_0}\frac{3\cos\theta\sin\theta}{r^3}\,\hat{\mathbf{x}} \tag{19}$$

$$\tag{20}$$

3 Magnetic Fields from Currents

How do we verify our solution of the magnetic dipole? Are we sure that a current loop really looks like a dipole? We can figure out both questions pretty quickly, once we remember how to get magnetic fields from currents. In electrostatics, we start with the scalar potential of the charge distribution, and then get $\vec{\mathbf{E}}$. In magnetostatics[iii], we start with the *vector potential* and then get $\vec{\mathbf{B}}$.

Probably you are not used to doing this – it is not done in a lot of E&M courses in favor of the Biot-Savart law or some such thing, and using the vector potential is just hard sometimes. This is unfortunate, but easily correctable. We *like* the vector potential, since in magnetostatics it is neatly related to the current density distribution just as in electrostatics the scalar potential is related to the charge distribution.

To start with, take Maxwell's equation for the curl of $\vec{\mathbf{B}}$, assuming that $\vec{\mathbf{E}}$ does not vary with time:

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \frac{\vec{\mathbf{j}}}{\epsilon_0 c^2} \tag{21}$$

Notice again that when we compare the corresponding $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ equations, we pick up a factor c^2 from the Lorentz contraction. We can roughly substitute in the following way *in the static case* to get from electric

[iii]Static in the sense that $\vec{\mathbf{B}}$ and $\vec{\mathbf{M}}$ are not changing, but obviously we need moving charges for there to be a current.

to magnetic formulas:

$$\vec{\mathbf{E}} \to \vec{\mathbf{B}} \qquad \rho \to \frac{\vec{\mathbf{j}}}{c^2} \tag{22}$$

$$\cdot \to \times \qquad \times \to \cdot \tag{23}$$

Don't read too much into this. The only point is that charge density and current density are analogous, cross products for one are dot products for the other, and Lorentz transforming from one field to the other picks up a factor c^2.

That aside: our vector potential is defined by $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$. Substituting this into Eq. 21 ...

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}\right) = \frac{\vec{\mathbf{j}}}{\epsilon_0 c^2} = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}\right) - \nabla^2 \vec{\mathbf{A}} \tag{24}$$

Here we used the identity for $\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}}\right)$ without telling you first. This looks formidable, but remember that with $\vec{\mathbf{A}}$ we have a gauge choice[iv]– we can add the divergence of any other field to $\vec{\mathbf{A}}$ and not affect the resulting field $\vec{\mathbf{B}}$, since the curl of the divergence of any function is identically zero. Put more simply, we have to *pick* specifically what we want $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}$ to be for our solution to be unique, since *all* choices are perfectly valid. Based on what we have above, the most convenient choice is just to choose $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$. We always take the easy way out when we can! This is known as the Coulomb gauge, and it is particularly convenient because it decouples the equations for the scalar and vector potentials.

Within the 'Coulomb gauge,' our equation becomes, on its face, nice and simple:

$$\nabla^2 \vec{\mathbf{A}} = \frac{-\vec{\mathbf{j}}}{\epsilon_0 c^2} = -\mu_0 \vec{\mathbf{j}} \tag{25}$$

We now have a Poisson's equation relating *magnetic* vector potential and *current* density, just like the one we have relating *electric* scalar potential and *charge* density. The same equations have the same solutions, so we can immediately write down a general (magnetostatic) solution for $\vec{\mathbf{A}}$:

$$\vec{\mathbf{A}}(r) = \frac{1}{4\pi\epsilon_0 c^2} \int \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3r' = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3r' \tag{26}$$

We get electric potential by integrating the charge density over all space, whereas the magnetic potential comes from integrating the *current* over all space. The main difference is that current density is a vector, while charge density is a scalar.

3.1 A Long, Straight Wire

Now let's consider the magnetic vector potential from a long current-carrying wire, a segment of which is shown in Fig. 3. The wire of cross-section $2a$ carries a current I in the z direction. Cylindrical coordinates

[iv] Choosing a gauge is just a mathematical procedure for coping with redundant degrees of freedom, in this case, coping with the non-uniqueness of $\vec{\mathbf{A}}$ by just *choosing* what the divergence of $\vec{\mathbf{A}}$ should be.

will be natural in this case. First, of all we cannot make the wire *infinitely* long, since that would make $\vec{\mathbf{A}}$ infinite. We will consider wire of length $2l$, made up of two segments of length l.

