PH 125 / LeClair   Spring 2009

A Short Math Guide

Contents

1   Coordinate systems

Note that we use the convention that the cartesian unit vectors are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, rather than $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\mathbf{k}}$, using the substitutions $\hat{\mathbf{x}} = \hat{\boldsymbol{\imath}}$, $\hat{\mathbf{y}} = \hat{\boldsymbol{\jmath}}$, and $\hat{\mathbf{z}} = \hat{\mathbf{k}}$.

1.1   2D systems

In two dimensions, commonly use two types of coordinate systems: cartesian $(x-y)$ and polar $(r-\theta)$. A point of interest specified in one system can easily be expressed within the other system through basic trigonometry, as shown below.
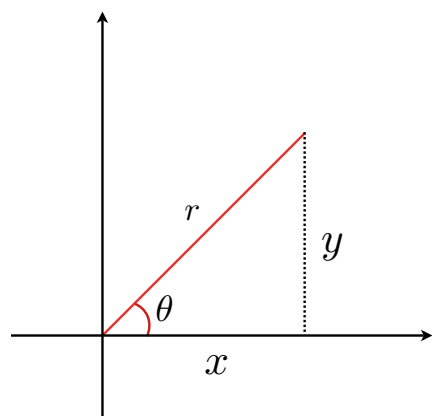
Relationships between 2D systems.	
cartesian (x, y)	**polar (r, θ)**
$r = \sqrt{x^2 + y^2}$	$x = r \cos \theta$
$\theta = \tan^{-1}(y/x)$	$y = r \sin \theta$

For example, we can describe the point $\mathcal{P}$ equivalently in both coordinate systems:

$$\mathcal{P} = (x, y) = (r \cos \theta, r \sin \theta) \tag{1}$$

Note that we have implicitly chosen a *right-handed* coordinate system, where with x running to the right, y points upward. This is in contrast to a *left-handed* system, in which y would run downward if x ran to the right. Left-handed and right-handed systems cannot be interchanged by a pure rotation (convince yourself of this!), but require an *inversion* in conjunction with a rotation, or a *mirror plane*.[i] We will always choose a right-handed system.

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[i]Inversion means changing the signs of all of the coordinates, e.g., $(x, y) \to (-x, -y)$. Inversion with rotation is called an *improper rotation*. An improper rotation of an object produces a rotation of its mirror image, and changes left-handed to right-handed or vice versa. A mirror plane means switching the sign of only one of the coordinates, e.g., $(x, y) \to (x, -y)$, which also changes left-handed into right-handed.

1.2   3D systems

In three dimensions, there are three common coordinate systems we will use: cartesian (x, y, z), cylindrical (R, φ, z), and spherical (r, φ, θ) systems. Note that the cartesian system is still a right-handed one – if your fingers point along the x axis ($\hat{\mathbf{x}}$), and your thumb along the y axis ($\hat{\mathbf{y}}$), the z axis ($\hat{\mathbf{z}}$) points out of the back of your hand for a right-handed system. Note that some authors use different conventions for specifying angles and distance, particularly in the cylindrical and spherical systems!
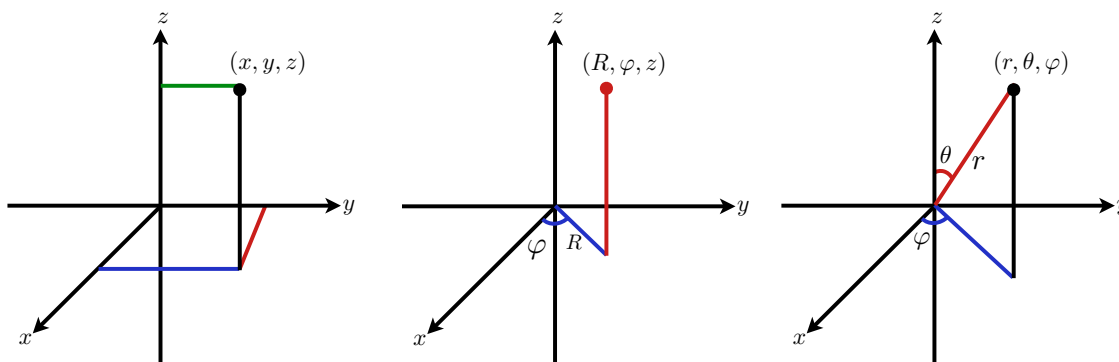

Figure 1: *Common 3-D coordinate systems.* **left:** *Cartesian* (x, y, z), **center:** *cylindrical* (R, φ, z), **right:** *spherical* (r, φ, θ).

The interrelationships between these three coordinate systems also follow from basic geometry, provided one takes care in the definitions of the angles φ and θ ... Below we list the range of the angular variables within the conventions we will adopt, as well as the interrelationship between different coordinate systems.

Table 1: *Range of angular variables in 3D coordinate systems*

system	variable	range	relevant plane
cylindrical	φ	$0 - 2\pi$	xy
spherical	φ	$0 - 2\pi$	xy
spherical	θ	$0 - \pi$	inclination from xy

1.3   Rotation of coordinate systems

1.3.1   Two dimensions

Take a normal (x, y) system, and pick a point $P(x, y)$. A *counterclockwise* rotation about the origin by θ (rotating x toward y in a right-handed system) means that the point $P(x, y)$ is described by the coordinates $P'(x', y')$ in the rotated frame, as shown below.

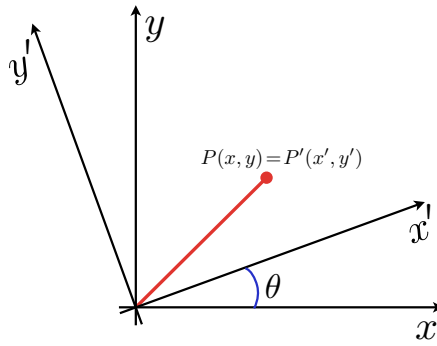|	**cartesian** (x, y, z)	**cylindrical** (R, φ, z)	**spherical** (r, φ, θ)
(x, y, z)	$x = x$ $y = y$ $z = z$	$x = R\cos\varphi$ $y = R\sin\varphi$ $z = z$	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$
(R, φ, z)	$R = \sqrt{x^2 + y^2}$ $\varphi = \tan^{-1}\frac{y}{x}$ $z = z$	$R = R$ $\varphi = \varphi$ $z = z$	$R = r\sin\theta$ $\varphi = \varphi$ $z = r\cos\theta$
(r, θ, φ)	$r = \sqrt{x^2 + y^2 + z^2}$ $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$ $\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$	$r = \sqrt{R^2 + z^2}$ $\varphi = \varphi$ $\theta = \tan^{-1}\left(\frac{R}{z}\right)$	$r = r$ $\varphi = \varphi$ $\theta = \theta$

