PH253: Accelerating charges, radiation, and Planck's hypothesis

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I

Thermal Radiation

1.1 Physical model and ingredients

Our task is no small one: we wish to figure out how accelerating charges emit radiation in general, and specifically find the spectrum of radiation emitted from a hot object. Why should hot objects emit radiation? In short, individual charges in atoms acquire random thermal energy, which causes them to oscillate, which causes them to radiate. We aim to calculate the spectrum of radiation emitted, within a simple toy model.

Our procedure goes something like this:

1. Figure out the field from moving charges
2. Find the radiation emitted from accelerating charges, particularly for simple harmonic motion
3. From the power emitted by this radiation, find the radiation reaction force that must be present
4. Use this effective damping force to compute the equation of motion and energy of oscillating, radiating charges
5. Model a hot object as a collection of random oscillators excited by thermal energy
6. Realize the result is silly, and resort to Planck's hypothesis ...

It sounds like a lot, but we will really need nothing more than standard introductory electrodynamics and a good knowledge of the harmonic oscillator. As it turns out, we really only need to figure out what happens for a single charge in harmonic motion.

Subsequent sections marked with a * may be treated as optional. These sections derive formulas required for later sections (e.g., the power radiated by an accelerating charge) from more basic principles, and develop the background necessary a bit further. If you are willing to accept a few new formulas (e.g., power radiated by an accelerating charge) without derivation, they may be safely skipped.

1.2 Electric fields in different reference frames*

First, we must figure out the field due to charges in motion. Unlike length or time, *the amount of charge present is independent of reference frame.* That is, if one observer sees a charge q, all other observers will see the same charge q, independent of their frame of reference. With that in mind, consider the situation in Fig. 1.1 below, where we have a capacitor in reference frame O creating an electric field E due to a charge density σ on its plates of area A. In reference frame O' we have an observer traveling either parallel or perpendicular to the capacitor's electric field at velocity v. What electric field does the observer see?

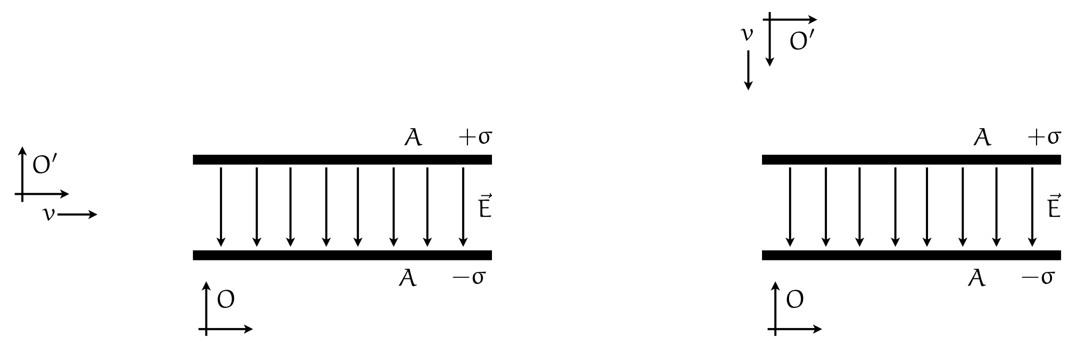

Figure 1.1: (left): *An observer in* O' *travels at velocity* v *perpendicular to the electric field created by a capacitor in frame* O. **(right)** *An observer in* O' *travels at velocity* v *parallel to the electric field created by a capacitor in frame* O.

In the capacitor's reference frame O, we know that the field between the plates is

$$E = \frac{\sigma}{\epsilon_o} = \frac{Q}{A\epsilon_o} \tag{1.1}$$

since the total charge on each plate Q is just σA. Consider now the case where the observer travels perpendicular to the electric field. From the observer's point of view, the dimensions of the capacitor along the direction of motion must be shortened by a factor γ. That means the area of the plates from the point of view of the observer in O' must be smaller by a factor γ. If the total amount of charge Q is invariant, then smaller plates means *a larger apparent charge density*! Thus, the observer in O' must see a charge density

$$\sigma' = \gamma\sigma \tag{1.2}$$

Meaning the electric field in the observer's frame must be

$$E' = \frac{\sigma'}{\epsilon_o} = \gamma\frac{\sigma}{\epsilon_o} = \gamma E \qquad \left(\vec{v} \perp \vec{E}\right) \tag{1.3}$$

The electric field for the observer moving perpendicular to the field is *enhanced* by a factor γ. Now consider the second situation, relative motion parallel to the field. In this case, the *spacing* of the capacitor is contracted according to the moving observer, but the area of the plates remains the same and thus so does the charge density. Since the field between the plates doesn't depend on the spacing,[i] but only the charge density, the field in this case is the same!

$$E' = E \qquad \left(\vec{v} \parallel \vec{E}\right) \tag{1.4}$$

[i]The *capacitance* does depend on the spacing of the plates, but the field does not!

In fact, there is nothing special about the field created by the capacitor, it is just like any other electric field. What we have derived, then, is the transformation of the electric field between different reference frames:

$$\begin{aligned} E'_{\perp} &= \gamma E_{\perp} \\ E'_{||} &= E_{||} \end{aligned} \tag{1.5}$$

Components of the electric field perpendicular to the velocity are increased by a factor γ, components parallel to the velocity are unaffected. This result holds only for charges that are stationary in one of the two frames, if charges are in motion in both frames, we will also have to consider the *magnetic* field present. Incidentally, the *force* must transform the same way, since in any frame the electric force is qE:

$$\begin{aligned} F'_{\perp} &= \gamma F_{\perp} \\ F'_{||} &= F_{||} \end{aligned} \tag{1.6}$$

Again, with the restriction that the charges in question must be at rest in at least one of the two frames. In one of the appendices to this chapter, we show how you can derive the magnetic field from the electric field of moving charges, and state the general field transformation rules when both E and B are present.

1.2.1 Field from a moving point charge*

Armed with the rules for transforming electric fields, we can now consider what the electric field of a moving point charge looks like.[ii] We will imagine that we have a charge q traveling at velocity v along the x axis as measured by an observer in frame O', and the charge's own frame of reference will be O. Thus, in frame O the charge is at rest, while from the point of view of frame O' the charge is in motion at constant velocity v. Since we know that the perpendicular (z)and parallel (x) components of $\vec{E}$ transform differently, we also know that both the magnitude and orientation of the field will be different in O'.

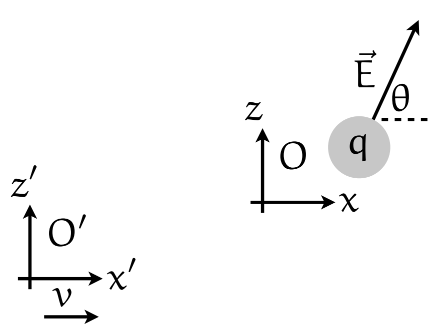

Figure 1.2: *A charge is at rest in frame* O*, while frame* O' *moves with velocity* v *and angle* θ

[ii]In this section we follow the treatment of Purcell[1] closely.

Let us assume that the origins of the two reference frames coincide at $t=0$. In frame O, the charge is at rest, so the field at a distance r from the origin measured in O is:[iii]

$$E = \frac{kq}{r^2} \tag{1.7}$$

Broken down by components, we have

$$E_x = \frac{kq}{r^2}\cos\theta = \frac{kq}{x^2+z^2}\frac{x}{\sqrt{x^2+z^2}} = \frac{kqx}{\left(x^2+z^2\right)^{3/2}} \tag{1.8}$$

$$E_z = \frac{kqz}{\left(x^2+z^2\right)^{3/2}} \tag{1.9}$$

In frame O′, the charge is moving at constant velocity. In order to find the field in O′ we will first need to "translate" the distances as measured in O via the Lorentz transformations:

$$x = \gamma\left(x' - vt'\right) \tag{1.10}$$

$$z = z' \tag{1.11}$$

$$t = \gamma\left(t' - \frac{vx'}{c^2}\right) \tag{1.12}$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{1.13}$$

Based on the previous section, we also know that the component of the field parallel to the relative motion (E_x) will remain constant, but the component of the field perpendicular to the relative motion (E_z) will be augmented by a factor γ:

$$E'_x = E_x \tag{1.14}$$

$$E'_z = \gamma E_z \tag{1.15}$$

Using the field transformation and the Lorentz transformations, we can write down the field according to an observer in O′ for each component:

$$E'_x = E_x = \frac{kqx}{\left(x^2+z^2\right)^{3/2}} = \frac{kq\gamma\left(x'-vt'\right)}{\left(\gamma^2\left(x'-vt'\right)^2+z'^2\right)^{3/2}} \tag{1.16}$$

$$E'_z = \gamma E_z = \frac{kq\gamma z}{\left(x^2+z^2\right)^{3/2}} = \frac{kq\gamma z'}{\left(\gamma^2\left(x'-vt'\right)^2+z'^2\right)^{3/2}} \tag{1.17}$$

[iii] For convenience, we use $k=1/4\pi\epsilon_o$ in this section.

This is something of a mess. However, our main interest here is to find the difference between the electric field observed by the moving and stationary observer at the same location (i.e., when their origins overlap). We aren't particularly worried about time dependence, issues of simultaneity, or time delays in the propagation of electromagnetic influences. Thus, we can transform the fields at time $t = t' = 0$ only, which simplifies things to

$$E'_x = \frac{kq\gamma x'}{\left(\gamma^2 x'^2 + z'^2\right)^{3/2}} \tag{1.18}$$

$$E'_z = \frac{kq\gamma z'}{\left(\gamma^2 x'^2 + z'^2\right)^{3/2}} \tag{1.19}$$

We can already notice that the angle of the field in frame O' is

$$\tan\theta' = \frac{E'_z}{E'_x} = \frac{z'}{x'} \tag{1.20}$$

This tells us that the field in O' points along the radial direction, or that E' makes the same angle with the x' axis that the radial vector r' does. Thus, E' points radially outward from the *instantaneous position* of q. Given both components of the field in E', finding the magnitude of the field is just algebra:[iv]

$$E'^2 = E'^2_x + E'^2_z = \frac{k^2q^2\gamma^2x'^2}{\left(\gamma^2x'^2+z'^2\right)^3} + \frac{k^2q^2\gamma^2z'^2}{\left(\gamma^2x'^2+z'^2\right)^3} = k^2q^2\gamma^2\left[\frac{x'^2+z'^2}{\left(\gamma^2x'^2+z'^2\right)^3}\right] \tag{1.21}$$

$$= k^2q^2\gamma^2r'^2\left[\frac{1}{\left(\gamma^2x'^2+z'^2\right)^3}\right] = \frac{k^2q^2\gamma^2r'^2}{\gamma^6}\left[\frac{1}{\left(x'^2+z'^2/\gamma^2\right)^3}\right] \quad \left(\text{note } \frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2}\right) \tag{1.22}$$

$$= \frac{k^2q^2r'^2}{\gamma^4}\left[\frac{1}{\left(x'^2+z'^2-\left(v^2/c^2\right)z'^2\right)^3}\right] = \frac{k^2q^2r'^2}{\gamma^4}\frac{1}{\left(x'^2+z'^2\right)^3}\frac{1}{\left[1-\frac{v^2}{c^2}\frac{z'^2}{x'^2+z'^2}\right]^3} \tag{1.23}$$

$$\tag{1.24}$$

Still a mess, but we can note that $z'/\sqrt{x'^2+z'^2} = \sin\theta'$, and again use $r'^2 = x'^2 + z'^2$;

$$E'^2 = \frac{k^2q^2}{\gamma^4r'^4}\frac{1}{\left[1-\frac{v^2}{c^2}\sin^2\theta'\right]^3} = \frac{k^2q^2}{r'^4}\frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left[1-\frac{v^2}{c^2}\sin^2\theta'\right]^3} \tag{1.25}$$

$$\Longrightarrow \qquad E' = \frac{kq}{r'^2}\frac{1-\frac{v^2}{c^2}}{\left(1-v^2/c^2\sin^2\theta'\right)^{3/2}} = \frac{q}{4\pi\epsilon_o r^2}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}} \tag{1.26}$$

Finally, we have the field in the frame in which the charge is moving at velocity v. What ends up happening is that the field lines end up being "squashed" along the direction of motion, so the field is

[iv]Note that $r'^2 = x'^2 + z'^2$.

much higher along the perpendicular (z') direction compared to the parallel direction (x'). Below are the field lines for a point charge moving at $0, 0.75c, 0.9c, 0.99c$ to illustrate this "relativistic compression" of field lines.

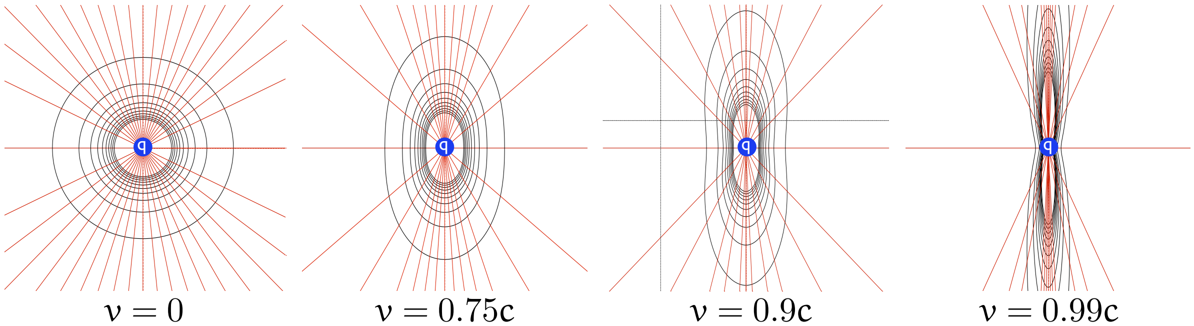

Figure 1.3: *Electric field lines (red) and contours of constant electric field (black) for a point charge moving at various velocities. At all speeds the law is an inverse square, but it is only isotropic for very low speeds.*

As the charge's relative velocity approaches c, the field becomes more and more directional. Along the horizontal axis ($z'=0$, $\theta=0°$), the field is reduced by a factor γ^2 compared to what it would be for a stationary charge,

$$E' = \frac{kq}{\gamma^2 r'^2} \qquad \text{(along } x'\text{)} \tag{1.27}$$

while along the vertical axis ($x'=0$, $\theta=90°$), the field is *enhanced* by a factor γ:

$$E' = \frac{kq\gamma}{r'^2} \qquad \text{(along } z'\text{)} \tag{1.28}$$

It is also the case that no static charge distribution could produce this electric field, or the electric field lines in the figure above. You can convince yourself of that by noticing that the integral of $\vec{E} \cdot d\vec{l}$ around closed paths in the figure above (say, a circle centered on the charge) is *not* zero as it must in electrostatics. Since the line integral of $\vec{E}$ around a closed path is not zero, Maxwell's equations imply a time-varying magnetic flux. Associated with our moving charge is not just an electric field, but also a magnetic field.

1.2.2 Fields of charges that start and stop*

So far, we can figure out the fields from stationary charges, and charges in motion at constant velocity. What about charges that start or stop moving?[v] In order to find the fields in those situations, we need to remember that in free space, electromagnetic influences travel at the speed of light (you saw this in deriving the wave equation in introductory physics). This "cosmic speed limit" implies the existence of electromagnetic radiation, as it turns out.

Let us imagine we have a charge q which is initially at rest, and at time $t=0$ it is suddenly accelerated to a constant velocity v along the x axis. We'll assume a constant acceleration a, and a duration of accelerated

[v] In this section we follow the treatment of Purcell[1] closely.

motion τ, where τ is very short compared to the time scale over which we observe the charge. What does the field look like surrounding the charge?