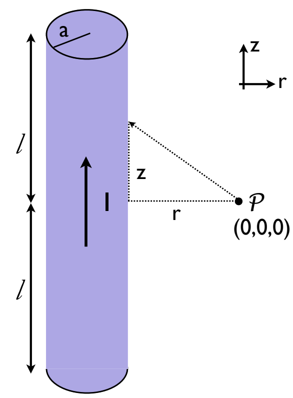

Figure 3: *A segment of a long, straight wire of length $2l$ carrying a current I. The current flows in the z direction.*

The two segments of wire are joined in the $z=0$ plane, as is our point $\mathcal{P}$, and we will make the point $\mathcal{P}$ at a distance r our origin as well. Our task is to find the vector potential at point $\mathcal{P}$, a distance r from the midpoint of the two segments, as shown in Fig. 3. In the end, we can impose the condition $l \gg r$, which is practically as good as making the wire infinitely long.

Since the current is only in the z direction, by symmetry the vector potential only has a z component – A_x and A_y must be zero since j_x and j_y are zero. Let us only worry about the potential from the upper segment of wire first, and we can add in that of the lower segment by superposition once we are finished – clearly, they will give the same contribution. The distance from the point $\mathcal{P}$ to any point on the upper segment of wire is just $\sqrt{r^2+z^2}$, so already we can write down the integral for A_z due to the upper half wire:

$$\vec{\mathbf{A}} = \int_0^l \frac{1}{4\pi\epsilon_0 c^2} \frac{j_z\,\hat{\mathbf{z}}}{\sqrt{r^2+z^2}}\,dz \tag{27}$$

You can look this integral up in any halfway decent table.[v]

$$\vec{\mathbf{A}} = \frac{j_z\,\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2} \ln\left|z+\sqrt{r^2+z^2}\right|\Bigg|_0^l = \frac{j_z\,\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2}\left[\ln\left|l+\sqrt{r^2+l^2}\right| - \ln\left|r\right|\right] = \frac{j_z\,\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2}\ln\left|\frac{l+\sqrt{r^2+l^2}}{r}\right| \tag{28}$$

This is the vector potential from the *upper half* wire. The vector potential from the lower half wire can be found in the same way, excepting that we integrate from $-l$ to 0, which gives us exactly the same result as above. No surprise, the potential due to the whole wire is just double that from half a wire! Now, if we assume that $l \gg r$, we can approximate $\sqrt{r^2+l^2} \sim l$, and make things a bit simpler:

$$\vec{\mathbf{A}} \approx \frac{j_z\,\hat{\mathbf{z}}}{2\pi\epsilon_0 c^2}\ln\left|\frac{2l}{r}\right| = \frac{j_z\,\hat{\mathbf{z}}}{2\pi\epsilon_0 c^2}\left(\ln 2l - \ln r\right) \tag{29}$$

No problem. Now, since the field $\vec{\mathbf{B}}$ is given by the *curl* of $\vec{\mathbf{A}}$, we have another sort of freedom in finding

[v] Or, substitute $z=r\tan\theta$ and just grind through it. It isn't so bad.

$\vec{\mathbf{A}}$, *viz.*, we can add any constant or constant vector we like to $\vec{\mathbf{A}}$ and get the same $\vec{\mathbf{B}}$.[vi] Our gauge choice just requires that $\vec{\nabla}\cdot\vec{\mathbf{A}}=0$, remember, so adding a constant vector is completely within our freedom. In that case, we choose to add precisely $\frac{-j_z\,\hat{\mathbf{z}}}{4\pi\epsilon_0 c^2}\ln 2l$ to our existing $\vec{\mathbf{A}}$, which gives us a nice neat form for the vector potential:

$$\vec{\mathbf{A}} = -\frac{j_z}{2\pi\epsilon_0 c^2}\ln r\,\hat{\mathbf{z}} = -\frac{I}{2\pi\epsilon_0 c^2}\ln r\,\hat{\mathbf{z}} = -\frac{\mu_0 I}{2\pi}\ln r\,\hat{\mathbf{z}} \qquad l \gg r \tag{30}$$