Figure 2: *Rotation of a 2D coordinate system.*

Using basic trigonometry, we can readily find a relationship between the two coordinate representations:

$$x' = x\cos\theta - y\sin\theta \qquad\qquad x = x'\cos\theta + y'\sin\theta \qquad\qquad (2)$$

$$y' = x\sin\theta + y\cos\theta \qquad\qquad y = x'\sin\theta - y'\cos\theta \qquad\qquad (3)$$

Or, writing this as a matrix equation,

$$P'_\theta = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} P \equiv \mathbf{R}(\theta)P \tag{4}$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \equiv \mathbf{R}'(\theta)P'_\theta \tag{5}$$

Here, the matrix $R(\theta)$ acting on the column vector P is known as a *rotation matrix*, quite sensibly. Rotation matrices must have a determinant of ± 1 in order to preserve length.[ii]

On the other hand, if make a *clockwise* rotation of θ about the origin (rotating y toward x in a right-handed system), the new coordinates after rotation are

$$x' = x\cos\theta + y\sin\theta \tag{6}$$

$$y' = -x\sin\theta + y\cos\theta \tag{7}$$

In terms of the rotation matrix,

$$P'_{-\theta} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} P \equiv \mathbf{R}(-\theta)P \tag{8}$$

$$\tag{9}$$

Note that as expected $R(\theta)$ and $R(-\theta)$ are the same if one substitutes $-\theta$ for θ, and vice versa.[iii]

This much follows from basic trigonometry. In Sect. 2.10 we will consider how vectors transform under coordinate rotations. Below are a few additional examples of specific rotations in two dimensions.

$$R(90°) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad\qquad \text{Counterclockwise by } 90°$$

$$R(180°) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2 \qquad\qquad 180°; \ I_2 = \text{identity matrix}$$

$$R(-90°) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -R(90°) \qquad\qquad \text{Clockwise by } 90° \tag{10}$$

Note that $R(180°) = R(-90°)R(-90°) = R(90°)R(90°)$, or that two successive $90°$ rotations is the

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[ii] As a result, rotations also preserve the angles between vectors.
[iii] One can equivalently say that $R(\theta)$ is the matrix transpose of $R(-\theta)$.

same as a 180° rotation (you can think of successive multiplications by rotation matrices as successive rotations). Further, $R(-90°)R(90°)=I_2$ and $R(-90°)R(90°)=I_2$ give no net rotation, and one obtains the identity matrix I_2 as required.

Another point to note is that for a rotation about a fixed origin, distances from the origin must be preserved. This requires, in two dimensions,

$$x^2 + y^2 = x'^2 + y'^2$$

which you can verify is true for relationships above. This is equivalent to requiring that the rotation matrices have a determinant of ± 1 ($\det R = \pm 1$), where $+$ and $-$ correspond to proper and improper rotations, respectively (see below). This also makes the rotation matrices *orthogonal transformations*.

1.3.2 Three dimensions

In three dimensions, we have three types of rotations about a fixed origin to consider: rotation of the x, y, and z axes in a counterclockwise direction while looking at the origin. The three basic rotation matrices (for a right-handed coordinate system) are

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \text{rotates } y \text{ axis toward } z \text{ axis} \tag{11}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & 1 \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \text{rotates } z \text{ axis toward } x \text{ axis} \tag{12}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{rotates } x \text{ axis toward } y \text{ axis} \tag{13}$$

Any rotation can be given as a composition of rotations about these three axis, and can thus be represented by a 3×3 matrix operating on a column vector describing position.

1.3.3 Proper and Improper Rotations: Handedness

We have stated already that we always choose a *right-handed* coordinate system. What this means is that if in two dimensions we choose an x axis, have two possible choices for an orthogonal y axis (and in 3D, picking x and y leaves two choices for z). The two possible choices are left-handed, and right-handed. In the two-dimensional right-handed case, if your right index finger points along x,

your right thumb points along y (or, if x increases to the right, then y increases upward). This is in contrast to a *left-handed* system, in which y would run downward if x ran to the right. Left-handed and right-handed systems cannot be interchanged by a pure rotation (convince yourself of this!), but require an *inversion* in conjunction with a rotation, or a *mirror plane*.

Inversion means changing the signs of all of the coordinates, e.g., $(x, y, z) \rightarrow (-x, -y, -z)$. Inversion followed by rotation is called an *improper rotation*. An improper rotation of an object produces a rotation of its mirror image, and thus changes left-handed to right-handed or vice versa. A mirror plane means switching the sign of only one of the coordinates, e.g., $(x, y, z) \rightarrow (x, y, -z)$, which also changes left-handed into right-handed. Inspection of the rotation matrices is enough to tell you whether you are dealing with a proper or improper rotation. An (orthogonal matrix) A is classified as proper (corresponding to pure rotation) if $\det(A) = 1$, where $\det(A)$ is the determinant of A, or improper if $\det(A) = -1$. For example, using the rotation matrix in 2D given above,

$$\det R = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \tag{14}$$

On the other hand, the following matrix must be an improper rotation:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{15}$$

$$\det A = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -\cos^2\theta - \sin^2\theta = -1 \tag{16}$$

For the special case $\theta = 0$, you can see that this matrix carries (x, y) into $(x, -y)$, which can also be viewed as a mirror operation (reflecting y about a mirror placed along the x axis). For $\theta = 90°$, this matrix swaps the x and y axis (reflection about $y = x$).

2 Vectors

2.1 What is a vector?

So what is a vector? Probably you've heard the most common definition of a vector already: it is an object that has both magnitude and direction. That is certainly not precise enough to do anything with – we haven't actually defined either magnitude or direction. Moreover, it is a bit misleading to think of a vector as just an amalgamation of direction and magnitude. A course heading is one example, such as "20 km/h due north." Our heading has a magnitude (20 km/h) and a direction (due north), but either part alone is insufficient. One might better say that *a vector*

is a quantity that can be usefully represented with an arrow. The character of the quantity is such that it requires both an amount and a certain directionality to have full meaning and utility, and the whole is more than the sum of its parts. We will characterize vectors by their magnitude and direction, but while keeping in mind that our vector is really just an arrow, and magnitude and direction is just one way of representing it.