For an observer at a distance r from the origin at time T after the charge begins accelerating, it depends on whether enough time has passed for the influence of the charge's motion to travel at the speed of light over a distance r. If $r > cT$, then not enough time has passed for the "news" of the charge's motion to have reached the observer, since the news can only travel at the speed of light. Thus, for distances from the origin $r > cT$, the observer at r is unaware that the charge has now been set in motion, and the field still appears as that of a point charge! Moreover, since observers at these distances are unaware that the charge has started moving, outside a spherical shell of radius cT from the origin *the field still appears to be emanating from the charge's position at time* $t = 0$, the origin.

On the other hand, for observers within $r < c(T - \tau)$, enough time has passed that the news of the charge's acceleration has had time to reach the observer, so observers within this radius see the field of a moving point charge. Since observers at these distances are aware of the charge's motion, they also see the charge as having moved forward by an amount $x_o = v(T - \tau) + \frac{1}{2}a\tau^2$. Thus, observers inside a sphere of radius $c(T - \tau)$ see the field of a moving point charge centered at position x_o along the x axis. Figure 1.4 below illustrates the field inside and outside the "sphere of information."

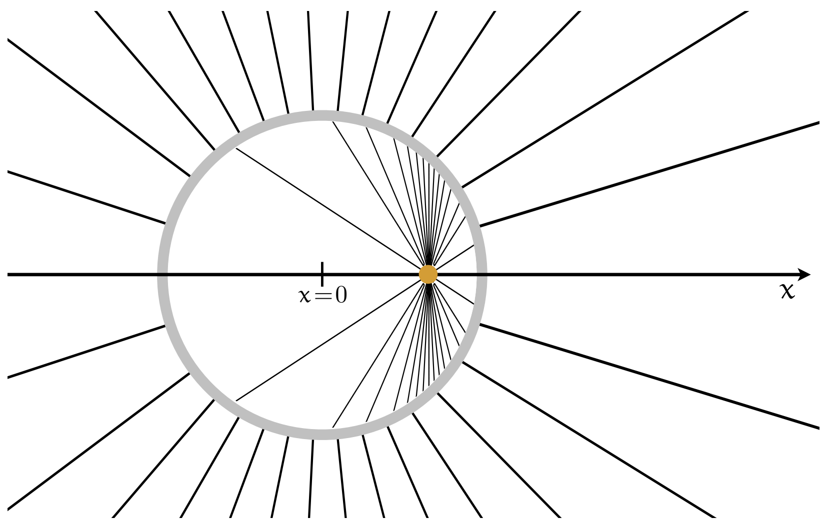

Figure 1.4: *An electron initially at rest in the lab frame is suddenly accelerated at* $t = 0$ *and moves with constant velocity* $0.8c$ *thereafter. Outside a sphere of radius* ct *from the origin, news of the charge's acceleration has not yet arrived, and the field is that of a point charge at rest. Inside the sphere, the field is that of a charge in motion at* $0.8c$. *In the grey region in between, the field lines between the two regions join.*

As time passes, the spherical shell corresponding to the duration of the acceleration moves outward from the origin, and observers at progressively larger distances from the origin begin to see the dramatic change in the field. What happens inside the spherical shell? We know that field lines cannot cross, and that the number of field lines must remain the same so long as the amount of charge q remains constant (field lines can't stop or start in empty space). Thus, the field lines inside and outside the shell must connect to each other within the shell. These connecting lines will no longer be purely radial (either from the origin or the charge's later position), implying that within the shell the field has a *transverse* component as well. In essence, as the charge accelerates it "sheds" part of its electric field within the spherical shell, which travels outward at c. The presence of an electric field in the shell implies that energy is being carried away from the charge, what we usually call *electromagnetic radiation.* This means that the charge is losing the energy contained in the electric field within the shell, and if it is losing energy it must be experiencing a force due to the emission of radiation. We will derive this resistive force in later sections.

As another example, we could consider a charge which suddenly *stops* instead, as shown in Fig. 1.5. In this case, our charge is moving with velocity $v = 0.8c$ until reaching the origin at $t = 0$, and which point it suddenly stops. Outside a sphere of radius ct, the news of the charge's deceleration has not been received, and thus the field appears to be that of a point charge in motion at $0.8c$, emanating from a point vt past the origin on the x axis. Within the spherical shell, information of the charge's deceleration has had sufficient time to propagate, and the field appears as that of a point charge at rest at the origin.

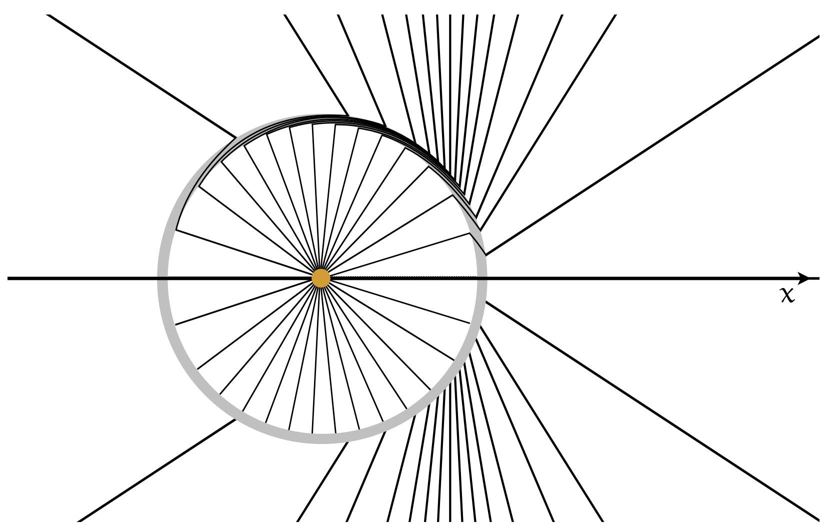

Figure 1.5: *An electron that was moving with constant velocity* $0.8c$ *reaches the origin at* $t = 0$, *suddenly stops, and remains stationary thereafter. Outside a sphere of radius* ct, *the field lines are those of a charge in motion at* $0.8c$, *while inside the sphere the field is that of a point charge at rest. In the spherical shell corresponding to the duration of the deceleration, field lines from inside and outside the sphere connect (shown for the upper portion of the figure only).*

Inside the spherical shell representing the deceleration period, we have shown how the field lines connect in the upper half of the figure. The precise shape of the kinks depends on the details of the acceleration, and are of little interest here. What is important is that they are *transverse* with almost no radial component, and this field within the shell propagates outward as a pulse. Further, given that the electric field is a function of time, there will also be a magnetic field associated, and together the two fields make up an electromagnetic pulse. Figure 1.6 below shows contours of constant power for charge undergoing uniform acceleration along the horizontal axis. In the next section, we'll derive the formula for the radiated power.

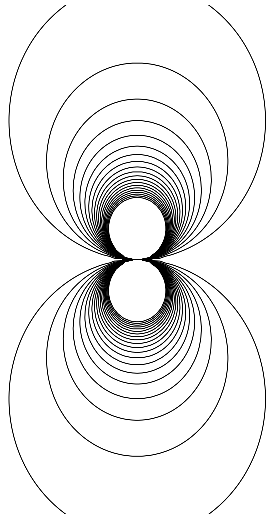

Figure 1.6: *Radiation pattern of a charge accelerating to along the horizontal axis. The different curves are contours of constant emitted power per unit area, in decreasing magnitude further from the origin.*

1.3 Radiation by accelerating charges

In your introductory physics course, you learned about the electromagnetic waves produced by an antenna, and the general fact that accelerating charges emit electromagnetic radiation. In the previous section, we established that a charge that suddenly (but smoothly) accelerates "sheds" part of its electric field as a spherical shell of radiation. The question we wish to answer now is *how much* radiation is emitted by an accelerating charge?[vi]

1.3.1 A uniformly accelerating point charge

We will now consider a charge q which has been traveling at velocity v_o along the x axis for a long time, and suddenly at time $t=0$ it decelerates smoothly for a time τ (implying acceleration $a=v_o/\tau$) until it comes to rest as shown in Fig. 1.7.[vii]

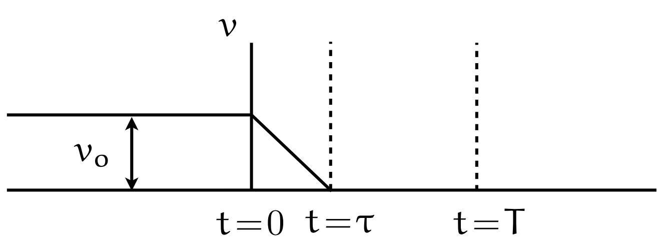

Figure 1.7: *Velocity versus time for the point charge. It travels with velocity* $v_o \ll c$ *until time* $t=0$*, at which point it smoothly decelerates to rest over a time* $t=\tau$.

[vi]This section closely follows Appendix B of [1].
[vii]In this section we follow the treatment of Purcell[1] closely.

From the time the particle begins its deceleration until it stops, it will have moved a distance $x = \frac{1}{2}v_o\tau$ further along the x axis where it comes to rest. Since we presume $v_o \ll c$, this distance is tiny compared to the other relevant distances, viz., the distance traveled by light over the time scales given. At a given position from the charge at some time $t = T \gg \tau$, what does the field look like? We have to be careful again to take into account the fact that the influence of the charge's motion travels outward from the charge at $v = c$, so an observer at a distance d doesn't 'get the news' that the charge stopped until a time $\delta t = d/c$ later! At a time T after the start of the deceleration, observers farther away than $R > cT$ cannot know that the charge has stopped yet, since that would imply communication faster than the speed of light. On the other hand, observers within a radius $R < c(T-\tau)$ will already see the charge as stationary. Within a thin shell of width $c\tau$ at a distance $cT < R < (T\text{–}\tau)$, observers see the charge in the midst of its deceleration.

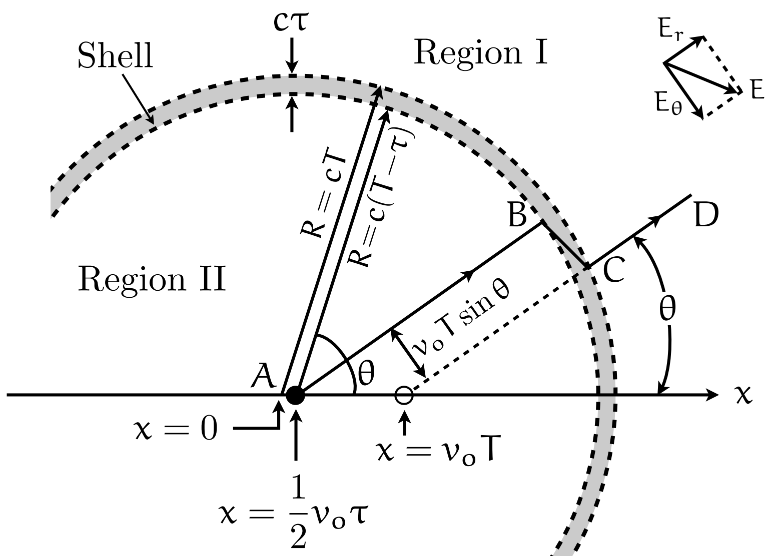

Figure 1.8: *Schematic space diagram of the instant $t = T \gg \tau$, a long time after the particle has decelerated. Inside a radius $R < cT$ (Region II), observers see a particle at rest at position $x = \frac{1}{2}v_o\tau$. Outside a radius $R < c(T-\tau)$ (Region I), observers still see a charge in motion at constant velocity v_o located at $x = v_oT$. A shell of width $c\tau$ between the two regions represents the transition to the field of a moving to a stationary charge. One electric field line ($ABCD$) is shown through all three regions. The diagram in the upper right shows the electric field and its component inside the shell along segment BC.*

We've tried to depict this situation in Fig. 1.8, there are three distinct regions of space categorized by what an observer would see:

Region I: Outside the thin shell, $R > cT$, news of the charge's deceleration has not reached observers, so *the field must still look like that of a charge moving at constant velocity v_o!* In fact, it must appear that nothing has changed, so the field is as though the charge is still moving at v_o, and at position $x = v_oT$ after a time T. At any given time in Region I, the field appears to emanate from the present position of the charge *as if it were still in motion*, $x = v_oT$, but compressed along the axis perpendicular to the direction of motion compared to those of a point charge. One such field line is shown as segment CD in the figure above.

Region II: In this region, time enough has passed for information about the charge's deceleration to reach observers. Inside a radius $R < c(T-\tau)$, see the charge at rest, and the field is simply that of a stationary point charge at position $x=\frac{1}{2}v_o\tau$. The field lines emanate radially from the charge's position, like segment AB in the figure.

The Shell: Between regions I and II, $cT < R < (T-\tau)$ observers are at just the right distance to see the charge in the midst of its deceleration. Our task is to find the field in this region, since we already know the field in the other two regions! Since field lines cannot start and stop in empty space, the field must be represented by segment BC.

What should the field look like in the transition region? Gauss' law provides an answer. Consider a field line like the one connecting points A and B (which would actually form a cone around the x axis). This cone contains a certain amount of the flux from the charge q. If another field line, like CD, makes the same angle with the x axis, the cone it defines must contain exactly the same amount of flux. Since field lines can never cross, it must be true that AB and CD are part of the same field line, connected by segment BC.

What of the field in the shell? It must be along the segment BC connecting the field lines AB and CD, and therefore has radial and tangential components. In region II, we have a simple stationary point charge, with a purely radial field. Gauss' law tells us that the flux through the surface defining the inner surface of the shell can only depend on the charge enclosed within, and the flux itself is determined purely by the radial portion of the field. Since we only ever have the single charge q inside region II, the *radial portion of the field cannot change when going from region II to the shell.* In region II the radial component of the field is just that of a point charge, and it must be the same inside the shell:

$$E_r = \frac{q}{4\pi\epsilon_o R^2} = \frac{q}{4\pi\epsilon_o c^2 T^2} \tag{1.29}$$

Noting from the geometry of the figure[viii]

$$\tan\theta = \frac{E_r}{E_\theta} = \frac{c\tau}{v_o T \sin\theta} \tag{1.30}$$

This gives us the tangential portion of the field:

$$E_\theta = E_r \frac{v_o T \sin\theta}{c\tau} = \frac{q v_o \sin\theta}{4\pi\epsilon_o c^3 T\tau} \tag{1.31}$$

We can also remember that v_o/τ is just the acceleration and $R=cT$, which gives us

$$E_\theta = \frac{q a \sin\theta}{4\pi\epsilon_o c^2 R} \tag{1.32}$$

[viii] Look at a little triangle with BC as a hypotenuse.

One striking thing about this result is that the tangential field goes as $1/R$, not $1/R^2$! As time goes on, meaning R increases, the tangential field will eventually be much stronger than the radial one, owing to its slower $1/R$ decay. Just to review: in region II we have the field of a point charge at constant velocity, which has both radial and tangential components. In region I, we have the purely radial field of a stationary point charge. During the deceleration, the tangential component of the field is 'lost' as radiation, and this radiation emanates outward from the charge at velocity c making up a thin shell of width $c\tau$.