Here we also made use of the fact that $j_z=\frac{I}{\pi a^2}$. How about the field $\vec{\mathbf{B}}$? No problem, we just have to take some derivatives, since $\vec{\nabla}\times\vec{\mathbf{A}}=\vec{\mathbf{B}}$. This is the nice thing about finding fields from potentials – usually finding the potential is fairly straightforward, and getting the field from that is just differentiation. Overall, we save on the number of integrals. Anyway: since $\vec{\mathbf{A}}$ has only a $\hat{\mathbf{z}}$ component, and that depends only on r, its curl is particularly simple even in cylindrical coordinates:

$$\vec{\mathbf{B}} = \vec{\nabla}\times\vec{\mathbf{A}} = -\frac{\partial A_z}{\partial r}\,\hat{\boldsymbol{\varphi}} = \frac{\mu_0 I}{2\pi}\frac{\partial}{\partial r}\left(\ln r\right)\hat{\boldsymbol{\varphi}} = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\varphi}} \tag{31}$$

After all that, it is reassuring that we recover the usual form for the field surrounding a current-carrying wire. We could have found this trivially with, *e.g.,* Ampere's law, but the point is to become a bit more comfortable with the vector potential, as it will make our discussion of dipoles considerably more straightforward.

3.1.1 Electrostatic Analogy

The electrostatic analogy to this problem is just finding the field around an infinite charged cylinder (with linear charge density λ. You have probably already solved this problem, most likely with Gauss' law for $\vec{\mathbf{E}}$ and then finding φ, and know the answer to be:

$$\varphi = -\frac{\lambda}{2\pi\epsilon_0}\ln r \tag{32}$$

It is curious to compare this to the vector potential we found above. If we make the substitutions

$$\lambda \to \frac{\pi a^2 j_z}{c^2} = \frac{I}{c^2} \tag{33}$$

$$\varphi \to \vec{\mathbf{A}} \tag{34}$$

and add the appropriate unit vector to $\vec{\mathbf{A}}$, they are the same. This shouldn't be surprising – a current carrying wire *is* a charged rod from the point of view of a test charge moving alongside the wire. This is how we derived the magnetic field from the electric field, and we should not be terribly surprised that charge density just becomes current, picking up a factor c^2 from the Lorentz transformation. This trick will work pretty generally for magnetostatics, because after all, the same equations have the same solutions. Our task in many cases is just to find the right electrostatics problem we've already solved, and make the appropriate substitutions.

[vi] Just like with the electric potential, we can choose zero potential anywhere we like.

3.2 A current loop

We need to solve one more problem before we get into real magnetic phenomena, that of a small current loop. Figure 4 show the geometry for the small current loop we will consider. We place the current loop of radius a carrying current I at the origin, and using cylindrical coordinates, we will find the vector potential at a point $\mathcal{P}(R, \varphi, z)$. From there, it will be a simple matter to find the field $\vec{\mathbf{B}}$.

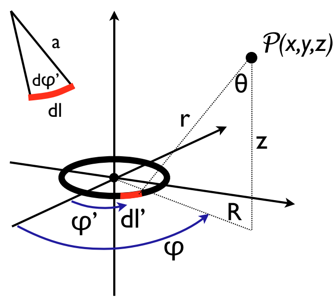

Figure 4: *Geometry for our current loop.*

3.2.1 Vector Potential

First, consider a small element of the loop $d\vec{\mathbf{l}}'$, at coordinates $\vec{\mathbf{P}}' = (a, \varphi', 0)$. The primed coordinates will refer to the tiny element. The vector from $d\vec{\mathbf{l}}'$ to $\mathcal{P}$ we will just call $\vec{\mathbf{r}}$. The vector defining the small element $d\vec{\mathbf{l}}'$ itself can be most simply written:

$$d\vec{\mathbf{l}}' = a\,d\varphi'\,(-\sin\varphi'\,\hat{\boldsymbol{\imath}} + \cos\varphi'\,\hat{\boldsymbol{\jmath}}) = a\,d\varphi'\,\hat{\boldsymbol{\varphi}} \tag{35}$$

which you should be able to verify easily. This is just the gradient of the position on the circle of radius a, since we want $d\vec{\mathbf{l}}'$ to lie along the tangential direction. The vector potential at $\mathcal{P}$ is now just an integral over all $d\vec{\mathbf{l}}'$ around the loop, noting that since the current density is constant $\vec{\mathbf{j}}\,dV = I\,d\vec{\mathbf{l}}'$:

$$\vec{\mathbf{A}}(\vec{\mathbf{P}}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\mathbf{j}}(\vec{\mathbf{P}}')}{|\vec{\mathbf{P}} - \vec{\mathbf{P}}'|}\,dV \tag{36}$$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{\mathbf{l}}'}{|\vec{\mathbf{r}}|} \tag{37}$$