2.2   Representing vectors

Our working definition, then, is that a vector is a physical quantity that can be represented by an arrow of a certain length and orientation. Usually, we will want to represent vectors in relation to some coordinate system. This does not change the vector one bit, it only gives you an easy way to describe it mathematically. For instance, if we choose an $x - y$ cartesian coordinate system, we can define vectors in terms of their *components*. What do we mean by that? Basically, that we can define an arrow by saying "starting at a particular point, move over so many units in the x direction, and up so many in the y direction." Concretely, we could describe an arrow of length $\sqrt{2}$ pointing $45°$ above the x axis by moving one unit along the x axis, one unit along the y axis, and connecting the origin to that point:
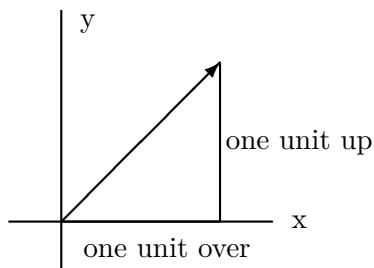
We need some nicer mathematical way to describe this definition if it is going to be useful. Therefore, we will define some new symbols, which basically mean "move one unit along the given axis." We'll call these things *unit vectors*, and there will be one for each axis in our coordinate system. You will know when you see a unit vector, because they wear hats:

$\hat{\imath}$ or $\hat{\mathbf{x}}$ move one unit along $+x$

$\hat{\jmath}$ or $\hat{\mathbf{y}}$ move one unit along $+y$

$\hat{\mathbf{k}}$ or $\hat{\mathbf{z}}$ move one unit along $+z$

Thus, our earlier vector could be written $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ – move one unit along the x axis, and then one along the y axis to build the arrow. Generally, the unit vectors are tied to the coordinate axes in the coordinate system you are using, one unit vector per axis. In most cases we will deal with, the coordinate axes will be mutually orthogonal (perpendicular), so the unit vectors will be uniquely

defined by going one unit in the positive direction along each axis.[iv]

Now that we can compactly describe a vector, it would also be nice to name the vector itself, rather than always writing out the components. Why? We should have a symbol for the arrow itself because the arrow is independent of whatever coordinate system we choose, but its description is not! It is the arrow that is important, not the components. If we change coordinate systems (say, from $x - y$ to $r - \theta$), the description will change, but the vector itself will not. We will therefore, first and foremost, name our arrows, and we will signify them with a bolded symbol with an arrow, such as $\vec{\mathbf{v}}$. Once we have chosen a particular coordinate system, we can represent $\vec{\mathbf{v}}$ in that system. In cartesian coordinates, our earlier vector would be $\vec{\mathbf{v}} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$. Keep in mind, however, that we do not even *need* a coordinate system, it is just a way to put our arrow in context. A coordinate system is a matter of choice, and often convenience. The vector itself is the fundamental thing.

For now, however, we will stick to 'normal' cartesian $x - y$ coordinates just to keep things concrete. Let's say we write a new vector, $\vec{\mathbf{v}} = 4\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$. This means our vector $\vec{\mathbf{v}}$ is described by moving 4 units along the x axis and 3 along the y; this forms a 3-4-5 triangle, so our $\vec{\mathbf{v}}$ must be of length 5. In a simple two-dimensional $x - y$ coordinate system, it is easy to see that finding vector components really just requires solving right triangles. Now consider a generic vector, $\vec{\mathbf{r}} = r_x\,\hat{\mathbf{x}} + r_y\,\hat{\mathbf{y}}$ (move r_x units along the x axis, and r_y along the y axis). We now know how to draw this and label the components:
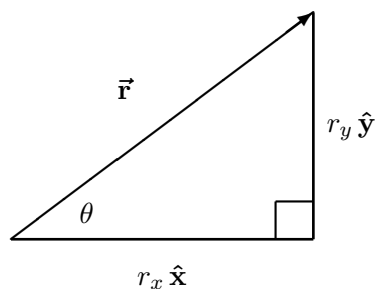
For our generic vector $\vec{\mathbf{r}} = r_x\,\hat{\mathbf{x}} + r_y\,\hat{\mathbf{y}}$, we can readily find the length as $|\vec{\mathbf{r}}| = \sqrt{r_x^2 + r_y^2}$ and the angle with respect to the x axis $\theta = \tan^{-1}(r_y/r_x)$.

This way of describing vectors is known as *component notation*, meaning we describe our arrows by building them out of individual components (unit vectors) along each of the coordinate axes. Later, we will also do this for other coordinate systems as well. Does it matter which unit vector you

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[iv]In spherical coordinates, for example, this is more subtle. A tiny *infinitesimal* rotation and a tiny *infinitesimal* radial displacement are orthogonal, even though rotation and radial displacements do not commute for finite angular displacements. When we use unit vectors in a spherical coordinate system, we should think of the $\hat{\theta}$ direction as being a unit vector pointing in the direction of an infinitesimal increase in θ, not a vector pointing from one point to another after rotating through a finite angle. More on this later!

consider first? Should one move along the x axis first, or along the y? In any orthogonal coordinate system (which we deal with exclusively), it does not matter. If the vector is $\vec{v} = 4\hat{x} + 3\hat{y}$, we can first move 4 units along x and then 3 units along y, or vice versa, it makes no difference.

Why did we represent our vectors in terms of x and y components? Convenience, mostly. There is no reason why we could not represent the same vector in terms of polar coordinates – move so many units in the radial direction (which we'll call $\hat{r}$), and then so many units in the direction of increasing angle (which we'll call $\hat{\theta}$).[v] Just as any point can be represented in either polar or cartesian coordinates, the same vector can be represented in different coordinate systems, or in a different *basis*. In a cartesian basis, we might have a vector $\vec{r} = x\hat{x} + y\hat{y}$, the same vector in a polar basis would just be $\vec{r} = r\hat{r}$, where $r = \sqrt{x^2 + y^2}$ and the angle with respect to the x axis is still $\theta = \tan^{-1}(y/x)$. The transformation is essentially the same as for single points in space, and in any system, we decompose vectors according to the unit vectors which define our *basis*[vi] for that space.

Before moving onward, let's review our notation quickly (and add a few new things) before we move on to manipulating vectors.

Table 3: *Vector notation*

quantity	remark
$\vec{A}$	a vector
$\hat{i}$ or $\hat{x}$	unit vector in the $+x$ direction
$\hat{j}$ or $\hat{y}$	unit vector in the $+y$ direction
$\hat{k}$ or $\hat{z}$	unit vector in the $+z$ direction
$\vec{A} = 3\hat{i} + 2\hat{j}$	specifying a vector by components
$|\vec{A}|$ or A	magnitude (length) of the vector $\vec{A}$

2.3 Adding vectors

As we will find out soon, vectors are terribly useful little things. Before we can appreciate this, however, we will need to know a few more properties of vectors and how to manipulate them.