Our next question is then how much energy must be 'lost' by the charge during deceleration, i.e., how much energy is carried away by radiation? This amounts to finding the energy stored in the tangential field within the spherical shell, since the radiation happens only during the deceleration of the charge. The energy density (energy per unit volume) is readily calculated:

$$u_\theta = \frac{1}{2}\epsilon_o E_\theta^2 = \frac{q^2 a^2 \sin^2\theta}{32\pi^2\epsilon_o c^4 R^2} \tag{1.33}$$

The volume of the shell is just surface area times thickness, or $4\pi R^2 c\tau$, so the energy carried away in the tangential electric field shed by the charge is:

$$U_\theta = \frac{q^2 a^2 \tau \sin^2\theta}{8\pi\epsilon_o c^3} \tag{1.34}$$

Of course, in addition to the tangential electric field there must also be a magnetic field whenever charge is in motion. We know that the magnetic field carries the same amount of energy as the electric field, so we can simply double the result above:

$$U_\theta = \frac{q^2 a^2 \tau \sin^2\theta}{4\pi\epsilon_o c^3} \tag{1.35}$$

This is the energy emitted at an angle θ with respect to the x axis. More convenient is the total emitted energy, which means we should average over all θ. You can convince yourself that the average value of $\sin^2\theta$ over a sphere is $\frac{2}{3}$, giving a total energy of

$$\langle U_\theta \rangle = \frac{q^2 a^2 \tau}{6\pi\epsilon_o c^3} \tag{1.36}$$

Here the angle brackets just remind us that we are dealing with an average quantity. Note that since the shell represents the *entire* deceleration process over time τ, what we have just found is the energy dissipated by radiation during the whole deceleration process. What is striking about this result is that the dependence on R has cancelled entirely. This much energy simply emanates outward from the charge at speed c from the site of the charge's deceleration. The deceleration happens over a time τ, so we could also define a power radiated during the deceleration process:

$$P_{rad} = \frac{\langle U_\theta \rangle}{\tau} = \frac{q^2 a^2}{6\pi\epsilon_o c^3} \qquad \text{total emitted power, E and B fields} \tag{1.37}$$

This is the famous Larmor formula for the power radiated by an accelerating charge. If you wanted the power emitted at a particular angle, you could skip the averaging step above and find the angle-resolved power:

$$P_{rad} = \frac{q^2 a^2 \sin^2\theta}{4\pi\epsilon_o c^3} \qquad \text{emitted power from E at angle } \theta \tag{1.38}$$

If we divide that by $4\pi r^2$, we have the power per square meter of surface area at a distance r radiated in direction θ:

$$P_{rad} = \frac{q^2 a^2 \sin^2\theta}{16\pi^2 \epsilon_o c^3 r^2} \qquad \text{emitted power from E per unit area at angle } \theta \tag{1.39}$$

What is interesting about these results is that is the *square* of acceleration that enters the equation for power, meaning the sign of the acceleration is irrelevant. Acceleration and deceleration give the same result, consistent with relativity – after all, what is deceleration in one reference frame is acceleration in another. As it turns out, P_{rad} is also independent of reference frame. Finally, and we've saved the best for last, the Larmor formula above is much more general than we have a right to expect. It works not only for instantaneous acceleration, as we derived it above, but for variable acceleration as well, such as simple harmonic motion.

1.3.2 Oscillating Charges*

As an example, let's consider a charge in simple harmonic motion,[ix] following the trajectory $x(t) = x_o \cos\omega_o t$. We know that in this case the acceleration is $a = -\omega^2 x = -\omega_o^2 x_o \cos\omega_o t$ at any instant, where $\omega_o = 2\pi f_o$ is the natural (angular) frequency of oscillation, x the instantaneous position, and x_o the amplitude of oscillation.[x] Can we just square this and plug it in the Larmor equation? We should be a bit more careful than that – plugging in this acceleration would give us the instantaneous power, but what is more useful is the average power emitted over one full cycle of oscillation. For that we want the average of a^2 over one full cycle. Since the average of $\cos^2\omega t$ is $1/2$, the average squared acceleration per cycle is[xi]

$$\langle a^2 \rangle = \langle -\omega_o^4 x_o^2 \cos^2\omega_o t \rangle = -\omega_o^4 x_o^2 \langle \cos^2\omega_o t \rangle = \frac{1}{2}\omega_o^4 x_o^2 \tag{1.40}$$

Thus the total emitted power must be

$$P = \frac{q^2 \omega_o^4 x_o^2}{12\pi\epsilon_o c^3} \tag{1.41}$$

The fact that the oscillator is emitting power means that it is losing energy, and it must therefore be losing amplitude. Even in empty space, a freely-oscillating charge would eventually stop oscillating due to radiation losses – there is no friction, viscosity, or drag, but nevertheless dissipation occurs via radiation. Physically, our accelerating charge emits radiation – electromagnetic waves – at its resonance frequency

[ix] In this section we follow the treatment of Feynman[2] closely.

[x] Here we're ignoring any irrelevant phase factor, since we're only talking about a single oscillator. More generally, we should write $x(t) = x_o e^{i\omega_o t}$.

[xi] Since ω and x_o are constants, we can bring them out of the averaging brackets.

ω_o.[xii] In fact, the charge could have begun its oscillation in the first place by being excited by incident radiation! One accelerating charge emits EM waves for a time (which we will determine below), until the radiative dissipation fritters away too much of its energy. These EM waves can be absorbed by another nearby charge, which will set *it* in oscillatory motion at the same frequency, leading to further emission of radiation, which can excite another charge ... and now we have propagation of radiation through a medium.

For a "lossy" oscillator, such as the mathematically-equivalent RLC circuit or a mass-spring-damper system, we typically calculate the quality factor Q, a measure of the rate of energy loss through viscous damping. It is defined as[xiii]

$$Q = 2\pi \frac{\text{total energy of oscillator}}{\text{rate of energy loss per radian}} = \omega_o \frac{\text{energy stored}}{\text{power loss}} = \omega_o \frac{\mathcal{E}}{d\mathcal{E}/d\theta} = \frac{\omega_o \mathcal{E}}{P} \tag{1.42}$$

Another equivalent definition of Q is $Q = \omega/\Delta\omega$, where $\Delta\omega$ is the width of the resonance curve. Using $d\mathcal{E}/dt = P$, for a given Q, the rate of energy loss (power dissipation) of the oscillator can be found in terms of Q, $\mathcal{E}$, and ω_o:

$$P = -\frac{d\mathcal{E}}{dt} = -\frac{\omega \mathcal{E}}{Q} \tag{1.43}$$

$$\Longrightarrow \qquad \mathcal{E} = \mathcal{E}_o e^{-\omega_o t/Q} \tag{1.44}$$

where the initial energy of the oscillator is $\mathcal{E}_o$ at $t=0$. The minus sign is just there to signify that energy is being lost, not gained, so $d\mathcal{E}/dt$ must be negative. The energy of the oscillating charge exponentially decays with a time constant of Q/ω_o, just as we would find for an RLC circuit.[xiv] Great, but what is Q for our oscillating charge?

The average energy of a simple harmonic oscillator, you may recall, is always half kinetic and half potential, for a total of

$$\langle \mathcal{E} \rangle = \frac{1}{2} m \omega_o^2 x_o^2 \tag{1.45}$$

for an oscillator of mass m. If our oscillator is vibrating at its natural frequency ω_o, this gives us

$$\frac{1}{Q} = \frac{P}{\omega_o \mathcal{E}} = \frac{q^2 \omega_o^4 x_o^2}{12\pi\epsilon_o c^3} \left(\frac{1}{\frac{1}{2} m \omega_o^2 x_o^2} \right) \left(\frac{1}{\omega_o} \right) = \frac{q^2 \omega_o}{6\pi\epsilon_o c^3 m} \tag{1.46}$$

In terms of wavelength $\lambda_o = 2\pi c/\omega$,

[xii] There will be a spread in the emitted frequencies dictated by the degree of dissipation and the "quality factor" of the oscillator, which we discuss below.

[xiii] In this section we will use $\mathcal{E}$ for energy to avoid confusion with the electric field E.

[xiv] For a series RLC circuit, $Q = (1/R)\sqrt{L/C}$

$$\frac{1}{Q} = \frac{q^2}{3\epsilon_o mc^2\lambda_o} = \left(\frac{q^2}{4\pi\epsilon_o mc^2}\right)\left(\frac{1}{\lambda_o}\right)\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3}\frac{r_e}{\lambda_o} \tag{1.47}$$

The combination $r_e = q^2/4\pi\epsilon_o mc^2$ has units of length, and is known as the *classical electron radius* if the charges we are dealing with are individual electrons of charge $q = e$. The Q factor depends only on the ratio of the classical electron radius to the wavelength of radiation under consideration, which makes Q dimensionless overall as it must be. For $q = e$, the numeric value of r_e is

$$r_e = \frac{e^2}{4\pi\epsilon_o mc^2} \approx 2.8 \times 10^{-15}\,\text{m} \tag{1.48}$$

The electron is, as far as we can tell, a point particle. The classical electron radius is based on an (incorrect) model of the electron, in which the electron is imagined as a uniform sphere of charge. In this model, r_e is roughly the size an electron would need to be for its rest energy to be completely due to electrostatic potential energy, ignoring quantum mechanics. We know now that the electron's rest energy is not electrostatic in nature, and quantum mechanics is required to understand the behavior of electrons on small distance scales. Still, r_e sets a semi-classical length scale for problems involving electrons, below which subtle quantum effects become extremely important.

Armed with this information, what is the Q value for a typical atom? For a sodium discharge lamp, the dominant emission is at a wavelength of about $\lambda = 600\,\text{nm}$ (in the yellow region of the spectrum), so

$$Q = \frac{4\pi r_e}{\lambda_o} = \frac{3\epsilon_o mc^2\lambda_o}{e^2} \sim 10^8 \tag{1.49}$$

The Q for a typical atom emitting visible light is $\sim 10^7 - 10^8$, meaning an atomic oscillator will oscillate for $10^7 - 10^8$ radians or $\sim 10^7$ cycles before the energy is reduced by a factor $1/e \approx 1/2.718 \approx 0.37$.[xv] A wavelength of 600 nm implies a period of $\sim 10^{-15}$ s, so it takes about 10^{-8} s for the energy of a freely-oscillating atom in empty space to decay by a factor of $1/e$. It doesn't seem like much, but this is an eternity for an atom! Of course, for atoms in a solid or liquid, we have bonding and interactions between atoms to worry about, not to mention collisions, so there are additional sources of damping that decrease this time (and make it temperature-dependent).

Finally, we should note that the Q factor can be related to the *damping constant* γ of an oscillator, which is mathematically the coefficient of the 'viscous' force proportional to velocity:[xvi]

$$\frac{1}{Q} = 2\gamma \tag{1.50}$$

Evidently,

[xv] Compare this to $Q = R\sqrt{C/L} \sim 10 - 100$ for typical circuit applications, possibly up to 10^6 for very precise circuits! For a laser cavity, one can achieve $Q \sim 10^{11}$.

[xvi] For a series RLC circuit, $\gamma = (R/2)\sqrt{C/L}$.

$$\gamma = \frac{q^2 \omega_o}{12\pi\epsilon_o c^3 m} \tag{1.51}$$

In the following sections, we will derive the damping factor by considering the forces on an oscillating charge, but we will of course come to the same result.

Knowledge of the damping factor or Q factor also allow us to find the width of the resonance $\Delta\omega$, since $\Delta\omega = \omega_o/Q$. More useful is typically the linewidth $\Delta\lambda$ as a function of wavelength. Since $\lambda_o = 2\pi c/\omega_o$,[xvii] the variation in λ_o is

$$\Delta\lambda = \frac{2\pi c \Delta\omega}{\omega_o^2} = \frac{2\pi c}{Q\omega_o} = \frac{e^2}{3\epsilon_o mc^2} = \frac{e^2}{4\pi\epsilon_o mc^2}\frac{4\pi}{3} = \frac{4\pi r_e}{3} \tag{1.52}$$

For our sodium atom, this is amounts to $\Delta\lambda \sim 10^{-14}$ m. The relative linewidth (the "sharpness" of the line) is then

$$\frac{\Delta\lambda}{\lambda_o} = \frac{4\pi r_e}{3\lambda_o} \sim 10^{-8} \tag{1.53}$$

1.3.3 Charges in Circular Motion*

Another useful example of accelerated motion of charges is uniform circular motion.[xviii] If we put a charge q traveling at velocity v in a magnetic field B perpendicular to v, we know that the charge q will follow a circular path. We can find the radius of that path by noting that the magnetic force on the particle must provide the centripetal force to maintain the circular path:

$$qvB = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{mv}{qB} \tag{1.54}$$

The acceleration is just v^2/r, which is the magnetic force per unit mass:

$$a = \frac{v^2}{r} = \frac{v^2 qB}{mv} = \frac{qvB}{m} \tag{1.55}$$

Of course, for circular motion we also know that the charge will repeat its motion with an angular frequency $\omega = v/r$:

$$\omega = \frac{v}{r} = \frac{a}{v} = \frac{qB}{m} \equiv \omega_c \tag{1.56}$$

This frequency ω_c is called the *cyclotron frequency*, and the radius of the path is called the *cyclotron radius*. With this in hand, we can use the Larmor formula to find the power radiated by the charge:

[xvii] Since $\Delta\lambda$ is an essentially an uncertainty, we must use propagation of uncertainty to find it. This is something you will encounter in your laboratory classes if you have not already; if it is unfamiliar, see, for example, `http://en.wikipedia.org/wiki/Propagation_of_uncertainty`. Basically, you differentiate both sides, giving $d\lambda = 2\pi c\, d\omega/\omega_o^2$, and presume small enough changes to turn the differentials into discrete changes.

[xviii] In this section we follow the treatment of Bekefi and Barrett[3] closely.

$$P = \frac{q^2 a^2}{6\pi\epsilon_o c^3} = \frac{q^2 \omega_c^2 v^2}{6\pi\epsilon_o c^3} \tag{1.57}$$

We can go even further than this, however: in uniform circular motion the potential and kinetic energies are equal. Since the kinetic energy is $K = \frac{1}{2}mv^2$, the total energy is $\mathcal{E} = mv^2$. We can then substitute $v^2 = \mathcal{E}/m$ into the Larmor equation above, and replace ω_c with qB/m:

$$P = \frac{q^2 \omega_c^2 \mathcal{E}}{6\pi\epsilon_o mc^3} = \frac{q^4 B^2 \mathcal{E}}{6\pi\epsilon_o m^3 c^3} \tag{1.58}$$

This is the radiated power, which is just the rate at which energy is lost: $P = -d\mathcal{E}/dt$, where the minus sign signifies that energy is being *lost* by the charge. Using this, we can find the energy of the charge as a function of time:

$$P = -\frac{d\mathcal{E}}{dt} = \frac{q^4 B^2}{6\pi\epsilon_o m^3 c^3} \tag{1.59}$$

$$\frac{d\mathcal{E}}{\mathcal{E}} = \frac{-q^4 B^2}{6\pi\epsilon_o m^3 c^3} \tag{1.60}$$

We can integrate both sides easily to solve for $\mathcal{E}$. For convenience, let $\tau \equiv 6\pi\epsilon_o m^3 c^3/q^4 B^2$, and let the charge's initial energy at $t=0$ be $\mathcal{E}_o$. Then the energy as a function of time is:

$$\mathcal{E} = \mathcal{E}_o e^{-t/\tau} \tag{1.61}$$

The energy of the charge decays exponentially with time constant τ, which means the charge will *not* maintain circular motion, but will follow shrinking spiral path until it eventually stops.

Incidentally, if we use the electric force of a nucleus in place of the magnetic force to keep an electron in circular motion, we have the classical planetary model of the atom ... which we can already see cannot possibly be stable. This problem is worked out in more detail in the following section.