Already we can see that the vector potential from each element of the ring is in the direction of the current, so the overall vector potential will have only a φ component. Of course, just from the symmetry of the problem, this is also clear. Using the fact that $\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\boldsymbol{\imath}} + \cos\varphi\,\hat{\boldsymbol{\jmath}}$, we can write the φ component of the vector potential:

$$A_\varphi = \vec{\mathbf{A}} \cdot \hat{\boldsymbol{\varphi}} \tag{38}$$

$$= \hat{\boldsymbol{\varphi}} \cdot \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\mathbf{l}}'}{|\vec{\mathbf{r}}|} \tag{39}$$

$$= \frac{\mu_0 I}{4\pi} \oint \frac{\hat{\boldsymbol{\varphi}} \cdot d\vec{\mathbf{l}}'}{|\vec{\mathbf{r}}|} \tag{40}$$

Finding $\varphi \cdot d\vec{\mathbf{l}}'$ is straightforward, if we remember a handy identity for $\cos(a+b)$

$$\hat{\boldsymbol{\varphi}} \cdot d\vec{\mathbf{l}}' = a\, d\varphi' \left(\sin\varphi \sin\varphi' + \cos\varphi\cos\varphi'\right) \tag{41}$$

$$= a\, d\vec{\mathbf{l}}' \cos\left(\varphi - \varphi'\right) \tag{42}$$

Things brings us finally to something that *seems* manageable enough:

$$A_\varphi = \frac{\mu_0}{4\pi} \int \frac{a\cos\left(\varphi - \varphi'\right) d\varphi'}{|\vec{\mathbf{r}}|} \tag{43}$$

$$|\vec{\mathbf{r}}| = \sqrt{\left(R\cos\varphi - a\cos\varphi'\right)^2 + \left(R\sin\varphi - a\sin\varphi'\right)^2 + z^2} \tag{44}$$

There is no reason to use the angle φ any more, we can just set it to zero, which simplifies $\vec{\mathbf{r}}$ a bit:

$$|\vec{\mathbf{r}}| = \sqrt{(R - a\cos\varphi') + (a\sin\varphi') + z^2} \tag{45}$$

$$= \sqrt{R^2 + 2aR\cos\varphi' + a^2\cos^2\varphi' + a^2\sin^2\varphi' + z^2} \tag{46}$$

$$= \sqrt{R^2 + a^2 + z^2 - 2aR\cos\varphi'} \tag{47}$$

This is nice, but it just leads to an annoying elliptical integral:

$$A_\varphi = \frac{\mu_0 I}{4\pi} \int \frac{a\cos\varphi'\, d\varphi'}{\sqrt{R^2 + a^2 + z^2 - 2aR\cos\varphi'}} \tag{48}$$

One can look them up in tables, or use any number of numerical analysis packages (*e.g.,* Mathematica), but this is useless for our purposes. From now on, we will assume the current loop is far away, such that $R^2 + z^2 \gg a^2$. In this case, we can approximate a bit:

$$\sqrt{R^2 + a^2 + z^2 - 2aR\cos\varphi'} = \frac{1}{\sqrt{R^2 + a^2 + z^2}} \frac{1}{\sqrt{1 - \frac{2aR\cos\varphi'}{R^2 + a^2 + z^2}}} \tag{49}$$

$$\approx \frac{1}{\sqrt{R^2 + a^2 + z^2}} \left(1 + \frac{aR\cos\varphi'}{R^2 + a^2 + z^2}\right) \tag{50}$$

Now we have something a bit more manageable:

$$A_\varphi = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a\cos\varphi'}{\sqrt{R^2+a^2+z^2}} \left(1 + \frac{aR\cos\varphi'}{R^2+a^2+z^2}\right) d\varphi' \tag{51}$$