Let us imagine that we use vectors to define how far we have hiked in a given day, in reference to our base camp. The first day, we travel $3\,\text{km}$ north, and $2\,\text{km}$ east, while on the second day, we travel $1\,\text{km}$ south and $3\,\text{km}$ east. On the third day, base camp wants to know a heading to our

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[v]Again, for a given radial position, the 'direction of increasing angle' should be thought of as the direction you would travel on a circle of that radius. The angular unit vector changes its orientation for different points in the plane.

[vi]Quoth the Wikipedia, "A basis is a set of vectors that, in a linear combination, can represent every vector in a given vector space or free module, and such that no element of the set can be represented as a linear combination of the others." For a cartesian system, our basis is $\hat{x}$, $\hat{y}$, $\hat{z}$.

position. How can we specify it?

We can start by defining each day's travel by a vector. In order to do that, we first specify a coordinate system with which to anchor the arrows. Our base camp can be the origin, and we will let east be the $+x$ direction, and north the $+y$ direction (so west is $-x$ and south is $-y$). With these definitions, each day's travel vector is:

$$\text{first day} \quad d_1 = (2\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}})\,\text{km} = d_{1x}\,\hat{\mathbf{x}} + d_{1y}\,\hat{\mathbf{y}}$$
$$\text{second day} \quad d_2 = (3\,\hat{\mathbf{x}} - 1\,\hat{\mathbf{y}})\,\text{km} = d_{2x}\,\hat{\mathbf{x}} + d_{2y}\,\hat{\mathbf{y}}$$

Notice that our vector has units, since we are talking about distance, and that moving west the second day means moving in the $-x$ or $-\hat{\boldsymbol{\imath}}$ direction. Graphically, our travel looks something like this:
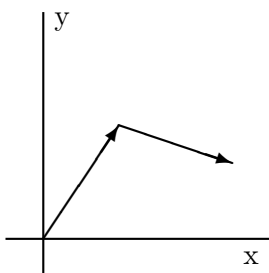
The heading to our current position on the third day has to be the sum of our previous two day's travel in some way: we have to add the two arrows together. How do we do this? We realize again that it does not matter whether we first travel along x and then y, displacements in orthogonal coordinate systems (coordinate systems in which the different axes are perpendicular) *commute*. Thus, for the whole trip, we can add together the total x distances and the total y distances, and say that we went a total of $2+3=5$ km along the $\hat{\mathbf{x}}$ (east) direction, and a total of $3-1=2$ km along the $\hat{\mathbf{y}}$ direction. We can then use the distance formula to figure out that our total displacement from base camp is $d_{\text{tot}} = \sqrt{5^2 + 2^2} \approx 5.4$ km.

What we just did was to find the length of the total trip geometrically, which basically find the length of a third arrow which connects the tail of our first arrow to the head of the second. The third arrow, describing the distance from base camp after two days, is 5 units along $\hat{\mathbf{x}}$ and 2 units along $\hat{\mathbf{y}}$, or $d_{\text{tot}} = 5\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$. We did this by adding the distances in the orthogonal (perpendicular) directions separately, and then applied the Pythagorean theorem (distance formula) to find the net length. This is precisely the same as the length of a new arrow whose $\hat{\mathbf{x}}$ component is the sum of those the first two arrows and whose $\hat{\mathbf{y}}$ component is the sum of those of the first two arrows! We can add our vectors by component, and we get the same result that we would geometrically. More

generally, we have hit on the method by which one adds vectors – we just add the like components to make a new vector. In the particular case we have presented,

$$\vec{\mathbf{d}}_1 = (2\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}})\,\text{km} = d_{1x}\,\hat{\mathbf{x}} + d_{1y}\,\hat{\mathbf{y}}$$
$$\vec{\mathbf{d}}_2 = (3\,\hat{\mathbf{x}} - 1\,\hat{\mathbf{y}})\,\text{km} = d_{2x}\,\hat{\mathbf{x}} + d_{2y}\,\hat{\mathbf{y}}$$
$$\vec{\mathbf{d}}_{\text{tot}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 = (5\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}})\,\text{km} = (d_{1x} + d_{2x})\,\hat{\mathbf{x}} + (d_{1y} + d_{2y})\,\hat{\mathbf{y}}$$
$$|\vec{\mathbf{d}}| = \sqrt{5^2 + 2^2}\,\text{km} = \sqrt{(d_{1x} + d_{2x})^2 + (d_{1y} + d_{2y})^2} \approx 5.4\,\text{km}$$

How about subtracting vectors? Same thing, but we reverse the component of one vector so $\vec{\mathbf{A}} - \vec{\mathbf{B}}$ is the same as $\vec{\mathbf{A}} + \left(-\vec{\mathbf{B}}\right)$, and $-\vec{\mathbf{B}}$ is just $\vec{\mathbf{B}}$ with the signs of all components reversed. The general rules for addition are then:

$$\vec{\mathbf{A}} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$$
$$\vec{\mathbf{B}} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}$$
$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = (a_x + b_x)\,\hat{\mathbf{x}} + (a_y + b_y)\,\hat{\mathbf{y}}$$
$$\vec{\mathbf{A}} - \vec{\mathbf{B}} = (a_x - b_x)\,\hat{\mathbf{x}} + (a_y - b_y)\,\hat{\mathbf{y}}$$

2.4 Unit vectors

In order to easily specify vector quantities in two dimensions, we need a consistent set of orthonormal unit vectors. A unit vector is denoted by a (typically lowercase) letter with a caret or "hat" over it, e.g., $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\imath}}$, as we discussed above. A unit vector $\hat{\mathbf{a}}$ can be defined from any vector $\vec{\mathbf{a}}$ by dividing $\vec{\mathbf{a}}$ by its length, termed *normalizing* the vector:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} \tag{17}$$

This gives a vector of unit length pointing in the same direction as $\vec{\mathbf{a}}$. Within any coordinate system (space), we wish to choose a *basis*, or a set of vectors that, in a linear combination, can represent every vector in the space uniquely. Typically, our basis is a set of orthonormal vectors, meaning a set of unit vectors which are mutually perpendicular. In a normal cartesian (x, y, z) system, our orthonormal vectors are simple unit vectors along the x, y, and z axes, or $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

The most commonly encountered bases are cartesian, polar, and spherical coordinates, as shown above. Each uses different unit vectors according to the symmetry of the coordinate system, but the basic rule is one unit vector per coordinate. Unit vectors representing "distance" coordinates, such as $\hat{\mathbf{r}}$, $\hat{\mathbf{z}}$, or $\hat{\mathbf{y}}$ simply point in the direction of increasing distance. Unit vectors representing

"angular" coordinates, such as $\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\theta}}$ point toward increasing angle, counterclockwise in our right-handed systems.