1.3.4 Orbiting Charges: Classical Atoms*

In a hydrogen atom an electron of charge $-e$ orbits around a proton of charge $+e$. The electron must be constantly accelerating to stay in circular motion, which means it is radiating. This loss of energy implies a decaying orbit, which means after some time the electron will simply crash into the proton. An approach to finding out how long it will take might be as follows:

(a) Find the total energy E as a function of r, the distance between the electron and proton.
(b) Calculate the energy radiated per unit time as a function of r.
(c) Using $dr/dt = (dr/dE)(dE/dt)$, find the time it takes for a hydrogen atom to collapse from a radius of 10^{-9}m to a radius of 0.

The total energy is kinetic plus potential. The potential energy is that of two point charges e and $-e$ separated by a distance r. If we take the frame of reference that the (much heavier) proton is at rest, the kinetic energy is just that of the electron, to which we will assign mass m and velocity v:

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_o r} \tag{1.62}$$

This equation has the electron velocity present, and we wish to find the energy as a function of radius only. We can eliminate the velocity by noting that the electric force between the proton and electron is constrained to equal the centripetal force required to maintain circular motion. That is,

$$\frac{-e^2}{4\pi\epsilon_o r^2} = -\frac{mv^2}{r} \qquad \Longrightarrow \qquad mv^2 = \frac{e^2}{4\pi\epsilon_o r} \tag{1.63}$$

Substituting into our first equation,

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_o r} = \frac{e^2}{8\pi\epsilon_o r} - \frac{e^2}{4\pi\epsilon_o r} = -\frac{e^2}{8\pi\epsilon_o r} \tag{1.64}$$

Just like gravitational orbits, the total energy is half of the potential energy. Given that the electron is in circular motion, it is accelerating, which means it must be radiating. The Larmor formula gives us the average radiated power, or energy per unit time:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{1.65}$$

Here we have inserted the minus sign because we know that the electron is losing energy by radiating. The acceleration a can be found from our force balance above, diving through by mass m:

$$a = -\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_o mr^2} \tag{1.66}$$

Using the right-most form, we can find the power in terms of radius alone:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\epsilon_o c^3} = -\frac{e^2}{6\pi\epsilon_o c^3}\left(\frac{e^2}{4\pi\epsilon_o mr^2}\right)^2 = -\frac{e^6}{96\pi^3\epsilon_o^3 m^2 c^3 r^4} \tag{1.67}$$

If the electron is radiating, it is losing energy, which means its orbit must be decaying. With the power in hand, we can calculate the rate at which the radius of the electron's orbit decays and figure out how long such an atom would be stable. Using the chain rule

$$\frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \frac{dE}{dt}\bigg/\frac{dE}{dr} \tag{1.68}$$

Since dE/dt is the power we just found, we need only dE/dr:

$$\frac{dE}{dr} = \frac{d}{dr}\left(-\frac{e^2}{8\pi\epsilon_o r}\right) = \frac{e^2}{8\pi\epsilon_o r^2} \tag{1.69}$$

Putting it together,

$$\frac{dr}{dt} = \frac{dE}{dt} \Big/ \frac{dE}{dr} = -\frac{e^6}{96\pi^3\epsilon_o^3 m^2c^3r^4}\left(\frac{8\pi\epsilon_o r^2}{e^2}\right) = -\frac{e^4}{12\pi^2\epsilon_o^2 m^2c^3r^2} = -\left(\frac{e^4}{12\pi^2\epsilon_o^2 m^2c^3}\right)\frac{1}{r^2} \quad (1.70)$$

For convenience, let $C = \frac{e^4}{12\pi^2\epsilon_o^2 m^2c^3}$. This hideous combination is just a constant anyway, lumping it all together means we just have to keep track of one constant instead of 6. Our equation then reads

$$\frac{dr}{dt} = -\frac{C}{r^2} \quad (1.71)$$

This equation is separable[xix]:

$$r^2\,dr = -C\,dt \quad (1.72)$$

Integrating both sides, and noting that we start at time $t=0$ at radius $r_i = 10^{-9}$ m and end at time t with radius zero,

$$\int_{r_i}^{0} r^2\,dr = -\frac{1}{3}r_i^3 = \int_0^t -C\,dt = -Ct \quad (1.73)$$

$$t = \frac{r_i^3}{3C} \quad (1.74)$$

Substituting our definition of C, the time for the electron to reach the proton is

$$t = \frac{4\pi^2\epsilon_o^2 m^2c^3}{e^4} r_i^3 \quad (1.75)$$

With the given radius of $r_i = 10^{-9}$ m, $t \sim 10^{-7}$ s. Using a more realistic radius for the lowest energy state of a hydrogen atom, $r_i \approx 5\times10^{-11}$ m, one finds $t \sim 10^{-11}$ s.

Moral of the story: classical atoms are not stable.

1.3.5 Radiation Reaction Force

From classical electrodynamics, we know that accelerating charges, such as oscillating charges, radiate electromagnetic waves, and therefore lose energy. If the oscillating charge is losing energy, it is also losing amplitude, and thus the radiation loss by the charge amounts to an effective damping force. In effect, the act of radiating acts as a recoil force on the accelerating charge, or a dissipation mechanism in some ways similar to viscous drag on a mechanical oscillator. The Larmor formula derived above relates the radiated power to the acceleration of the charge:

$$P = \frac{e^2a^2}{6\pi\epsilon_o c^3} \quad (1.76)$$

[xix] If we close our eyes and manipulate the differentials like fractions, we would cross multiply to separate the equation.

The power of a mechanical system can be found in general from a knowledge of force and velocity:

$$P = \int \vec{F} \cdot \vec{v}\, dt \tag{1.77}$$

Let us consider the power emitted by our oscillator from time t_1 to time t_2, and let this time interval correspond to exactly one period of motion for our oscillator, i.e., $t_1 - t_2 = T = 1/f$. We will consider our oscillating charge to be a simple point charge of mass m and charge e with a natural resonance frequency of $\omega_o = 2\pi f$.[xx] Conservation of energy dictates that the power radiated away by the charge integrated over time must equal the mechanical power lost by the oscillator:

$$0 = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt + \int_{t_1}^{t_2} P\, dt \qquad \text{or} \qquad \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = -\int_{t_1}^{t_2} P\, dt \tag{1.78}$$

Here we have restricted ourselves to non-relativistic velocities ($v \ll c$) since we used the classical form of momentum for mechanical power and force. Using Eq. 1.37, and noting $a = dv/dt$,

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = -\int_{t_1}^{t_2} P\, dt = -\int_{t_1}^{t_2} \frac{e^2 a^2}{6\pi\epsilon_o c^3}\, dt = -\int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt}\, dt \tag{1.79}$$

We can integrate by parts, yielding[xxi]

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d\vec{v}}{dt} \cdot \vec{v}\Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt \tag{1.80}$$

Since we are integrating over a full cycle of oscillation, the first term vanishes because $\frac{d\vec{v}}{dt} \cdot \vec{v}$ has the same value for equivalent points in the cycle of oscillation. Thus,

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt \tag{1.81}$$

We can readily identify

$$\vec{F} = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^3\vec{x}}{dt^3} \tag{1.82}$$

This is the effective damping force acting the oscillating charge due to the fact that it is radiating. This "recoil" force is known as the *Abraham-Lorentz force.* Physically, the emitted radiation carries away

[xx] This is equivalent to saying our mass m is connected to a spring of spring constant k, if you like.

[xxi] Recall $\int f \frac{dg}{dx}\, dx = fg - \int g \frac{df}{dx}\, dx$. Use $f = g = dv/dt$.

momentum (since we know EM radiation carries momentum), and conservation of momentum dictates that the charge must be pushed in the direction opposite the direction of the emitted radiation. This is an unusual force, in that the charge is feeling a force in response to its own radiation! Essentially, we have just calculated one special case of the effect a charge has on itself - an odd problem to consider, in light of what we know of Newton's third law, but it is this problem which led to the development of quantum electrodynamics (QED), perhaps the most accurately-tested theory in all of physics.

1.3.6 Equation of motion for an oscillating charge

Our oscillating charge will experience a damping force due to the radiation it emits, and this damping force will act on the oscillatory motion in much the same way as a viscous fluid drag would on a mechanical oscillator. Not *exactly* the same, but within certain (reasonable) limits, we can reduce the problem of our oscillating charge to the familiar one of a damped harmonic oscillator.[xxii]

Without damping, the equation of motion for a simple harmonic oscillator of resonant frequency ω_o is[xxiii]

$$F = ma = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} = -kx = -m\omega_o^2 x \tag{1.83}$$

In the present situation, we must also include the radiation reaction force derived above, which acts as the same direction as the restoring force: $m\omega_o^2 x$:

$$\begin{aligned} F = m\frac{d^2x}{dt^2} &= -m\omega_o^2 x - \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} \\ 0 = m\frac{d^2x}{dt^2} &+ m\omega_o^2 x + \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} \end{aligned} \tag{1.84}$$

With the radiation reaction force present, the amplitude of oscillation will decay with time, as a would be the case for a mechanical oscillator (though in a somewhat more complicated way, given that the form of the damping force is different). We are not interested in the isolated case of a single oscillator, however, but rather the case where the oscillator is interacting with an electric field, particularly that due to thermal radiation in subsequent sections. That is, we wish to consider a *driven* oscillator.

The simplest possible case would be to consider what happens when our oscillating charge is exposed to a monochromatic electric field, i.e., an electric field which varies sinusoidally with time with a single frequency $\omega = 2\pi f$:

$$|\vec{E}| = E_o \cos\omega t \tag{1.85}$$

[xxii] In this section we follow portions of the treatment by Feynman[2].
[xxiii] Again, for a mass-spring system, $\omega_o = \sqrt{k/m}$.

where in general $\omega \neq \omega_o$, i.e., the frequency of the driving electric field is not necessarily identical to the resonance frequency of the oscillating charge. This time-varying electric field, the electric portion of an EM wave, will produce a time-varying force $e|\vec{E}|$ on our charge, which is the driving force for our oscillator. Adding this driving force to our already-damped oscillator (Eq. 1.84):

$$m\frac{d^2x}{dt^2} + m\omega_o^2 x + \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} = eE_o \cos\omega t \tag{1.86}$$

This is a tough equation, more than we wish to handle. What we would really like is to somehow make this equation look like the driven harmonic oscillator we already know and love.[xxiv] But what to do with that ugly third derivative?

The situation is not so bad as it seems. In most cases of interest, the radiation resistance force is small compared to the restoring force giving rise to the oscillation (the atomic bonds).[xxv] In this case of small damping, the acceleration is *approximately* the same as it is without damping, or $a \sim \omega_o^2 x$. If this is the case,

$$\frac{d^2x}{dt^2} \sim \omega_o^2 x \qquad \text{or} \qquad \frac{d^3x}{dt^3} = \frac{da}{dt} \sim \omega_o^2 \frac{dx}{dt} \tag{1.87}$$

The basic idea is this: the damping term with the third derivative is small in Eq. 1.86, so for that term we will use the substitution above as a good approximation. The other terms we will leave alone, since we have no reason to presume they are small, and we know how to deal with them anyway. This gives us:

$$m\frac{d^2x}{dt^2} + m\omega_o^2 x + \frac{e^2\omega_o^2}{6\pi\epsilon_o c^3}\frac{dx}{dt} = eE_o \cos\omega t \tag{1.88}$$

or

$$\frac{d^2x}{dt^2} + \omega_o^2 x + \frac{e^2\omega_o^2}{6\pi\epsilon_o mc^3}\frac{dx}{dt} = \left(\frac{eE_o}{m}\right)\cos\omega t \tag{1.89}$$

If we define a "damping constant" γ

$$\gamma = \frac{e^2\omega_o}{12\pi\epsilon_o mc^3} \tag{1.90}$$

we can make our equation of motion just like that of a driven harmonic oscillator with a viscous damping proportional to velocity, or an LC resonant circuit with resistance included.[xxvi]

[xxiv] See `http://en.wikipedia.org/wiki/Harmonic_oscillator` for a quick review.

[xxv] We can make an order-of-magnitude estimate from Eq. 1.82: presuming an amplitude of vibration of 0.1 nm (very large for an atom!), incident red light ($\omega_o/2\pi = f_o \sim 5\times10^{14}$ Hz), and a maximum acceleration of $\omega_o^2 A$ over a time of $1/f \approx 10^{-15}$ s, we find a force in the 10^{-18} N range. Using as an example the force constant for an HCl molecule, $k \sim 500$ N/m, and a displacement of 0.1 nm from equilibrium we find a restoring force of order 10^{-8} N, a comfortable ten orders of magnitude larger than the radiation resistance. This is consistent with our estimate of $Q \sim 10^8$ for an oscillating atom in empty space, another way of saying the dissipation is small.

[xxvi] Note that this is the same damping constant we found in Eq. 1.51!

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{eE_o}{m}\cos\omega t \tag{1.91}$$

$$\implies \qquad x(t) = A\cos(\omega t + \varphi) \tag{1.92}$$

The table below shows the analogous quantities for series and parallel RLC circuits and a mechanical oscillator

	Series RLC	**Parallel RLC**	**Mechanical**
restoring	inverse capacitance $1/C$	inverse inductance $1/L$	spring constant k
"mass"	inductance L	capacitance C	mass m
friction	R	$1/R$	damping coefficient c
damping γ	$\frac{1}{2}R\sqrt{C/L} = \frac{1}{2}RC\omega_o = R/2L\omega_0$	$\frac{1}{2R}\sqrt{L/C} = \frac{1}{2R}L\omega_o = 1/2RC\omega_o$	c/m
ω_o	$\sqrt{1/LC}$	$\sqrt{1/LC}$	$\sqrt{k/m}$
$Q = 1/2\gamma$	$\frac{1}{R}\sqrt{L/C}$	$R\sqrt{C/L} = RC\omega_o = R/L\omega_o$	$m/2c$

The steady-state solution to this equation given above can be found readily with complex exponentials; we will presume you have done this sort of thing before. If not ... you will. Many, many times. The solution gives us the amplitude A and phase φ of vibration of the oscillator as a function of the driving frequency ω and the damping constant γ:

$$A = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{1.93}$$

$$\varphi = \tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2}\right) \tag{1.94}$$

From the amplitude, one can also find the resonance frequency $\omega_r = \omega_o\sqrt{1-2\gamma^2}$, which for small damping reduces[xxvii] to $\omega_r \approx \omega_o(1-\gamma^2) \approx \omega_o$. First, from the phase we can see that for $\omega < \omega_o$, low driving frequencies, the phase angle is small and the charge will oscillate in sync with the driving field. However, when $\omega > \omega_o$, the displacement is in the opposite direction from the driving force, 180° degrees out of phase with the field. Consequently, the amplitude strongly decreases above ω_o, and more gradually below ω_o. The amplitude displays a sharp peak in the region where the driving frequency matches the oscillator's resonance frequency, $\omega = \omega_r$, as shown in Fig. 1.9

Given the amplitude, we can also find the potential energy of the oscillator, $U = \frac{1}{2}m\omega_o^2A^2$. Averaged over a whole cycle, the kinetic and potential energies of the oscillator are the same, so the total average energy is just $m\omega_o^2A^2$. We will make use of this in the following sections.

[xxvii] Using $(1+x)^n \approx 1 + nx$.