Now note the following two facts:

$$\int_0^{2\pi} \cos\varphi'\, d\varphi' = 0 \qquad \text{and} \qquad \int_0^{2\pi} \cos^2\varphi'\, d\varphi' = \pi \tag{52}$$

Pull everything not depending on φ' out of the integral, and we are nearly done:

$$A_\varphi = \frac{\mu_0 I}{4\pi} \frac{a^2 R}{(R^2+a^2+z^2)^{\frac{3}{2}}} \cdot \pi \tag{53}$$

$$= \frac{\mu_0 I a^2 R}{4\,(R^2+a^2+z^2)^{\frac{3}{2}}} \tag{54}$$

At this point, it is easier to switch to spherical polar coordinates. Keeping in mind that we are far away from the loop, so we will make the approximation $\sqrt{R^2+a^2+z^2} \approx \sqrt{R^2+z^2}$, and thus $r^2 = R^2+z^2$ and $\sin\theta = R/\sqrt{R^2+z^2}$. This gives A_φ a nicer form:

$$A_\varphi = \frac{\mu_0 I}{4} \frac{a^2 R}{\sqrt{R^2+z^2}\,(R^2+z^2)} = \frac{\mu_0 I a^2 \sin\theta}{4r^2} \qquad R^2+z^2 \gg a^2 \tag{55}$$

Now we will define a magnetic dipole moment in terms of the current carried by the loop and its area, $|\vec{\mu}| = IA = \pi a^2 I$. The direction of the magnetic dipole moment points normal to the area of the loop, $\vec{\mu} = IA\,\hat{n}$.

$$A_\varphi = \frac{\mu_0 I a^2 \sin\theta}{4r^2} = \frac{\mu_0}{4\pi} \frac{|\vec{\mu}|\sin\theta}{r^2} \tag{56}$$

This just begs to be rewritten as a cross-product! In fact, there is little reason we can't write it as one. All that matters is that the relative orientation of $d\vec{\mathbf{l}}'$ and the radius vector $\vec{\mathbf{r}}$ to the point $\mathcal{P}$ where we want to find the potential and field. You can check that this is correct if you think about the appropriate directions:[vii]

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \hat{\mathbf{r}}}{r^2} \tag{57}$$

Lo and behold, this is just like the formula for the potential from an electric dipole, if we use our substitutions $\vec{\mathbf{p}} \to \vec{\mu}/c^2$ and $\cdot \to \times$. Our current loop is the magnetic equivalent of an electric dipole, a *magnetic dipole.*

[vii] While you're at it, you can derive the Biot-Savart law by thinking just about the current element itself. Purcell and Feynman do this well.

3.2.2 The Field

Since $\vec{\mathbf{A}}$ has only a $\hat{\varphi}$ component, finding $\vec{\mathbf{B}}$ from $\vec{\mathbf{B}}=\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}$ is simple. First, the r component (we are in spherical coordinates now, remember):

$$\begin{aligned}
B_r &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin\theta A_\varphi\right) && (58)\\
&= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}r\sin\theta\frac{\mu_0}{4\pi}\frac{|\vec{\mu}|\sin\theta}{r^2}\right) && (59)\\
&= \frac{\mu_0|\vec{\mu}|}{4\pi r^3\sin\theta}\left(\frac{\partial}{\partial\theta}\sin^2\theta\right) && (60)\\
&= \frac{\mu_0|\vec{\mu}|}{4\pi r^3\sin\theta}\left(2\sin\theta\cos\theta\right) && (61)\\
&= \frac{\mu_0|\vec{\mu}|\cos\theta}{2\pi r^3} && (62)
\end{aligned}$$

Clearly, $B_\varphi=0$, but we don't get off quite so easily with B_θ:

$$\begin{aligned}
B_\theta &= -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right) && (63)\\
&= -\frac{1}{r}\left(\frac{\partial}{\partial r}r\frac{\mu_0}{4\pi}\frac{|\vec{\mu}|\sin\theta}{r^2}\right) && (64)\\
&= -\frac{\mu_0|\vec{\mu}|\sin\theta}{4\pi r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right) && (65)\\
&= \frac{\mu_0|\vec{\mu}|\sin\theta}{4\pi r^3} && (66)
\end{aligned}$$