The relationships between unit vectors in 2D polar and cartesian systems is the same as the relationships between the coordinate systems, subjected to the condition of unit length. Given $\vec{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$, and the fact that $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ should be perpendicular ($\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$), one can derive the relationships below (presuming a right-handed coordinate system).

Table 4: *Relationships between unit vectors in cartesian and polar systems.*

cartesian (x, y)	polar (r, θ)
$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$	$\hat{\mathbf{x}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$
$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}$	$\hat{\mathbf{y}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$

One can also transform between unit vectors using rotation matrices, which we will return to in Sect. 2.10. Note again that the one rotation matrix is simply the transpose of the other, and that both have a determinant of 1.

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix} \qquad (18)$$

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix} \qquad (19)$$

In three dimensions, the transformations are similar. Note that in our conventions, the angle in the xy plane is φ in cylindrical and spherical coordinates and θ in spherical coordinates is the inclination above the xy plane.

Table 5: *Relationships between unit vectors in different 3D systems.*

	cartesian (x, y, z)	cylindrical (R, φ, z)	spherical (r, θ, φ)
(x, y, z)	$\hat{\mathbf{x}} = \hat{\mathbf{x}}$ $\hat{\mathbf{y}} = \hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \cos\varphi\,\hat{\mathbf{R}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\varphi\,\hat{\mathbf{R}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{x}} = \sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}} = \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$
(R, φ, z)	$\hat{\mathbf{R}} = \frac{x}{R}\hat{\mathbf{x}} + \frac{y}{R}\hat{\mathbf{y}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$ $\hat{\boldsymbol{\varphi}} = -\frac{y}{R}\hat{\mathbf{x}} + \frac{x}{R}\hat{\mathbf{y}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{R}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{R}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$
(r, θ, φ)	$\hat{\mathbf{r}} = \frac{1}{r}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$ $\hat{\mathbf{r}} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \frac{1}{r}\left(R\,\hat{\mathbf{R}} + z\,\hat{\mathbf{z}}\right)$ $\hat{\boldsymbol{\theta}} = \frac{1}{r}\left(z\,\hat{\mathbf{R}} - R\hat{\mathbf{z}}\right)$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	$\hat{\mathbf{r}} = \hat{\mathbf{r}}$ $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$

Note that unlike the cartesian unit vectors, $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ are not uniquely defined! The 3D transfor-

mations can also be written as matrix equations using rotation matrices; for example, transforming from cartesian to spherical polar coordinates, or cartesian and cylindrical coordinates

$$
\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\varphi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{bmatrix}
\tag{20}
$$

$$
\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} \end{bmatrix} = R_z(\varphi) \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} \end{bmatrix}
\tag{21}
$$

The remaining transformations can also be written as matrix equations ... which we leave as an exercise to the reader!

2.5   Scalar multiplication

We can also scale the length of our arrows by a constant amount. Say we have a vector $\vec{\mathbf{A}}$, and we want a second vector $\vec{\mathbf{B}}$ which is twice as long as $\vec{\mathbf{A}}$ but pointing in the same direction. Simply done: if we multiply both the x and y components of $\vec{\mathbf{A}}$ by 2, the length is twice as much, but the angle is the same.

$$
\vec{\mathbf{A}} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} \qquad \vec{\mathbf{B}} = 2a_x\hat{\mathbf{x}} + 2a_y\hat{\mathbf{y}}
$$
$$
|\vec{\mathbf{A}}| = \sqrt{a_x^2 + a_y^2} \qquad \theta_A = \tan^{-1}\left(a_y/a_x\right)
$$
$$
|\vec{\mathbf{B}}| = 2\sqrt{a_x^2 + a_y^2} \qquad \theta_B = \tan^{-1}\left(a_y/a_x\right)
$$

When we multiply all components of a vector by a plain number c, the length of the resulting vector is just c times bigger. Scalar multiplication is distributive, associative, and commutative:

Table 6: *Algebraic properties of scalar multiplication*

formula	relationship
$\vec{\mathbf{a}}c = c\vec{\mathbf{a}}$	commutative
$c \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = c\vec{\mathbf{a}} + c\vec{\mathbf{b}}$	distributive
$cd\vec{\mathbf{a}} = dc\vec{\mathbf{a}} = c\vec{\mathbf{a}}d = d\vec{\mathbf{a}}c = \vec{\mathbf{a}}cd = \vec{\mathbf{a}}dc$	associative

2.6 Scalar products

Say you have two vectors, $\vec{a}$ and $\vec{b}$:

$$\vec{a} = a_x\,\hat{\mathbf{x}} + a_y\,\hat{\mathbf{y}} + a_z\,\hat{\mathbf{z}} \tag{22}$$

$$\vec{b} = b_x\,\hat{\mathbf{x}} + b_y\,\hat{\mathbf{y}} + b_z\,\hat{\mathbf{z}} \tag{23}$$

What if we want to find the *projection* of vector $\vec{a}$ onto vector $\vec{b}$, or the part of $\vec{b}$ which is parallel to $\vec{a}$? A type of vector multiplication known as the 'dot' or scalar product of these two vectors gives the magnitude of $\vec{a}$ times the portion of $\vec{b}$ which is parallel to $\vec{a}$, or $\vec{a}$ times the projection of $\vec{b}$ onto $\vec{a}$. A good physical example of this type of product is *work*, which is the magnitude of a force parallel to the a displacement – we pick the part of the force parallel to the displacement, or the part of the displacement parallel to the force, and multiply the two. The result is a plain number, a scalar. Physical examples include work (force dotted into displacement), electric potential (electric field dotted into displacement), and power (force dotted into velocity).

We find this number for two arbitrary vectors by multiplying like components:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \sum_{i=1}^{n} a_i b_i \tag{24}$$

where the latter sum runs over the number of dimensions in your problem (so from 1 to 3 for a normal 3-dimensional problem). The dot product of a vector with itself thus gives the length of the vector:

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{25}$$

This gives us a geometric interpretation of the dot product of a vector $\vec{a}$ with itself: it is the length of a square of side $\vec{a}$. If we wish to find the component of a vector $\vec{b}$ which is parallel to $\vec{a}$, we may simply write $\vec{b} \cdot \hat{\mathbf{a}}$ – since $\hat{\mathbf{a}}$ has unit length, this simply 'picks out' the projection of $\vec{b}$ along the direction defined by $\hat{\mathbf{a}}$. Thus, if we have a force $\vec{F}$, $\vec{F} \cdot \hat{\mathbf{x}} = F_x$ picks out the x component of the force.