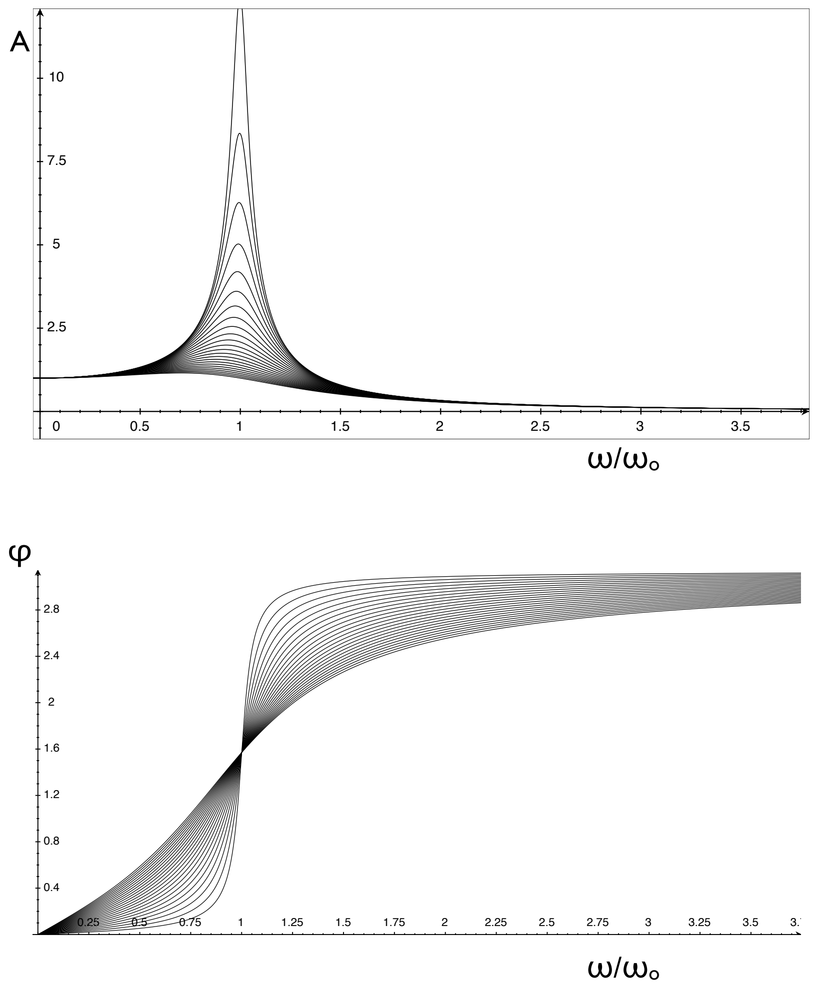

Figure 1.9: (upper) *Relative amplitude of oscillation versus driving frequency with* γ *ranging from* 0.04 *(top curve) to* 0.5 *(bottom curve) in steps of* 0.02. *The linewidth of the resonance curve is* $\omega_o/2Q$. **(lower)** *Phase in radians versus driving frequency with* γ *ranging from* 0.04 *(sharpest curve) to* 0.5 *(smoothest curve) in steps of* 0.02.

As a quick sanity check on our answer, we can check that our result makes sense reducing our result to the case of no damping, $\gamma \to 0$, which gives

$$A = \frac{eE_o}{m\left(\omega_o^2 - \omega^2\right)} \qquad (\gamma \to 0) \tag{1.95}$$

This is just what we expect for a driven oscillator without damping. If we remove the periodicity of the driving force ($\omega \to 0$) just have a free oscillator in a static electric field:

$$A = \frac{eE_o}{m\omega_o^2} \qquad (\omega \to 0) \tag{1.96}$$

Note that this is the same as $m\omega_o^2 A = kA = eE$, just a simple force balance at the oscillator's extremal points.

What have we learned over all? Our charged oscillator is driven by a periodic electric field, and this field 'feeds' energy into the oscillator,[xxviii] which is in turn drained away by radiation damping. That is, the charge absorbs energy from the electric field, and reemits it as radiation at the same frequency. This leads to a steady-state equilibrium, in which the energy gained from the field balances the energy lost by radiation.

More importantly, we are slowly building up a model of the interaction of radiation and matter. We can imagine that our oscillating charges are not bare electrons, but perhaps the most weakly-bound electrons in the atoms of a gas. What we are really doing is trying to figure out how radiation – light – is emitted and absorbed by matter.

1.3.7 Scattering of Light*

What if instead of a single oscillating charge in a single atom, we have many? In a nice crystal, we would expect that we have constructive and destructive interference of emitted radiation due to the regular, periodic arrangement of atoms. If we consider a *random* collection of atoms with oscillating charges, however, overall there is no net constructive or destructive interference, and the total intensity is just the sum of the intensities of all the individual atoms. Even in a regular crystal, random thermal motion of the atoms means that at any given moment strict periodicity is broken, and so the strict condition for interference is also broken. Essentially, we assume that all the atoms incoherently emit radiation, and so we can just figure out the radiative properties of a single atom and multiply by the number of atoms. Physically, what we have is incident light in a single direction falling on an atom, and being reemitted over a range of angles, or what we usually call *scattering*.[xxix]

What we wish to figure out now is what happens when an incident beam of light (an EM wave) strikes at atom. We know the incident light beam has an electric field component like $\vec{E} = \vec{E}_o e^{i\omega t}$, and when it strikes an atom an electron in the atom will feel a periodic force $q\vec{E}$ and begin to vibrate up and down. Thus, the charge accelerates, and re-radiates some of the energy it received from the incident electric field. This is *scattering* of light, and more importantly, it is again our driven harmonic oscillator. We know the amplitude of vibration will be given by Eq. 1.93, so the position as a function of time is:

$$x(t) = \frac{eE_o/m}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2}} \cos\left(\omega t + \varphi\right) \tag{1.97}$$

Since we know that we have a large Q factor for an isolated atom, we will for the moment neglect damping ($\gamma \to 0$) to simplify matters. We could also try to take into account that the electron might act as an oscillator with several different frequencies, but we will also neglect this complication. Without damping, we have an amplitude

[xxviii] Sort of in the same way that in pushing a person on a swing you are the driving force, feeding periodic energy to maintain the oscillations. Similarly, the resultant amplitude is largest when your driving frequency matches the natural frequency of the person and the swing, and if your pushes are out of phase with the swing, oscillations are effectively reduced.

[xxix] In this section we follow the treatment of Feynman[2] closely.

$$x(t) = \frac{eE_o \cos \omega t}{m\left(\omega_o^2 - \omega^2\right)} \tag{1.98}$$

From this, we can find the acceleration and calculate the power re-radiated by the charge in any given direction using Eq. 1.39. A somewhat simpler task is to just find the total emitted power. We can use Eq. 1.37 and the acceleration determined from $x(t)$ above, which is in fact what we already did in deriving Eq. 1.41. All we need to do is replace the the amplitude for a free harmonic oscillator x_o with the amplitude of our driven harmonic oscillator given by Eq. 1.93 with the damping γ set to zero:

$$P = \frac{e^2\omega^4 A^2}{12\pi\epsilon_o c^3} = \frac{e^2\omega^4}{12\pi\epsilon_o c^3}\frac{e^2E_o^2}{m^2\left(\omega_o^2 - \omega^2\right)^2} = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{e^4}{6\pi\epsilon_o^2 c^3 m^2}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.99}$$

If we substitute for the classical electron radius (Eq. 1.48), we find

$$P = \left(\frac{1}{2}\epsilon_o E_o^2\right)\frac{e^4}{16\pi^2\epsilon_o^2 m^2 c^4}\left(\frac{8\pi c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{8\pi r_e^2 c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.100}$$

The most important result thus far is that the scattered energy goes as the *square* of the field, or as the (time-averaged) energy density of the incident field which is $\frac{1}{2}\epsilon_o E_o^2$. Since the *intensity* of electromagnetic radiation goes as E^2, the scattered radiation intensity is proportional to the incident radiation intensity. Basically: the brighter the source, the brighter the scattered light!

We can look at this in another way, however. Say we have light going through a surface of area σ. How much radiant energy passes through that surface in a given time t? It would be the energy density of the field, multiplied by the area σ, multiplied by the distance that light can travel during time t, or ct. The rate at which energy passes through the surface, the power transmission, is then just that energy divided by t, or $\frac{1}{2}\epsilon_o c E_o^2 \sigma$. Comparing that to what we have in Eq. 1.100 already, we notice for the scattered light

$$\sigma = \left(\frac{8\pi r_e^2}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.101}$$

Indeed, the right-hand side does have units of area! What is the meaning of this area? An atom scatters a certain total amount of radiation, which would then end up falling on a certain area, and it is this area σ that we just found. Our identification of σ above amounted to taking the ratio of the total energy scattered per second to the incident energy per square meter:

$$\sigma = \frac{P}{\frac{1}{2}\epsilon_o c E_o^2} = \frac{\text{total scattered energy per second}}{\text{incident energy per square meter per second}} \tag{1.102}$$

The area σ is usually called a *scattering cross section*, and it is a concept that is used frequently in physics. The idea is that the energy intercepted by the area σ is the same as that scattered by the atom. In other words, it is a measure of how much of the beam we would need to block to scatter away as much of the incident light as the atom does. In that way it is a sort of characteristic 'size' associated with scattering, and we could compare the these sizes for different scattering mechanisms to gauge their relative strengths[xxx] There isn't any real physical area to speak of – just oscillating point charges – but the *effect* is the same as if we made a tiny beam block of area σ to scatter away some of the incident light. Based on the definition above, and the fact that $\frac{1}{2}\epsilon_o E_o^2$ is just the average energy per unit volume of the incident electric field, the scattered power must be

$$P_{\text{scattered}} = \sigma c \langle u_E \rangle = \sigma I_{\text{incident}} \tag{1.103}$$

where I_{incident} is the *irradiance*, a common measure of radiation intensity. Irradiance is the energy flux per unit area, averaged over one period of oscillation, and it can be found from $I = c\langle u_{\text{field}} \rangle$. This is a sensible result: the scattered intensity is proportional to the incident intensity, so again the brighter the source, the brighter the scattered light!

Incidentally, the cross section we've found does not include radiation damping. If we repeat our derivation above without neglecting damping,[xxxi] things are only slightly more complex, and it is clear that non-zero damping *reduces* the cross section:

$$\sigma = \left(\frac{8\pi r_e^2}{3}\right) \frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2} \tag{1.104}$$

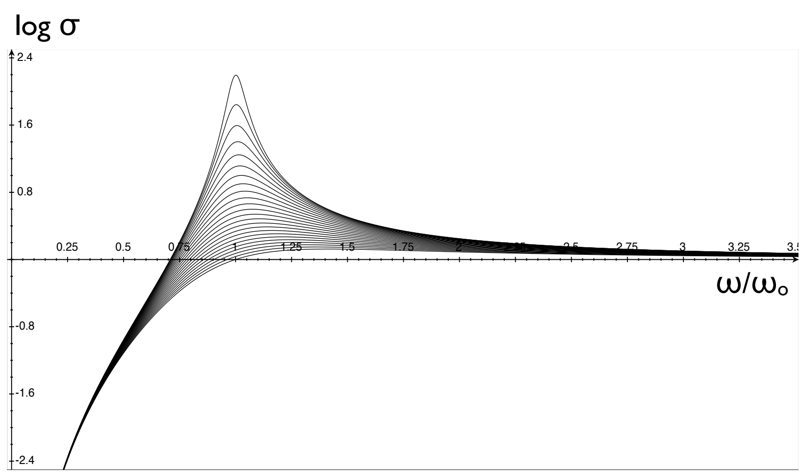

Figure 1.10: *Logarithm of the scattering cross section versus driving frequency with* γ *ranging from* 0.04 *(top curve) to* 0.5 *(bottom curve) in steps of* 0.02*. The width of the resonance curve at half maximum is* ω_o/Q.

What conclusions can we draw? One, the scattering depends strongly on ω. Since we have $\omega_o^2 - \omega^2$ in the denominator, the scattering cross-section becomes very large at the resonance frequency of an electron in an atom. This makes sense: the incident radiation can most efficiently transfer its energy to an electron

[xxx] The typical area unit used for scattering is the *barn*. It is commonly used in all fields of high energy physics to express the cross sections of scattering processes. A barn is $10^{-28}\,\mathrm{m}^2$ ($100\,\mathrm{fm}^2$), approximately the cross sectional area of a uranium nucleus. The term originated with American physicists during wartime research on the atomic bomb, scattering neutrons off of uranium nuclei. They described the uranium nucleus as "big as a barn".

[xxxi] Which really only amounts to replacing $\left(\omega_o^2 - \omega^2\right)^2$ with $\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2$

when its frequency matches the resonance frequency, and at resonance the electron will most efficiently re-radiate. Two, the numerator of the cross-section grows as ω^4, meaning that it is much larger above resonance than below. Three, the energy dependence of light scattering explains why the sky is blue! The constituents of the atmosphere have their relevant resonance frequencies well in the ultraviolet. Visible light is at much lower frequencies, so we are looking at the cross section at frequencies below the resonant peak. In this regime, higher frequency blue light is scattered more than lower frequency red light owing to the larger cross section. What you're seeing when you look away from the sun is the light which is scattered more by the atmosphere, which is more blue than red light. This also means that ultraviolet light is absorbed even more strongly, which is a good thing.[xxxii] Mathematically, if $\omega \ll \omega_o$ and damping is negligible, then Eq. 1.104 reduces to

$$\sigma \approx \left(\frac{8\pi r_e^2}{3}\right)\frac{\omega^4}{\omega_o^4} \tag{1.105}$$

The cross section grows as ω^4 (or decreases as λ^{-4} if you like), so higher frequency (smaller wavelength) radiation is scattered much more effectively. This is known as Rayleigh scattering,[xxxiii] though our analysis has left out some details, such as the angular distribution of the radiation (which we could recover easily enough from our derivation of the Larmor formula) and polarizability of the scattering medium (which is just an overall multiplying factor).

1.4 Radiation from hot objects

Finally, we are ready to address the subject of thermal radiation.[xxxiv] Our idea is the following: we know how to calculate the emission of radiation from oscillating charges, and how they scatter incident radiation generated by other charges. We will imagine that we have a hot object (say, a gas in a perfectly black box) made up of many identical atoms, each of which has electrons that can be induced to oscillate and radiate. Our hot atoms inside the box will acquire thermal energy, and random motion will be induced. This random motion will result in the atoms having many different frequencies of oscillation, which means that any given atom is being exposed to radiation with a wide range of frequencies added (incoherently) together. What we would like to do is figure out the energy emitted by a single atom in the box exposed to the radiation from all others over a spread range of frequencies. If we can figure out the energy re-emitted by a single atomic oscillator driven by thermally-induced radiation, we should be able to determine the spectrum of thermally-induced radiation since in principle we already know the amount of thermal energy present.

This might seem intractable at first, but we've already figured out the problem for a single incident frequency of light impinging on an oscillator in the previous section. In the case of non-zero but small damping, we can see from Fig. 1.9 that the only driving frequencies that really matter are those close

[xxxii] Ozone is particularly good at absorbing ultraviolet light, hence the importance of the ozone layer in the atmosphere.
[xxxiii] See `http://en.wikipedia.org/wiki/Rayleigh_scattering` for more details on scattering and why the sky is blue.
[xxxiv] In this section we follow the treatment by Fowler[4].

to the resonance frequency of the oscillator $\omega \approx \omega_r \approx \omega_o$, only those frequencies give rise to a large amplitude of oscillation.