If we had started from Eq. 57 and taken the curl directly, we could arrive at an ostensibly more convenient form for $\vec{\mathbf{B}}$, the 'elegant' one that is usually quoted without proof:

$$\begin{aligned}
\vec{\mathbf{B}} &= \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}} && (67)\\
&= \vec{\boldsymbol{\nabla}}\times\left[\frac{\mu_0}{4\pi}\frac{\vec{\mu}\times\hat{\mathbf{r}}}{r^2}\right] && (68)\\
&= \frac{\mu_0}{4\pi}\vec{\boldsymbol{\nabla}}\times\left[\left(\frac{1}{r^2}\right)\vec{\mu}\times\hat{\mathbf{r}}\right] && (69)\\
&= \frac{\mu_0}{4\pi r^2}\vec{\boldsymbol{\nabla}}\times\left(\vec{\mu}\times\hat{\mathbf{r}}\right)+\frac{\mu_0}{4\pi}\left[\vec{\boldsymbol{\nabla}}\frac{1}{r^2}\right]\times\left(\vec{\mu}\times\hat{\mathbf{r}}\right) && (70)\\
&= \frac{\mu_0}{4\pi r^2}\left[\vec{\mu}\left(\vec{\boldsymbol{\nabla}}\cdot\hat{\mathbf{r}}\right)-\hat{\mathbf{r}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\mu}\right)+\left(\hat{\mathbf{r}}\cdot\vec{\boldsymbol{\nabla}}\right)\vec{\mu}-\left(\vec{\mu}\cdot\vec{\boldsymbol{\nabla}}\right)\hat{\mathbf{r}}\right]-\frac{\mu_0}{4\pi r^3}\hat{\mathbf{r}}\times\left(\vec{\mu}\times\hat{\mathbf{r}}\right) && (71)
\end{aligned}$$

This is still fairly messy, until we realize that both the gradient and divergence of $\vec{\mu}$ are *zero* – $\vec{\mu}$ is just a constant vector. We can also use the identities $\left[\vec{\mathbf{a}}\cdot\vec{\boldsymbol{\nabla}}\right]\hat{\mathbf{r}}=\frac{1}{r}\left[\vec{\mathbf{a}}-\hat{\mathbf{r}}\left(\vec{\mathbf{a}}\cdot\hat{\mathbf{r}}\right)\right]$, $\vec{\boldsymbol{\nabla}}\cdot\hat{\mathbf{r}}=\frac{2}{r}$ and then it gets better:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi r^2}\left[\frac{2\vec{\mu}}{r} - 0 + 0 - \frac{1}{r}\left[\vec{\mu} - \hat{\mathbf{r}}\left(\vec{\mu}\cdot\hat{\mathbf{r}}\right)\right]\right] - \frac{\mu_0}{4\pi}\left[\frac{2}{r^3}\left[\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)\vec{\mu} - \left(\hat{\mathbf{r}}\cdot\vec{\mu}\right)\hat{\mathbf{r}}\right]\right] \tag{72}$$

$$= \frac{\mu_0}{4\pi}\left[\frac{2\vec{\mu} - \vec{\mu} + \hat{\mathbf{r}}\left(\vec{\mu}\cdot\hat{\mathbf{r}}\right)}{r^3}\right] - \frac{\mu_0}{4\pi}\left[\frac{2\vec{\mu} - 2\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\vec{\mu}\right)}{r^3}\right] \tag{73}$$

$$= \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}\left(\vec{\mu}\cdot\hat{\mathbf{r}}\right) - \vec{\mu}}{r^3} \tag{74}$$

If you are dubious of the enormous string of vector identities we have just employed, rest easy. We can easily verify that it is correct, and gives the same answer we already had:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{3\left(\vec{\mu}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \vec{\mu}}{r^3} \tag{75}$$

$$B_r = \vec{\mathbf{B}}\cdot\hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{3|\vec{\mu}|\cos\theta - |\vec{\mu}|\cos\theta}{r^3} = \frac{\mu_0|\vec{\mu}|\cos\theta}{2\pi r^3} \tag{76}$$

$$B_\theta = \vec{\mathbf{B}}\cdot\hat{\boldsymbol{\theta}} = \frac{\mu_0}{4\pi}\frac{0 + |\vec{\mu}|\sin\theta}{r^3} = \frac{\mu_0|\vec{\mu}|\sin\theta}{4\pi r^3} \tag{77}$$

Now we have the magnetic field of a current loop or magnetic dipole, at least fairly far from the dipole. Again, we have verified that we *could* have just used the solution to the analogous electric case, along with the appropriate substitutions. This is the tactic we employed during lecture, using a square current loop and comparing it to a set of charged rods.[viii]

3.3 Two Magnetic Charges

Now, what about two magnetic *charges* configured as a dipole? There is no need to solve this problem again, the solution is just like that for the electric dipole in Sect. 2.1. As we discovered in the previous lecture, if we assume the fields do not vary in time, we can use a *scalar* magnetic potential, and the solution really is identical.