We can also write the scalar product in terms of vector magnitudes, and the angle between $\vec{a}$ and $\vec{b}$:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = \frac{1}{2}\left(|\vec{u} + \vec{v}|^2 - |\vec{u}|^2 - |\vec{v}|^2\right) \tag{26}$$

Here the last form – which is simply the law of cosines in disguise – can be derived by making use of Eq. 25 above. Put another way, given two vectors, the angle between them can be found readily:

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|}\right) \tag{27}$$

Of course, this implies that if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal (at right angles), their dot product is zero:

$$\text{if} \quad \vec{\mathbf{a}} \perp \vec{\mathbf{b}}, \text{ then} \quad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0 \tag{28}$$

Moreover, two vectors are orthogonal (perpendicular) *if and only if* their dot product is zero, and they have non-zero length, providing a simple way to test for (or define) orthogonality. A few other properties are tabulated below.

Table 7: *Algebraic properties of the scalar product*

formula	relationship
$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	commutative
$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$	distributive
$\vec{\mathbf{a}} \cdot (r\vec{\mathbf{b}} + \vec{\mathbf{c}}) = r(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) + (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})$	bilinear
$(c_1\vec{\mathbf{a}}) \cdot (c_2\vec{\mathbf{b}}) = (c_1 c_2)(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$	multiplication by scalars
if $\quad \vec{\mathbf{a}} \perp \vec{\mathbf{b}}$, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	orthogonality

2.7 Scalar Products of Unit Vectors

For completion, we tabulate the scalar product between unit vectors in different coordinate systems. Given unit length, in a normal Euclidean space (the types of spaces we will deal with), the dot product between two unit vectors is simply the cosine of the angle between them. If unit vectors *within the same system* are orthogonal, the angle is $\pm 90°$ for unlike unit vectors and 0 for like unit vectors, and the dot product is either zero or one, respectively. If two vectors have unit length (normalized) and are normal to one another (orthogonal), we call them *orthonormal* vectors. The vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are orthonormal, and we would say they form an *orthonormal basis* for our cartesian coordinate system.

Between different systems, we need to first define the two unit vectors of interest in terms of a common basis – for example, to find $\hat{\mathbf{r}} \cdot \hat{\mathbf{x}}$ in two dimensions, we could first define $\hat{\mathbf{r}}$ in terms of $\hat{\mathbf{x}}$

and $\hat{\mathbf{y}}$, or $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$, and then calculate the dot product by the distributive property,

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = (\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}) \cdot \hat{\mathbf{x}} = \cos\theta\,\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = \cos\theta \tag{29}$$

Table 8: *Scalar products of unit vectors in two dimensions*

	Cartesian		Polar	
	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\theta}}$
$\hat{\mathbf{x}}$	1	0	$\cos\theta$	$-\sin\theta$
$\hat{\mathbf{y}}$	0	1	$\sin\theta$	$\cos\theta$

Table 9: *Scalar products of unit vectors in three dimensions*

	Cartesian			Spherical			Cylindrical		
	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\varphi}}$	$\hat{\mathbf{R}}$	$\hat{\boldsymbol{\varphi}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{x}}$	1	0	0	$\sin\theta\cos\varphi$	$\cos\theta\cos\varphi$	$-\sin\varphi$	$\cos\varphi$	$-\sin\varphi$	0
$\hat{\mathbf{y}}$	0	1	0	$\sin\theta\sin\varphi$	$\cos\theta\sin\varphi$	$\cos\varphi$	$\sin\varphi$	$\cos\varphi$	0
$\hat{\mathbf{z}}$	0	0	1	$\cos\theta$	$-\sin\theta$	0	0	0	1

2.8   Vector products

The 'cross' or vector product between two polar vectors results in a *pseudovector*, also known as an 'axial vector.'[vii] The cross product takes two vectors and generates a new vector *perpendicular to the plane formed by the first two vectors.* The cross product is therefore inherently three-dimensional, a point to remember. Physical examples of cross products are angular momentum (radial distance crossed into momentum), torque (radial distance crossed into force), magnetic force (velocity crossed into magnetic field), and the magnetic field (current crossed into displacement).

For instance, if we take the cross product of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, denoted $\hat{\mathbf{x}} \times \hat{\mathbf{y}}$, we generate a vector perpendicular to the xy plane. But which one? Both $\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ are perpendicular to the xy plane, so the whole z axis is a fair choice, in either direction. Thus, we have an additional choice (or 'degree of freedom') in defining the cross product. In fact, we had this same choice when picking the y axis. If the x axis is to the right, why do we pick the $+y$ axis as pointing upward (Sect. 1.1)? Consistency, and that is all. This choice is known as *handedness*, or *chirality*, since your choice picks whether

---

[vii]Pseudovectors act just like real vectors, except they gain a sign change under improper rotation. See for example, the Wikipedia page "Pseudovector." An improper rotation is an inversion followed by a normal (proper) rotation, just what we are doing when we switch between right- and left-handed coordinate systems. A proper rotation has no inversion step, just rotation.

your coordinate system is right-handed or left-handed. To illustrate this, pick $\hat{\mathbf{x}}$ along the index finger of your right hand, and $\hat{\mathbf{y}}$ along your right thumb. A right-handed system means $\hat{\mathbf{z}}$ points out of the back of your hand. Try it with the left hand – you don't generate the same coordinate system! The figure below will help illustrate this choice, we hope.
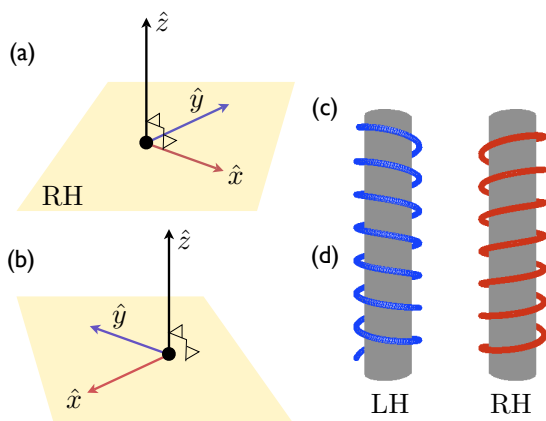

Figure 3: (a) *A right-handed coordinate system.* (b) *A left-handed coordinate system.* (c,d) *Right- and left-handed screws. No simple rotation can turn left-handed into right-handed, an inversion (mirror reflection) is also required.*

No amount of proper rotation alone will turn left-handed into right-handed, an *improper* rotation is required – rotating along with a inversion (changes the sign of all coordinates) or a mirror operation (changes the sign of one coordinate). We always pick the right-handed system, and there is a simple rule for picking the proper one: the aptly-named right hand rule.