Using our previous results, for a given mode of oscillation at resonance frequency ω_o driven by radiation at frequency ω, the *total* energy of the oscillator is

$$U_{osc} = m\omega_o^2 A^2 = m\omega_o^2 \frac{e^2E^2/m^2}{\left(\omega_o^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\omega_o^2} \tag{1.106}$$

If only frequencies near resonance, $\omega \approx \omega_o$ will lead to large amplitudes (and therefore significant radiated power), we can approximate the first part of the denominator in the equation above. First, a bit of factoring:

$$\left(\omega_o^2 - \omega^2\right)^2 = \left(\omega_o^2 - \omega^2\right)\left(\omega_o^2 - \omega^2\right) = \left(\omega_o - \omega\right)^2\left(\omega_o + \omega\right)^2 \tag{1.107}$$

If $\omega \approx \omega_o$, then $\omega_o + \omega \approx 2\omega_o$, and

$$\left(\omega_o^2 - \omega^2\right)^2 \approx 4\omega_o^2\left(\omega_o - \omega\right)^2 \tag{1.108}$$

This leads us to an expression for the oscillator energy as a function of the driving frequency of the incident radiation ω:

$$U_{osc} \approx \left(\frac{\omega_o^2}{m}\right)\frac{e^2E^2}{4\omega_o^2\left(\omega_o - \omega\right)^2 + 4\gamma^2\omega_o^4} = \left(\frac{e^2E^2}{4m}\right)\frac{1}{\left(\omega - \omega_o\right)^2 + \gamma^2\omega_o^2} \tag{1.109}$$

where we have also used $\omega \approx \omega_o$ for the damping term in the denominator.

This is still for a single precise frequency of incident radiation ω, but we wish to sum over all incident frequencies to find the total energy of the oscillator. If U_{osc} is the energy of the oscillator at frequency ω, then $U_{osc}\,d\omega$ is the energy contained in the narrow frequency range $\omega \in [\omega, \omega + d\omega]$.[xxxv] Summing over all such frequency ranges $d\omega$ amounts to integrating $U(\omega)\,d\omega$ over that same interval. In the case of small damping, it really won't matter much if we integrate $U(\omega)$ only around the peak at ω_o or over all frequencies from 0 to ∞ since $U(\omega)$ only has appreciable weight in a narrow region around ω_o. Thus, for an oscillator driven by a wide range of possible frequencies of incident radiation, the total energy is:

$$U_{osc,tot} \approx \int\limits_0^\infty \left(\frac{e^2E^2}{4m}\right)\frac{1}{\left(\omega - \omega_o\right)^2 + \left(\gamma\omega_o\right)^2}\,d\omega = -\frac{e^2E^2}{4m\gamma\omega_o}\tan^{-1}\left(\frac{\omega - \omega_o}{\gamma}\right)\Bigg|_0^\infty = \frac{\pi e^2E^2}{8m\gamma\omega_o} \tag{1.110}$$

Actually, we have missed one important detail: in considering the possible frequencies of driving radiation along a given axis, we have two possible polarizations of radiation to consider (i.e., oscillations along two

[xxxv] Think about slicing the area under the $U(\omega)$ curve into tiny rectangles of width $d\omega$.

possible directions perpendicular to the incident light propagation), so we must multiply by two. Doing that and using our definition of γ from Eq. 1.90:

$$U_{\text{osc,tot}} = \frac{\pi e^2 E^2}{4m\gamma\omega_o} = \left(\frac{1}{2}\epsilon_o E^2\right)\frac{6\pi^2 c^3}{\omega_o^2} \tag{1.111}$$

The term in brackets on the right is once again the total energy per unit volume contained in the electric field (a.k.a., the energy density):

$$u_{\text{field}} = \frac{1}{2}\epsilon_o E^2 = U_{\text{osc,tot}}\frac{\omega_o^2}{6\pi^2 c^3} \tag{1.112}$$

What we have now is a relationship between the total energy of a single oscillating charge and the energy contained in the electric field it is immersed in. This is of course only for a single component of the field, since we have thus far considered oscillations only in a single plane resulting from radiation incident from a single direction. The other two directions of the field and planes of oscillation will give the same result if the system is homogeneous and isotropic, so to account for the other directions we simply multiply by three:

$$u_{\text{field}} = \frac{\omega_o^2}{2\pi^2 c^3}U_{\text{osc,tot}} = \frac{2f^2}{c^3}U_{\text{osc,tot}} \tag{1.113}$$

You can think of this result in a slightly different way: the quantity $u_{\text{field}}(\omega)\,d\omega$ gives the energy per unit volume for radiation with angular frequency ω in the frequency range $[\omega, \omega + d\omega]$. This is a crucial result: what is says is that if we can find the total energy of a given oscillator by other means, we automatically know the energy contained in the radiation field at a given frequency. Clearly, the idea is that we should use thermodynamics to find the energy of an oscillator at temperature T and use it to find the radiation energy density and spectrum.

Here is where trouble starts!

1.4.1 Rayleigh-Jeans Law

From classical thermodynamics, we know that each oscillator has an average energy $\langle U_{\text{osc,tot}}\rangle = k_B T$ at a temperature T independent of the oscillator's frequency. Thus, for an oscillator at a given frequency, we would expect the energy density of the electric field to be

$$\langle u_{\text{field}}\rangle = \frac{2f^2}{c^3}\langle U_{\text{osc,tot}}\rangle = \frac{2k_B T f^2}{c^3} \tag{1.114}$$

A common measure of radiation intensity is the *irradiance* (often called simply "intensity"), the energy flux per unit area, averaged over one period of oscillation, and it can be found from $I = c\langle u_{\text{field}}\rangle$,[xxxvi] giving

[xxxvi]Intensity is power per unit area going through a patch of surface, which is just the energy density multiplied by the velocity at which the energy is moving through a given area, c. See http://en.wikipedia.org/wiki/Intensity_%28physics%29

$$I = \frac{2k_B T f^2}{c^2} \tag{1.115}$$

This is the famous Rayleigh-Jeans law, which says that the energy per unit volume of thermally emitted radiation should scale as T and f^2. It agrees reasonably well with experimental results at low frequencies (long wavelengths), but strongly disagrees at high frequencies. In fact, it predicts that the energy density should be arbitrarily large as frequency increases! Worse, since the energy is only an average it implies that *any* source of thermal energy should contain at least some high-frequency radiation. Since we know everyday hot objects don't emit X-rays, this is a problem, often called the "ultraviolet catastrophe" among those prone to hyperbole. It is no catastrophe in the grand scheme of things, it just means that our model has gone horribly wrong somewhere. In particular, it has gone wrong by assuming that oscillators of any frequency receive the same amount of energy.

1.4.2 Planck's Hypothesis

Where did we go wrong? We assumed that all oscillators receive the same $k_B T$ worth of thermal energy, no matter what their frequency of oscillation. This seems odd! Planck's ad hoc resolution to the problem was to assume that perhaps the oscillators cannot emit arbitrary amounts of energy, but only multiples of a smallest indivisible unit of energy. That is, we assume energy only comes in discrete bundles, rather than arbitrary amounts. This isn't totally crazy – the resonant standing modes of a vibrating string only have certain allowed energies, after all, owing to the geometric boundary conditions imposed. Perhaps energy is similarly discrete, owing to some yet-unforeseen boundary conditions on the smallest scales?[xxxvii]

Specifically, let us imagine that oscillators only emit energy in small bundles proportional to their frequency. After all, it makes some sense that the faster the oscillation, the more energy emitted by the oscillator. Planck proposed that energy is *quantized* and only comes in units of $\Delta E = hf$, where h is now known as *Planck's constant*. We now know that

$$h \approx 6.626 \times 10^{-34}\,\mathrm{J \cdot s} = 4.135 \times 10^{-15}\,\mathrm{eV \cdot s} \tag{1.116}$$

Planck's constant is *tiny*, which explains why we didn't notice the discretization of energy sooner – the "graininess" of energy is far too small to be noticed on the scale of everyday energies . . . but it is kind of a big deal for tiny things like atoms!

What this implies is that the allowed energies of our oscillators can only take on discrete integer multiples of hf. Thus, an oscillator can have energies of $E = \{0, hf, 2hf, 3hf, \ldots\}$ but not $E = 1.5hf$. The particular energy of an oscillator at any given moment can then simply be indexed by an integer n telling us how many units of energy it has: $E = nhf$, $n = \{0, 1, 2, 3, \ldots\}$. More formally, we could state the hypotheses of Planck as:[xxxviii]

[xxxvii] In this section we follow portions of the treatment by Feynman[2].
[xxxviii] Following Leighton[5] Ch. 2.

1. Each oscillator *absorbs* energy from the radiation field in a continuous fashion, following classical electrodynamics.
2. An oscillator can *radiate* energy only in *exact integral multiples* of energy proportional to its frequency. When an oscillator does radiate, it radiates *all* of its energy.
3. The radiation or non-radiation of an oscillator when it possesses an integral number of energy units is entirely governed by statistical chance. The ratio of the probability of nonemission to the probability of emission is proportional to the intensity of the radiation exciting the oscillator.

The first part we have already dealt with, the excitation of the oscillators by the radiation bath they are immersed in. The second is Planck's discretization hypothesis sketched above. The third point we may derive from classical thermal physics along with the discretization hypothesis, and taken together, we will be able to rescue our model of radiation.

What we must now ask ourselves is for a given temperature T, what is the average energy of our oscillators given these new constraints? At $T=0$, absolute zero, there would be no thermal energy, so all oscillators would have an energy of 0 and occupy the state $n=0$. At any nonzero temperature, our oscillators will be distributed over levels of various n, our task is to figure out how they are distributed. Since thermal energy is random, but with a well-defined mean, some oscillators will have the lowest possible energy, most in between, and a few will have relatively high energies. At low temperatures, when the thermal energy is small, most oscillators will have zero energy, and as temperature increases, more and more oscillators will be able to gain the right amount of thermal energy to have energies of hf, $2hf$, $3hf$, etc.

Given these equally-spaced energy levels, we can use the Boltzmann factor to find the probability that a given oscillator has a particular energy. Recall that the Boltzmann factor tells us the probability $P(E)$ of a given particle having an energy E at a temperature T:

$$P(E) = e^{-E/k_B T} \tag{1.117}$$

where

$$k_B = 1.38 \times 10^{-23}\,\text{J/K} = 8.617 \times 10^{-5}\,\text{eV/K} \tag{1.118}$$

is Boltzmann's constant. In the present case, the lowest possible energy is for $n=0$, corresponding to $E=0$. Let's say we have many oscillators, N_{tot} in total. We have an infinity of possible energy levels, $n=\{0,1,2,\ldots\}$ corresponding to $E_n=\{0,hf,2hf,\ldots\}$. Let the number of oscillators in each of these levels be $N_n=\{N_0,N_1,N_2,\ldots\}$. What is the average energy of all the oscillators? It must still be the total energy of all oscillators divided by the number of oscillators. The total energy is just the sum over all levels of the number of oscillators in each level times the energy of that level.

$$\langle E \rangle = \frac{\text{total } E}{\text{number of oscillators}} = \frac{\sum \text{(number per level) (energy of level)}}{\text{number of oscillators}} \tag{1.119}$$

The number of atoms in any given level is just the number of oscillators in total times the probability that a given level is occupied. The Boltzmann factor gives us the latter quantity. If there are N_0 atoms in the lowest energy level $n=0$, for a level n the number of oscillators with that energy is

$$N_n = N_0 e^{-E_n/k_B T} \tag{1.120}$$

The total energy of all oscillators together is just summing up the number in each level times the energy of that level:

$$E_{tot} = \sum_{n=0}^{\infty} N_n E_n = \sum_{n=0}^{\infty} N_0 e^{-nhf/k_B T} nhf \tag{1.121}$$

This is not so bad a sum as it looks; define a new variable $x = e^{-hf/k_B T}$, then the sum becomes a simple geometric series:

$$E_{tot} = \sum_{n=0}^{\infty} N_0 x^n nhf = N_0 hf \sum_{n=0}^{\infty} nx^n = N_o hf \frac{x}{(1-x)^2} \tag{1.122}$$

The number of oscillators in total is found by summing the number in each level, which results in another well-known series:

$$N_{tot} = \sum_{n=0}^{\infty} N_n = \sum_{n=0}^{\infty} N_0 e^{-E_n/k_B T} = N_o \sum_{n=0}^{\infty} x^n = N_o \frac{1}{1-x} \tag{1.123}$$

Thus, the average energy is

$$\langle E \rangle = \frac{E_{tot}}{N_{tot}} = hf \frac{x}{(1-x)^2}(1-x) = hf \frac{x}{1-x} \tag{1.124}$$

Recalling our definition of x,

$$\langle E \rangle = hf \frac{e^{-hf/k_B T}}{1 - e^{-hf/k_B T}} = \frac{hf}{e^{hf/k_B T} - 1} \tag{1.125}$$

This is the famous Planck formula for the average energy of the oscillators. It does not suffer from the divergences at high frequencies, and it is very different from the classical result $\langle E \rangle = k_B T$. In Fig. 1.11 below we have plotted $\langle E \rangle$ versus f for various values of $h/k_B T$, just to give you an idea of how $\langle E \rangle$ behaves compared to the classical prediction of $\langle E \rangle = k_B T$, independent of frequency.

Qualitatively, we see that at high temperatures, the oscillators are spread out over a wide range of energy levels, while at low temperatures they tend to occupy only the lowest levels. When $k_B T \gg hf$, i.e., when the thermal energy is much larger than the discrete spacing between energy levels, the discreteness of energy becomes unimportant, and our prior result holds at low frequencies or high temperatures. (You can show that this result is recovered in the case that $f \to 0$ or $T \to \infty$.) This is again a reason why the discreteness of energy took so long to notice: unless the system of interest is at very low temperatures, or very high frequency radiation is involved, the spacing of allowed energies is too small compared to the

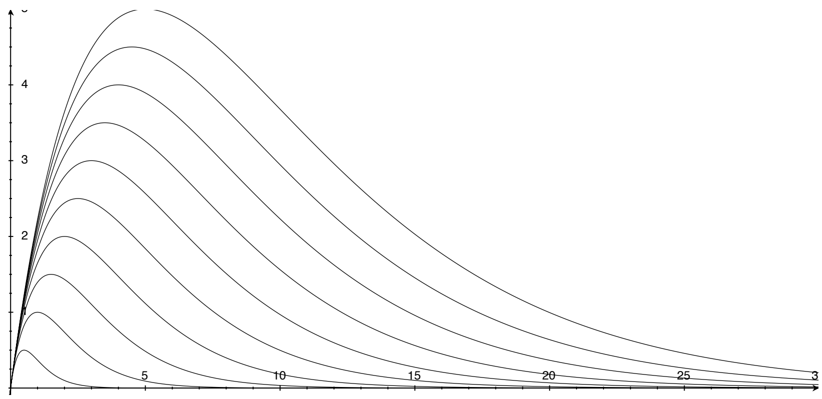

Figure 1.11: $\langle E \rangle$ *versus* f *for various values of* h/k_BT.

random thermal energy to be noticed.

1.4.3 The Radiation Spectrum

Armed with a new expression for the average energy of the oscillators, we can immediately apply our previous result:

$$\langle u_{\text{field}} \rangle = \frac{2f^2}{c^3} \langle U_{\text{osc,tot}} \rangle = \frac{2f^2}{c^3} \frac{hf}{e^{hf/k_BT} - 1} = \frac{2hf^3}{c^3} \frac{1}{e^{hf/k_BT} - 1} \tag{1.126}$$

The intensity (average power per unit area) is thus

$$I = c\langle u_{\text{field}} \rangle = \frac{2hf^3}{c^2} \frac{1}{e^{hf/k_BT} - 1} \tag{1.127}$$

This reproduces, with amazing accuracy, the observed emitted radiation energy per unit volume versus frequency. To find the intensity as a function of wavelength, the change of variable requires evaluating

$$I'(\lambda, T) = I(f, T) \left| \frac{df}{d\lambda} \right| = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_BT} - 1} \tag{1.128}$$

At this point, your textbook takes over fairly well in discussing the main features of thermal ("blackbody") radiation. In the problems below we derive Wein's displacement law relating the wavelength of peak radiation emission to temperature, show that the *total* emitted power over all wavelengths scales as T^4 (related to the Stefan-Boltzmann law), and consider some everyday near-blackbody sources (the sun, incandescent light bulbs).