3.4 Current Loops *vs.* Magnetic Charges

Comparing the magnetic dipole moment found for two magnetic charges to that found for a current loop gives us some insight into the many different units encountered in magnetism – there is essentially an entire system of units based on each approach.

First, for a magnetic dipole made up of two magnetic charges $\pm p$, such that $\vec{\mu} = p\vec{\mathbf{d}}$, the amount of magnetic charge is measured in Webers [Wb], so $|\vec{\mu}|$ must be Wb·m. Total magnetization is just the number of magnetic moments per unit volume, so magnetization is Wb/m^2, which is defined to be one Tesla. Within this approach, one usually writes $\vec{\mathbf{B}} = \mu_0\vec{\mathbf{H}} + \vec{\mathbf{M}}$, so both B and M have units of Tesla.

For a current loop, on the other hand, $\vec{\mu} = I\vec{\mathbf{A}} = IA\hat{n}$, so the dipole moment must be measured in A·m^2, and the moment per unit volume, the magnetization, must be in A/m. In this approach, it is customary to say $\vec{\mathbf{B}} = \mu_0(\vec{\mathbf{H}} + \vec{\mathbf{M}})$, so H and M have units of A/m, while B is in Tesla.

So this is why we have the complete and utter mess that we do in magnetism. This is why we use A/m and Tesla, and sometimes mix the two together. From all of this we can see that $pd = \mu_0 IA$, which means

[viii] The treatment we followed in class – handwaving at times, but essentially correct – is like that in vol. II of the *Feynman Lectures*.

that $1\,\mathrm{A/m} \rightarrow 4\pi \times 10^{-7}\,\mathrm{T}$, or $1\,\mathrm{kA/m} \rightarrow 4\pi \times 10^{-4}\,\mathrm{T}$. We attempt to summarize this mess in the table below.

MAKE TABLE

4 Force and Energy for Dipoles in a Field

Now, let us take one of our current loops and put it in a constant magnetic field. We will say our current loop carries a current I, and has area A (so $|\vec{\mu}| = IA\hat{n}$, where $\hat{n}$ is perpendicular to the loop surface), while the magnetic field is along the z direction, $\vec{\mathbf{B}} = B\,\hat{\mathbf{z}}$. The dipole moment of the current loop and the magnetic field make an angle θ.

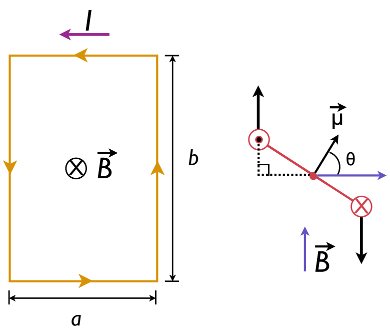

Figure 5: *Square current loop in a magnetic field.*

Just like an electric dipole, the presence of a magnetic field induces a torque on the magnetic dipole. This should already be familiar. In the electric case, the torque is $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$, and for the magnetic dipole, $\vec{\boldsymbol{\tau}} = \vec{\mu} \times \vec{\mathbf{B}}$.

The presence of a torque implies a change in potential energy. If the torque τ rotates the current loop through an angle $d\theta$, we can write the potential energy due to that rotation as

$$dU = -\tau d\theta = -|\vec{\mu}|B\sin\theta\,d\theta \tag{78}$$

Or, integrating over the total angular displacement, we can write the *change* in potential energy rotating through an angle θ as:[ix]

$$\Delta U = -\vec{\mu} \cdot \vec{\mathbf{B}} \tag{79}$$

Again, this is just like the electric dipole in an electric field.