> **Right-hand rule**
> 1. Point the fingers of your right hand along the direction of $\vec{\mathbf{a}}$.
> 2. Point your thumb in the direction of $\vec{\mathbf{b}}$.
> 3. The vector $\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ points out from the back of your hand.

The vector product of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ then returns a unit vector perpendicular to the xy plane consistent with this rule, $\hat{\mathbf{z}}$. If two vectors are parallel, there is no unique perpendicular axis or direction perpendicular to both, and their cross product is zero. For example, $\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$. We can already define the cross product for cartesian unit vectors, then:

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} \tag{30}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} \tag{31}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} \tag{32}$$

One key to remember is that the unit vectors in a orthogonal coordinate system follow a cyclical permutation of xyz. Break the order, and you pick up a minus sign (as that corresponds to an improper rotation, or mirror operation).

That settles the orientation of the vector product, what of its magnitude? The scalar product of two vectors returned a magnitude which was the magnitude of both vectors times the cosine of their interior angle. The vector product returns a magnitude which is the magnitude of both vectors times the *sine* of their interior angle. Rather than projecting one vector on to another, the magnitude of a cross product is equivalent to finding the area of a parallelogram defined by the two vectors, and the direction tells us the orientation of the resulting area. If we have two non-parallel vectors $\vec{a}$ and $\vec{b}$, their cross product is in magnitude $|\vec{a}||\vec{b}|\sin\theta$. For perpendicular vectors, this is just the area of a rectangle of sides $|\vec{a}|$ and $|\vec{b}|$.

Suppose we have two arbitrary vectors. How do we find the unique vector corresponding to their vector product, with both magnitude and direction? As it turns out, the rules for unit vectors above and brute-force multiplication is enough. Let our two vectors be $\vec{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\vec{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$. We can just multiply them, using the rules for cross products of unit vectors and our usual rules for scalar multiplication:

$$
\begin{aligned}
\vec{a} \times \vec{b} &= (a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}) \times (b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}) \\
&= a_x b_x \hat{\mathbf{x}} \times \hat{\mathbf{x}} + a_x b_y \hat{\mathbf{x}} \times \hat{\mathbf{y}} + a_x b_z \hat{\mathbf{x}} \times \hat{\mathbf{z}} \\
&\quad + a_y b_x \hat{\mathbf{y}} \times \hat{\mathbf{x}} + a_y b_y \hat{\mathbf{y}} \times \hat{\mathbf{y}} + a_y b_z \hat{\mathbf{y}} \times \hat{\mathbf{z}} \\
&\quad\quad + a_z b_x \hat{\mathbf{z}} \times \hat{\mathbf{x}} + a_z b_y \hat{\mathbf{z}} \times \hat{\mathbf{y}} + a_z b_z \hat{\mathbf{z}} \times \hat{\mathbf{z}} \\
&= 0 + a_x b_y \hat{\mathbf{z}} + a_x b_z (-\hat{\mathbf{y}}) + a_y b_x (-\hat{\mathbf{z}}) + 0 + a_y b_z \hat{\mathbf{x}} + a_z b_x \hat{\mathbf{y}} + a_z b_y (-\hat{\mathbf{x}}) + 0 \\
&= (a_y b_z - a_z b_y)\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\hat{\mathbf{z}}
\end{aligned}
$$

There is a nice symmetry to the thing (cyclic permutations!), but it is still a beast to remember. An easy way to remember how to calculate the cross product of these two vectors, $\vec{c} = \vec{a} \times \vec{b}$, is to write it as the determinant of the following matrix:

$$
\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \tag{33}
$$

Or, explicitly:

$$\vec{\mathbf{c}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{\mathbf{x}} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} \hat{\mathbf{y}} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{\mathbf{z}} = (a_y b_z - a_z b_y)\,\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\,\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\,\hat{\mathbf{z}}$$

(34)

From this, one can show that the magnitude of the cross product is just what we stated above,

$$|\vec{\mathbf{c}}| = |\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta \tag{35}$$

where θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. As we stated before, geometrically, the cross product can be viewed as the area of a parallelogram with sides defined by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ or the (signed) volume of a parallelepiped defined by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$. Again, note that if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are collinear (*i.e.,* the angle between them is either $0°$ or $180°$), the cross product is zero. We can also use the cross product to define a unit normal vector to a plane formed by vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:

$$\hat{\mathbf{n}} = \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}}{|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|} = \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta} = \frac{\hat{\mathbf{a}} \times \hat{\mathbf{b}}}{\sin\theta} \tag{36}$$

Check this by verifying that the z axis is perpendicular to the $x - y$ plane, since the angle between x and y axes is $90°$. Finally, this lets us write the cross product of two vectors in terms of their magnitudes and a mutually perpendicular unit vector:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta\,\hat{\mathbf{n}} \tag{37}$$

Unlike the scalar product, the vector product is *anticommutative* – the order of multiplication matters (as with matrices, a generalization of vector multiplication). It is also distributive over addition, and has numerous other algebraic properties listed in the table below. We also list decompositions of various higher-order products.

One higher order product has a particularly nice matrix representation:

$$\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}} \right) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (a_x b_y c_z - a_x b_z c_y) + (a_y b_z c_x - a_y b_x c_z) + (a_z b_x c_y - a_z b_y c_x) \tag{38}$$

formula	relationship
$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$	anticommutative
$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + (\vec{a} \times \vec{c})$	distributive over addition
$(r\vec{a}) \times \vec{b} = \vec{a} \times (r\vec{b}) = r(\vec{a} \times \vec{b})$	compatible with scalar multiplication
$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$	not associative; obeys Jacobi identity
$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$	triple vector product expansion
$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) = -\vec{a}(\vec{b} \cdot \vec{c}) + \vec{b}(\vec{a} \cdot \vec{c})$	triple vector product expansion
$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$	triple scalar product expansion[†]
$|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$	relation between cross and dot product
if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ then $\vec{b} = \vec{c}$ iff $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$	lack of cancellation

[†]Note that the parentheses may be omitted without causing ambiguity, since the dot product cannot be evaluated first. If it were, it would leave the cross product of a vector and a scalar, which is not defined

2.9 Polar and Axial Vectors

One defining feature of cross products is that they specify a mutually perpendicular *axis* which is unique, but the mutually perpendicular *direction* is a matter of choice – there is a 'handedness' built in. The fact that $\vec{a}$, $\vec{b}$, and $\vec{c}$ defined by $\vec{c} = \vec{a} \times \vec{b}$ are mutually perpendicular implies a unique *axis* for each, since in three dimensions there *are* only three mutually perpendicular axes. This fact alone does not determine a unique *direction* for all three, however. We have two possible choices for the convention of direction, corresponding to two senses of "handedness," or two possible coordinate systems, as shown in Fig. 3. You may recall the same problem when learning about torque and angular momentum. Or, if you are a chemist, you know this problem as *chirality*. An object is said to be 'chiral' if its mirror image cannot be superimposed on the original. No amount of rotation or translation will make the mirror image look exactly like the original. Your hands are good examples - no amount of rotation or manipulation will change a left hand into a right hand, hence the name. This is clearly the case for the the two coordinate systems in Fig. 3a and Fig. 3b, or the two helixes in Fig. 3c.