1.5 Problems related to thermal radiation*

1. *Leighton, 2.4[5]* As a function of wavelength, Planck's law states that the emitted power of a black body per unit area of emitting surface, per unit wavelength is

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} \tag{1.129}$$

That is, $I(\lambda, T)d\lambda$ gives the emitted power per unit area emitted between wavelengths λ and $\lambda + d\lambda$. Show by differentiation that the wavelength λ_m at which $I(\lambda, T)$ is maximum satisfies the relationship

$$\lambda_m T = b \tag{1.130}$$

where b is a constant. This result is known as *Wien's Displacement Law,* and can be used to determine the temperature of a black body radiator from only the peak emission wavelength. The constant above has a numerical value of $b = 2.9 \times 10^6$ nm-K. *Note: at some point you will need to solve an equation numerically.*

First, we must find $dI/d\lambda$. Strictly, we want $\partial I/\partial\lambda$, since we are presuming constant temperature, but that is only a formal point since T does not depend on λ. For convenience, define the following substitutions:

$$a \equiv 8\pi hc^2 \tag{1.131}$$

$$b \equiv \frac{hc}{kT} \tag{1.132}$$

Thus,

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} = \frac{a}{\lambda^5}\left[e^{\frac{b}{\lambda}} - 1\right]^{-1} \tag{1.133}$$

$$\frac{dI}{d\lambda} = \frac{-5a}{\lambda^6}\frac{1}{e^{\frac{b}{\lambda}} - 1} + \frac{-a}{\lambda^5}\left(\frac{1}{e^{\frac{b}{\lambda}} - 1}\right)^2\left(\frac{-be^{\frac{b}{\lambda}}}{\lambda^2}\right) = \left(\frac{a}{\lambda^7}\right)\frac{be^{\frac{b}{\lambda}} - 5\lambda e^{\frac{b}{\lambda}} + 5\lambda}{\left(e^{\frac{b}{\lambda}} - 1\right)^2} = 0 \tag{1.134}$$

Finding the maximum of $I(\lambda, T)$ with respect to λ means setting $dI(\lambda, T)/d\lambda = 0$.[xxxix] The denominator in the equation above is then irrelevant, as is the λ^{-7} prefactor, and we have

$$0 = be^{\frac{b}{\lambda}} - 5\lambda e^{\frac{b}{\lambda}} + 5\lambda \tag{1.135}$$

$$0 = be^{\frac{b}{\lambda}} + 5\lambda\left(1 - e^{\frac{b}{\lambda}}\right) \tag{1.136}$$

$$5 = \frac{be^{\frac{b}{\lambda}}}{\lambda\left(e^{\frac{b}{\lambda}} - 1\right)} \tag{1.137}$$

We can make another substitution to make things easier. Define $x \equiv \frac{b}{\lambda} = \frac{hc}{\lambda kT}$ and simplify:

[xxxix] Since we know the curve is concave downward, we won't bother with the second derivative test; we know very well we will find a maximum and not a minimum.

$$\frac{xe^x}{e^x - 1} - 5 = 0 \tag{1.138}$$

If we find the root of this equation, we have (after undoing our substitutions) the value of λ for which $I(\lambda, T)$ is maximum. Unfortunately, there is no analytic solution. Using Newton's method or something similar, we find the root is

$$x = \frac{hc}{\lambda kT} \approx 4.695 \tag{1.139}$$

Solving for λ, we obtain the desired result:

$$\lambda_{max} \approx \frac{hc}{4.965kT} \approx \frac{2.898 \times 10^6\,\text{nm} \cdot \text{K}}{T} \tag{1.140}$$

2. As a function of *frequency*, Planck's law states that the spectral energy density of a black body, the energy per unit volume per unit frequency, is given by[xl]

$$u(f, T) = \frac{8\pi hf^3}{c^3}\left[e^{\frac{hf}{k_bT}} - 1\right]^{-1} \tag{1.141}$$

If you think of a black body as an insulated, perfectly mirrored box with a tiny hot object inside, $u(f, T)$ would give the energy per unit volume of radiation with frequencies between f and $f + df$. Integrating this energy density over all frequencies, one obtains the *total* energy per unit volume V. Show that the total emitted power per unit volume is proportional to T^4. Specifically,

$$\frac{U(T)}{V} = \int_0^\infty u(f, T)\,df = \sigma T^4 \tag{1.142}$$

Here σ is a constant. This result is essentially the *Stefan-Boltzmann law.* The following integral may be useful:

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \zeta(4)\Gamma(4) = \frac{\pi^4}{90} \times 6 = \frac{\pi^4}{15} \tag{1.143}$$

A clever substitution might be to define a variable $x = hf/k_bT$.

All we need to do is integrate. It will be less messy with the following substitution:

$$x = \frac{hf}{kT} \tag{1.144}$$

$$\implies \quad f = \frac{xkT}{h} \tag{1.145}$$

$$\implies \quad df = \frac{kT}{h}\,dx \tag{1.146}$$

[xl] Slightly different units are used in this problem compared to the derivation in the previous section.

The limits of integration remain the same, 0 and ∞. You did remember the df, right? With these substitutions, we have

$$\frac{U(T)}{V} = \int_0^\infty u(f,T)\,df = \int_0^\infty u(x,t)\frac{kT}{h}\,dx \tag{1.147}$$

$$= \frac{8\pi\,(kT)^3}{c^3h^2}\int_0^\infty \frac{x^3}{e^x-1}\left(\frac{kT}{h}\right)dx = \frac{8\pi\,(kT)^4}{c^3h^3}\int_0^\infty \frac{x^3}{e^x-1}\,dx \tag{1.148}$$

The integral is now dimensionless (i.e., it has no units), and is in the end just some number. It happens to be $\pi^4/15$, but this has no real physical significance.[xli] We have already established the proportionality with T^4. Plugging in the value of the integral,

$$\frac{U(T)}{V} = \frac{8\pi^5\,(kT)^4}{15c^3h^3} = \frac{4\sigma T^4}{c} \tag{1.149}$$

Here σ is the Stefan-Boltzmann constant, which is the proportionality between *power* and T^4.

3. *Leighton, 2.8[5]* The wavelength of maximum intensity in the solar spectrum is about 500 nm, as some of you will verify in PH255. Assuming the sun radiates as a black body, compute its surface temperature.

The Wien displacement law from problem 1 is all we need:

$$T = \frac{2.898\times10^6\,\text{nm}\cdot\text{K}}{\lambda_{\text{max}}} \approx 5800\,\text{K} \tag{1.150}$$

4. In the figure below, the measured intensity as a function of wavelength is shown for a 60 W incandescent bulb at various supply voltages V. (You may ignore the smaller secondary peaks at higher wavelengths, they are due to a phosphorescent coating on the inside of the bulb.)

(a) Assuming the bulb filament radiates as a perfect black body, the wavelength at which peak intensity occurs should be inversely proportional to temperature, $\lambda_m = b/T$ with $b = 2.9\times10^6$ nm-K. Estimate the peak position for each curve. Plot the resulting estimated filament temperature versus the *relative* electrical power supplied to the filament. You may assume the bulb has constant resistance, such that the power supplied to the bulb is proportional to V^2. Do the results make sense?
(b) The total emitted power is proportional to the area under the intensity-wavelength graph. *Roughly estimate* the area under the curves for each voltage. This in turn should be proportional to the bulb temperature to the fourth power, T^4. Plot the estimated area versus for each curve versus T^4 using your temperature estimates from part a. Is the Stefan-Boltzmann law obeyed, within your margin of error?
(c) Is the bulb a reasonable approximation of a black body? You may want to check the melting point of

[xli] We provide a derivation in the appendix at the end of this section.

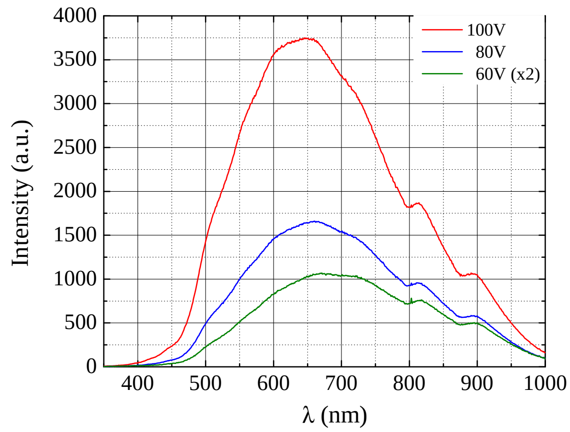

Figure 1.12: *Spectrum of a* 60 *W soft white incandescent bulb at three different supply voltages, measured in the PH255 lab. Note that the* 60 *V curve has been multiplied by a constant factor!*

the tungsten filament.

(something to think about) Compare this spectra qualitatively to the solar spectrum, e.g., `http://en.wikipedia.org/wiki/Sunlight`. Can you understand why incandescent bulbs at particular powers are favored for indoor lighting? Why is "color temperature" used to characterize such lighting sources?

In order to calculate the temperature of the bulb from the given spectra, we first assume that the bulb at least roughly behaves a blackbody radiator, and thus Wien's displacement law applies. From the wavelength at which the spectra peaks, we can find the temperature from $\lambda_{\text{peak}} = b/T$. In my case, I find

$$\lambda_{\text{peak}} \approx 648\,\text{nm} \qquad 100\,\text{V} \tag{1.151}$$

$$\lambda_{\text{peak}} \approx 660\,\text{nm} \qquad 80\,\text{V} \tag{1.152}$$

$$\lambda_{\text{peak}} \approx 670\,\text{nm} \qquad 60\,\text{V} \tag{1.153}$$

Assuming a constant bulb resistance, the electrical power supplied is V^2/R. Below, we plot the temperature obtained versus relative power (with 100 V arbitrarily defined as 100% power). The error bars represent the estimated error in peak wavelength determination $\delta\lambda$ propagated to a temperature uncertainty via

$$\left|\frac{\delta\lambda}{\lambda}\right| = \left|\frac{\delta T}{T}\right| \tag{1.154}$$

This result does make sense: the total radiated power should scale with the total applied electrical power! For verifying the Stefan-Boltzmann law, we numerically integrated the given curves from the raw data using a simple trapezoid rule in Excel, also estimating the relative error in the standard manner.[xlii] Un-

[xlii]See `http://en.wikipedia.org/wiki/Trapezoidal_rule#Error_analysis`

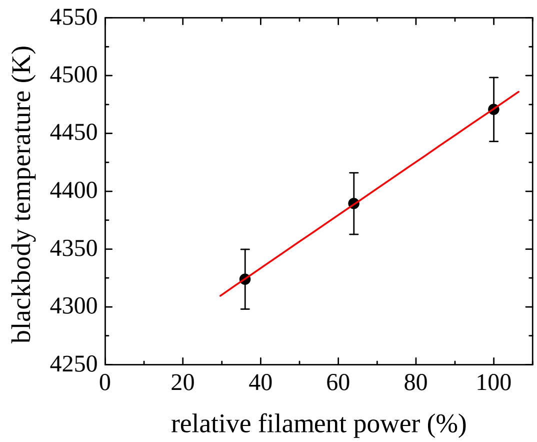

Figure 1.13: *Bulb temperature obtained from the Wien displacement law versus electrical power, assuming a constant bulb resistance. The red line is a best-fit using a weighted linear regression.*

certainty in T^4 was propagated according to

$$\left|\frac{\delta T^4}{T}\right| = 4\left|\frac{\delta T}{T}\right| \tag{1.155}$$

The uncertainty bars on T^4 are not visible on this scale. Within the limits of uncertainty, the Stefan-Boltzmann law is obeyed.

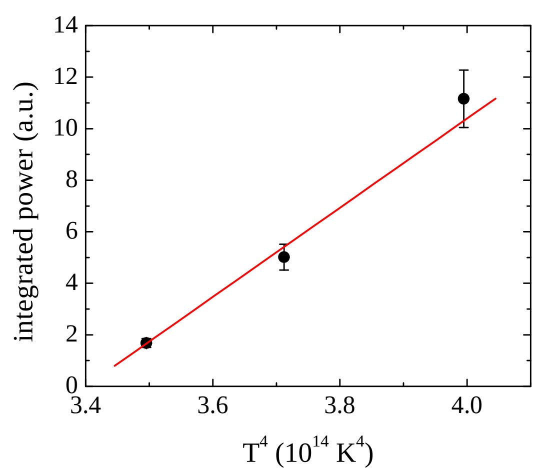

Figure 1.14: *Total radiated power, determined by finding the area under the* $I(\lambda)$ *curve, versus estimated temperature to the fourth power. The red line is a best-fit using a weighted linear regression.*

The bulb is *not* an ideal blackbody, for two very simple reasons: first, it is not black, and thus does not absorb all incident radiation; second, the estimated temperature is well above the melting point of the tungsten filament! One can come up with many other reasons, but either one of these two is sufficient ...

As for the last part, you may find `http://en.wikipedia.org/wiki/Color_temperature` interesting.

5. *Frank, 20.16* Compute the ratio of the increase of intensity of black-body radiation at a wavelength of 641 nm for an increase of temperature from 1200 to 1500 K.

Intensity versus wavelength is quoted in problem 1. The ratio of intensities for a given wavelength $\lambda = 641$ nm at temperatures $T_1 = 1500$ K and $T_2 = 1200$ K is then

$$\frac{I(\lambda, T_1)}{I(\lambda, T_2)} = \frac{\frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T_1}} - 1\right]^{-1}}{\frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T_2}} - 1\right]^{-1}} = \frac{e^{\frac{hc}{\lambda k_b T_1}} - 1}{e^{\frac{hc}{\lambda k_b T_2}} - 1} \approx 42 \tag{1.156}$$

As you can see, it is much more clever to solve the problem symbolically before using the numbers given. In the opposite case, one ends up performing a great deal of unnecessary calculations.

6. An accelerating charge loses electromagnetic energy at a rate of

$$\mathscr{P} = \frac{\Delta E}{\Delta t} = -\frac{2k_e q^2 a^2}{3c^3}$$

where k_e is Coulomb's constant, q is the charge of the particle, c is the speed of light, and a is the acceleration of the charge. Assume that an electron is one Bohr radius ($a_0 = 0.053$ nm) from the center of a Hydrogen atom, with the proton stationary. **(a)** Find the acceleration of the electron (hint: circular path). **(b)** Calculate the kinetic energy of the electron and determine within an order of magnitude how long it will take the electron to loose all of its energy, assuming a constant acceleration as found in part a. Be sure to point out whether you need to consider relativistic effects or not (hint: how big is v/c if you ignore relativity?).

The electron circulating around the proton has only one relevant force, the Coulomb interaction with the proton. This force must provide the centripetal force if the electron is to remain on a circular path:

$$qE = \frac{k_e q^2}{r^2} = ma_{\text{cent}}\frac{mv^2}{r} \qquad \Longrightarrow a_{\text{cent}} = \frac{kq^2}{mr} \sim 9 \times 10^{22}\,\text{m/s}^2 \tag{1.157}$$

Given that $a_{\text{cent}} = v^2/r$, we also readily find the velocity:

$$v = \sqrt{a_{\text{cent}} r} = \sqrt{\frac{k_e q^2}{mr}} \approx 2 \times 10^6\,\text{m/s} \tag{1.158}$$

Since $v \sim 0.01c$, we are justified in not using relativistic corrections. The kinetic energy is then simply

$$K = \frac{1}{2}mv^2 = \frac{k_e q^2}{2r} \approx 2.17 \times 10^{-18}\,\text{J} \approx 13.6\,\text{eV} \tag{1.159}$$

Basically, we have just reproduced the lowest energy level of the hydrogen atom - also known as the ionization energy.