5 Orbital Magnetism

Now, instead of a square current loop, imagine we have a single charge q orbiting a nucleus – a Bohr atom. The charge q has a constant speed v in its orbit, which means that the "current" corresponding to this single charge is just the amount of charge q divided by how long it takes to orbit the nucleus:

[ix] We pick up an arbitrary constant through integration, which we set to zero since we are only interested in *changes* in potential energy

$$I = \frac{\Delta Q}{\Delta t} = \frac{q}{2\pi r/v} = \frac{qv}{2\pi r} \tag{80}$$

The area of the "current loop" defined by the orbit is just πr^2, so the effective magnetic dipole moment is

$$|\vec{\mu}| = \pi r^2 I = \frac{1}{2}qvr \qquad \text{or} \qquad \vec{\mu} = \frac{1}{2}qvr\,\hat{\mathbf{z}} \tag{81}$$

The charge q also has an angular momentum J due to its orbit, which is just $J = mvr$. Since $\vec{\mathbf{J}}$ and $\vec{\mu}$ must be parallel – they are both perpendicular to the surface defined by the orbit, and parallel to $\hat{\mathbf{z}}$ – we can just as easily write μ in terms of $\vec{\mathbf{J}}$:

$$\vec{\mu} = \frac{q}{2m}\vec{\mathbf{J}}_{\text{orbit}} \tag{82}$$

Curiously enough, this depends on neither v or r, just the charge/mass ratio q/m.[x] For an orbiting electron, the $q = -e$, so the magnetic dipole moment of orbiting electrons is negative. Often, one sees this written in terms of the 'g-factor':

$$\vec{\mu} = -\frac{ge}{2m}\vec{\mathbf{J}} \tag{83}$$

Where $g = 1$ for orbiting electrons, while $g = 2$ for a pure spin moment, which we will come to in subsequent lectures.

6 Diamagnetism

What happens if we put our Bohr atom in an increasing magnetic field? Say we increase the field slowly from 0 to B in some amount of time t. Now we have a closed current loop with a time-varying magnetic field ... by Faraday's law, we must also have an electric field generated around the orbit! We integrate $\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ around the orbit, and the flux over the surface defined by the orbit:

$$\oint_{\text{orbit}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \tag{84}$$

$$2\pi r E = -\frac{\partial}{\partial t}\left(B\pi r^2\right) \tag{85}$$

$$E = -\frac{1}{2}r\frac{\partial B}{\partial t} \tag{86}$$

The time-varying B through the surface of the orbit gives rise to a circulating E field. This E field in turn acts on the single charge in the orbit. Since the charge is in uniform circular motion, the circulating

[x] A more rigorous quantum mechanical treatment gives basically the same answer. In this case, we are correct for the wrong reasons, but it is still worth proceeding.

electric field produces a *torque*:

$$
\begin{aligned}
|\vec{\boldsymbol{\tau}}| &= |\vec{\mathbf{r}} \times \vec{\mathbf{F}}| && (87) \\
&= |\vec{\mathbf{r}} \times q\vec{\mathbf{E}}| && (88) \\
&= \frac{1}{2} q r^2 \frac{\partial B}{\partial t} && (89)
\end{aligned}
$$

Now, if there is a torque present, we know that the torque must be equal to the time rate of change of the angular momentum.

$$
\tau = \frac{1}{2} q r^2 \frac{\partial B}{\partial t} = \frac{dJ}{dt} \tag{90}
$$

We can integrate this from 0 to B and 0 to t, and find the change in angular momentum due to the presence of the ramped B field. Since we can write magnetic moment in terms of angular momentum, this also gives us a change in magnetic moment:

$$
\Delta J = \frac{1}{2} q r^2 B \tag{91}
$$

$$
\Delta \mu = -\frac{q}{2m} \Delta J = -\frac{q^2 r^2}{4m} B \tag{92}
$$

Notice the minus sign – from Lenz's law – which means that the magnetic moment induced by the field B is *opposite* the applied field. Furthermore, the magnitude of the induced moment is proportional to the applied field. These are the classic signatures of **diamagnetism**, the first type of solid magnetism we will encounter.

The equation above misses a factor $\frac{2}{3}$, since we really only treated the problem two dimensionally. We will get to that in the next lecture.

More troubling, we got the 'right' answer for completely the wrong reasons, by ignoring quantum mechanics and treating the fictitious Bohr atom. Still, the approach above gives a useful insight to diamagnetism, and it is very nearly 'correct' for the most extreme example of diamagnetism, superconductivity.