How long is this classical atom stable? We should remember at this point that power is energy per unit time. The power in this case means how much energy we are loosing per unit time, hence the negative sign. What we want to find is how long it will take for the electron to loose *all* of its energy, the whole kinetic energy we just calculated. If, just to obtain an order of magnitude estimate, we assume that the rate of energy loss is constant,

$$\mathscr{P} = \frac{2k_e q^2 a^2}{3c^3} = \frac{2k_e^3 q^6}{3m^2c^3r^4} = \frac{\Delta E}{\Delta t} = \frac{K}{\Delta t}$$
$$\implies \quad \Delta t = \frac{K}{\mathscr{P}} = \frac{3m^2c^3r^3}{4k_e^2e^4} \sim 5 \times 10^{-11}\,\text{s}$$

Thus, if we can calculate the power - the rate of energy loss - using the (now) known acceleration a and various fundamental constants, we can use the kinetic energy to find out how long it takes the electron to loose all of its energy.

Of course, there are many *many* problems with this analysis. First, the whole orbiting electron model is a kludge of sorts, we know it to be deeply flawed (though still useful). Second, it is perhaps silly to think that the electron loses energy at a constant rate as it continues on its death spiral (particularly since we did already derive a more general related result). Finally, we will soon know better – atoms are indeed stable, and the answer lies in the wave-like nature of matter and the uncertainty principle.

7. Assuming that the human body has a surface area of 2 square meters and radiates like a black body at a temperature of 35°C, calculate the rate at which it loses heat in surroundings that have a temperature of 15°C.

According to the Stefan-Boltzmann law, a body at temperature T of emitting area A radiates a total power P given by

$$P = \sigma A T^4 \tag{1.160}$$

where σ is the Stefan-Boltzmann constant. How do we calculate the rate of heat loss? Heat is just another form of energy, and if no work is being done on our system of the human plus its surroundings, the change in heat Q is just the change in (internal) energy ΔU. The rate at which energy changes is power. Thus, we need only balance the power lost by the human due to its radiating at a temperature T_h against the power *gained* by absorbing power from its surroundings at a temperature T_s. Or, the net rate of energy loss, the net power, is just power in minus power out. Given that the human both emits and absorbs power over the same area A,

$$P_{\text{net}} = P_h - P_s = \sigma A T_h^4 - \sigma A T_s^4 = \sigma A \left(T_h^4 - T_s^4\right) \tag{1.161}$$

With the numbers given and $\sigma = 5.67 \times 10^{-8}\,\text{J/m}^2\,\text{s}\,\text{K}^4$, we find $P_{\text{net}} \approx 240\,\text{W}$.

1.5.0.1 Appendix: Evaluating $\int_0^\infty x^3\,dx/(e^x-1)$*

Pathologically, the best way to calculate the integral

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx \tag{1.162}$$

is to calculate a more general case and reduce it to the answer we require. Take the following integral

$$\int_0^\infty \frac{x^n}{e^x-1}\,dx = \int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}\,dx \tag{1.163}$$

The denominator is always less than one, and is in fact the sum of a geometric series with common multiplier e^{-x}:

$$\frac{1}{1-e^{-x}} = \sum_{k=0}^\infty e^{-kx} \tag{1.164}$$

If we substitute in this series, our integral becomes

$$\int_0^\infty x^n e^{-x} \sum_{k=0}^\infty e^{-kx}\,dx \tag{1.165}$$

We can bring the factor e^{-x} inside our summation, which only shifts the lower limit of the sum from 0 to 1, leaving:

$$\int_0^\infty x^n \sum_{k=1}^\infty e^{-kx}\,dx \tag{1.166}$$

Now make a change of variables $u=kx$, meaning

$$x^n = \frac{u^n}{k^n} \tag{1.167}$$

$$dx = \frac{du}{k} \tag{1.168}$$

With this change of variables, our integral is:

$$\int_0^\infty \frac{u^n}{k^n} \sum_{k=1}^\infty e^u \frac{du}{k} = \int_0^\infty u^n \sum_{k=1}^\infty e^u \frac{du}{k^{n+1}} \tag{1.169}$$

Each term in the sum represents an integral over u, all of which are convergent. This means we can interchange the order of summation and integration:

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \int_0^{\infty} u^n e^{-u} \, du \tag{1.170}$$

The integral on the right side is the definition of the Gamma function $\Gamma(n+1)$, while the summation is then the definition of the Riemann zeta function $\zeta(n+1)$. Thus,

$$\int_0^{\infty} \frac{x^n}{e^x - 1} \, dx = \zeta(n+1)\Gamma(n+1) \tag{1.171}$$

With $n=3$,

$$\zeta(n+1) = \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{1.172}$$

$$\Gamma(n+1) = n! = 3! = 6 \tag{1.173}$$

And finally,

$$\int_0^{\infty} \frac{x^3}{e^x - 1} \, dx = \zeta(n+1)\Gamma(n+1) = \frac{\pi^4}{15} \tag{1.174}$$

1.6 Appendix: Magnetism as a Consequence of Relativity*

As it turns out, the magnetic field we normally think of as a distinct physical phenomena is nothing more than a relativistic view of the electric field of moving charges.[xliii] In order to see the fundamental symmetry between the electric and magnetic fields, we will conduct a hypothetical experiment using a current-carrying wire and a moving test charge, as shown in Fig 1.15. We have a conducting wire with current flowing to the right when viewed from the laboratory reference frame (O). For simplicity, we will assume the current is due to the flow of *positive* charges, spaced evenly with an average separation l^O when viewed from the lab frame O.[xliv]

We know that our conducting wire must be electrically neutral in the laboratory frame, so in addition to the positive charges there must be an equal number of negative ions – the atoms making up the wire – also spaced at a distance l^O. Now (still in the laboratory frame) we place a positive test charge Q a distance R from the wire. Since the wire is electrically neutral, there is no force on the test charge. What happens if the test charge is moving? We will give the test charge Q a velocity $\vec{v}$ parallel to the wire, the same velocity with which the positive charges in the wire are moving for simplicity.

[xliii] This section follows the treatment of Purcell[1] closely.

[xliv] Even though we know now that negatively-charged electrons really carry the current, working with positive charges will make the discussion simpler (by avoiding a lot of pesky minus signs), and will not change the analysis in any way.

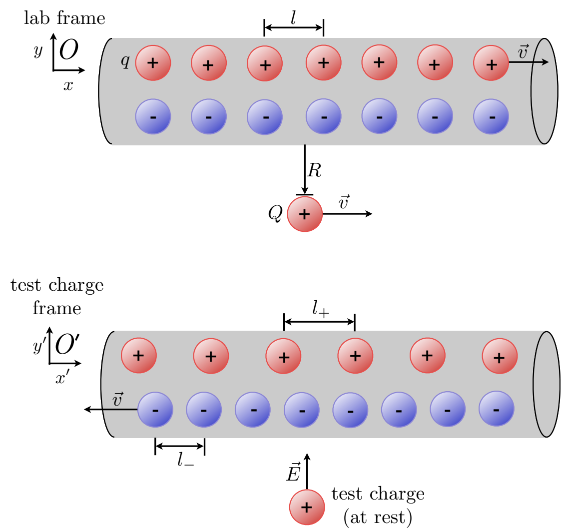

Figure 1.15: *An electric current in a wire viewed from the laboratory reference frame (O), and the reference frame of a moving test charge Q (O′). In the test charge frame, the spacing of the positive charges apparently increases while the spacing of the negative charges apparently decreases.*

What does the now moving test charge experience, viewed from its own reference frame (O′)? Since it is moving in the same direction, with the same velocity, as the positive charges in the wire, *it sees those positive charges as at rest relative to itself*, and *the negative charges as moving to the left with velocity* $\vec{v}$.

When the positive charges are viewed from the laboratory frame O, they appear to have an average spacing of l^{O}, moving at velocity $\vec{v}$. Once we switch to the test charge's frame, the positive charges appear to be at rest – in switching reference frames, the velocity of the positive charges goes from $\vec{v}$ to zero. From special relativity, we know that moving objects undergo a *length contraction*. When we view the spacing l^{O} of the positive charges in the lab frame O, *we are viewing the contracted length*. In the test charge's frame O′, we must un-contract the spacing l^{O} into the O′ frame to figure out what the test charge really sees. If we call the spacing of the positive charges that the moving test charge experiences in its frame O′ as $l_{+}^{O'}$, we can easily relate it to the spacing viewed from the lab frame O:

$$l_{+}^{O'} = l^{O}\gamma \tag{1.175}$$

$$l_{+}^{O'} = \frac{l^{O}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.176}$$

Since we know $\gamma \geqslant 1$, it is clear that the spacing the test charge sees is *larger* than what we see in the lab frame. Meanwhile, what about the negative charges, which are stationary in the lab frame? The test charge sees from its frame the negative charges moving to the *left* with velocity $\vec{v}$, so their spacing must be *contracted* to figure out the spacing of the negative charges $l_{-}^{O'}$ the test charge sees:

$$\gamma l_{-}^{O'} = l^{O} \tag{1.177}$$

$$l_{-}^{O'} = \frac{l^{O}}{\gamma} \tag{1.178}$$

$$l_{-}^{O'} = l^{O}\sqrt{1-\frac{v^2}{c^2}} \tag{1.179}$$

Again, since $\gamma \geqslant 1$, the positive test charge sees a *reduced* spacing of the negative charges. Since the positive and negative charges now no longer appear to have the same spacing when viewed from the test charge's frame, *the test charge sees a net negative charge density*, since there are effectively more negative charges per unit length than positive charges. The presence of a net negative charge density from the test charge's point of view means that it experiences a net attractive force from the wire. From the lab frame, we would not expect any force between the test charge and the wire, but sure enough, a proper relativistic treatment leads us to deduce that a force must in fact be present.

How big is the force? First, we need to figure out the charge density in the wire that the test charge sees. Since we don't want to restrict ourselves to any particular length of wire, we will calculate the number of charges per unit length as viewed in the test charge's frame, $\lambda^{O'}$. How do we find this? We know that all charges in the wire have charge q, and we know their average spacing. Dividing q by the average spacing for each kind of charge will give us the number of charges per unit length for both positive and negative charges, and subtracting those two will give use the net charge density:

$$\lambda^{O'} = \lambda_{+}^{O'} - \lambda_{-}^{O'} \tag{1.180}$$

$$= \frac{q}{l_{+}^{O'}} - \frac{q}{l_{-}^{O'}} \tag{1.181}$$

$$= \frac{q}{l^{O}}\sqrt{1-\frac{v^2}{c^2}} - \frac{q}{l^{O}}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.182}$$

$$= \frac{q}{l^{O}}\left(\sqrt{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{1.183}$$

This is a bit messy. However, we know that the drift velocity of charges in a conductor is *very* small compared to c ($v_d \sim 10^{-3}$ m/s. When $v \ll c$, we can use the following approximations:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \qquad v \ll c \tag{1.184}$$

$$\frac{1}{\gamma} = \sqrt{1-\frac{v^2}{c^2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2} \qquad v \ll c \tag{1.185}$$

Using these approximations in Eq. 1.183, we can come up with a simple expression for $\lambda^{O'}$:

$$\lambda^{O'} = \frac{q}{l^O}\left(\sqrt{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{1.186}$$

$$= \frac{q}{l^O}\left(1 - \frac{1}{2}\frac{v^2}{c^2} - \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\right) \tag{1.187}$$

$$= -\frac{q}{l^O}\frac{v^2}{c^2} \tag{1.188}$$

Now that we have the charge density of the wire as viewed from the test charge's frame, what is the electrostatic force? The problem is now to find the electric field at a distance R from a long, uniformly charged wire of charge density $\lambda^{O'}$, which is easily found from Gauss' law to be $E = 2k_e\lambda/R$. Substituting, we can immediately write down the electrostatic force experienced by the test charge in its reference frame:

$$|\vec{F}| = Q|\vec{E}| = Q\frac{2k_e|\lambda^{O'}|}{R} = \frac{2k_e Q q v^2}{Rlc^2} \tag{1.189}$$

We can simplify this a bit. The current in the wire is the charge q divided by the time it takes the charges to move a unit length, which is $\Delta t = l/v$.[xlv] Thus the current can be written as qv/l:

$$|\vec{F}| = Qv\left(\frac{2k_e I}{c^2 R}\right) \tag{1.190}$$

If we make the identification

$$|\vec{B}| = \frac{2k_e I}{c^2 R} = \frac{\mu_o I}{2\pi R} \qquad (c^2 = 1/\mu_o\epsilon_o) \tag{1.191}$$

then we have recovered the magnetic force law:

$$|\vec{F}| = Qv|\vec{B}| \qquad \text{with} \qquad |\vec{B}| = \frac{2k_e I}{c^2 R} \tag{1.192}$$

This is it. A test charge moving near a current-carrying wire experiences a net force proportional to its charge, velocity, and the current in the wire. We have managed to derive the existence of the magnetic field and magnetic force from nothing more than Coulomb's law and special relativity. In the laboratory frame, we typically consider a magnetic field created by the current in the wire, which acts on the test charge to produce a force qvB. What we have shown now is that we find exactly the same force on the test charge by considering it in its own reference frame, thus establishing that **a magnetic field is nothing more than the field of moving charges.**

In some sense, it is remarkable that we can measure magnetic forces due to currents at all. The drift velocity is *miniscule* compared to c, $\frac{v}{c} \sim 10^{-12}$ or so, and γ is barely different from 1, about $1.0 + 10^{-24}$. The magnetic force results from a tiny relativistic correction, certainly, but it is indeed a significant effect

[xlv] This just comes from kinematics, we know that the charge covers a distance l according to $l = v\Delta t$.

in the end because there are truly astronomical numbers of charges per unit length inside conductors. Even though the force per charge is miniscule, they make up for it in numbers. Before moving on, we note that if you repeat this analysis for the more complicated case that the test charge's velocity is *not* the same as the charges in the wire, and *not* parallel, you still arrive at the same result. It just takes quite a bit longer . . .

1.7 Appendix: General field transformation rules*

For the curious, we list here the general transformation between E and B fields in different (non-accelerating) frames.

Assume we have two reference frames O and O' whose coordinate axes are all parallel (i.e., x' parallel to x, y' parallel to y, etc.), with frame O' traveling at relative velocity v with respect to frame O along the x' axis. If we have fields E and B in frame O, the fields seen by an observer in frame O' are

$$E'_x = E_x \tag{1.193}$$

$$E'_y = \gamma\left(E_y - vB_z\right) \tag{1.194}$$

$$E'_z = \gamma\left(E_z + vB_y\right) \tag{1.195}$$

$$B'_x = B_x \tag{1.196}$$

$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \tag{1.197}$$

$$B'_z = \gamma\left(B_z + \frac{v}{c^2}E_y\right) \tag{1.198}$$

Note that the components of both E and B parallel to the motion remain unchanged. Two special cases are worth noting. If $E=0$ in O (purely magnetic field in one frame), then

$$\vec{E}' = \vec{v} \times \vec{B}' \tag{1.199}$$

If $B=0$ in O (purely electric field in one frame), then

$$\vec{B}' = -\frac{1}{c^2}\left(\vec{v} \times \vec{E}'\right) \tag{1.200}$$

If either E or B is zero in one frame, the fields in the other frame at a particular point are simply related.

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