The $\hat{\mathbf{z}}$ axis defined by a cross product $\hat{\mathbf{x}} \times \hat{\mathbf{y}}$ is in some sense chiral. If we were to reverse the direction of $\hat{\mathbf{x}}$, then we would also have to reverse the direction of $\hat{\mathbf{y}}$ but not $\hat{\mathbf{z}}$. Similarly, we could reverse both $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ and $\hat{\mathbf{y}}$ would be left unchanged. This is a result of the fact that the result of a cross product between two normal vectors is technically a *pseudovector*, not a true vector. Pseudovectors act just like real vectors, except they gain a sign flip under improper rotation. An improper rotation is an inversion followed by a normal (proper) rotation, just what we are doing when we switch between right- and left-handed coordinate systems. A proper rotation has no inversion step, just rotation. The diagram of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in Fig. 3 is not equivalent to its mirror image, and is hence chiral. A cross product of two vectors is not equivalent to its mirror image,

hence the anticommutative property

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}} \tag{39}$$

Because of this "handedness," cross products do not transform like true vectors under inversion. The lack of invariance under inversion (mirror reflection) defines a pseudovector. A normal vector, called a "polar" vector does not suffer this problem. Polar vectors are things like force, velocity, or electric field, and under either proper or improper rotations, the situation looks the same. Here is an example adapted from the Wikipedia, using the cartesian unit vectors:

> A simple example of an improper rotation in 3D (but not in 2D) is inversion through the origin: each (true) vector $\vec{\mathbf{v}}$ goes to $-\vec{\mathbf{v}}$. Under this transformation, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ go to $-\hat{\mathbf{x}}$ and $-\hat{\mathbf{y}}$ (by the definition of a vector), but $\hat{\mathbf{z}}$ clearly does not change. It follows that any improper rotation multiplies $\hat{\mathbf{z}}$ by -1 compared to the rotation's effect on a true vector.

How can you tell what is a real vector and what is not? A real vector displayed in its coordinate system will look the same in a mirror, and under arbitrary rotations of the underlying coordinate system, the vector will retain the same description. A pseudovector will flip its sign under a mirror reflection, and the physical situation will not look the same when reflected in a mirror. A pseudovector is often called an *axial vector* because it usually implies a unique *axis* but not a unique *direction*. This is the case for angular momentum. Think about a spinning a ball around your head on a string. If we reverse the direction of rotation (say, clockwise to counterclockwise), the angular momentum reverses direction even though the radial distance does not. If you do this in front of a mirror, the mirror image seems to be spinning the ball in the opposite sense (e.g., counterclockwise instead of clockwise). You don't behave like your mirror image because a pseudovector, angular momentum, is involved. More simply, we can consider the angular velocity of the ball

$$\vec{\boldsymbol{\omega}} = \frac{\vec{\mathbf{v}} \times \vec{\mathbf{r}}}{r^2} \tag{40}$$

Angular velocity, being proportional to a cross product of the polar vectors velocity and position, is a pseudovector. It has an associated axis defining the rotation, but the same axis corresponds to both positive and negative rotation, and there is always a choice in how to define the *direction* of the angular velocity. It is perpendicular to both $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$, but the direction is a matter of choosing the right-handed system. If we reverse both $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$, $\vec{\boldsymbol{\omega}}$ does *not* reverse. If you spin the ball in a clockwise direction, your mirror image spins the ball in a counter-clockwise direction. Contrast this with the case of simply throwing a ball to your right while looking in a mirror. Your mirror image throws the ball to its left, and the resulting velocity is also to the left in the mirror image. All relevant quantities reverse direction, and the physics is the same because distance, velocity and acceleration are normal "polar" vectors, which do transform properly under inversion. Whenever

you have an axial vector involved, there are going to be three quantities, and only two of them will look the same in a mirror.

Further, whenever we have a cross product, an axial vector is involved somewhere, and an axis of rotation is implied. The cross product of two polar vectors (such as distance and momentum) is an axial vector (such as angular momentum), whereas the cross product of an axial vector and a polar vector (such as velocity and magnetic field) is a polar vector. Below, we summarize the types of relationships:

Table 11: *Cross products between polar and axial vectors*

type	type	cross product	example
polar	polar	axial	$\vec{r} \times \vec{p} = \vec{L}$
axial	axial	axial	$\Omega \times \vec{L} = \tau$ (gyroscope)
polar	axial	polar	$\vec{v} \times \vec{B} = F_B$

Below are a few examples of polar and axial vectors. In each case, for the axial vectors you can think of a related axis of rotation.

Table 12: *Examples of polar and axial vectors*

quantity	vector	type
displacement	$\Delta\vec{x}$	polar
velocity	$\vec{v}$	polar
momentum	$\vec{p}$	polar
force	$\vec{F}$	polar
angular momentum	$\vec{L}$	axial
angular velocity	$\vec{\omega}$	axial
torque	$\vec{\tau}$	axial
magnetic field	$\vec{B}$	axial

Finally, the main message is this: if you find yourself needing the right-hand rule, you have an axial vector involved! Why go through all of this, since adding one simple rule (the right-hand rule) is all we need to make sense of everything? Mainly, because pseudovectors run rampant in electricity and magnetism, and it can be quite subtle to even figure out what *direction* a quantity of interest takes, much less finishing some involved calculation of magnitude. Understanding the difference between polar and axial vectors will help you figure out when you need to take care to apply the right-hand rule, and when you need not bother.
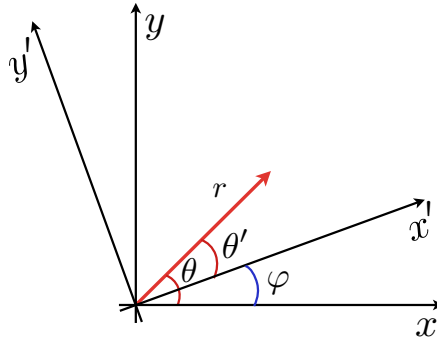
Figure 4: *Rotation of a 2D coordinate system.*

2.10 How Vectors Transform under Rotation, Inversion, and Translation

3 Vector Calculus

3.1 Derivatives of unit vectors in non-cartesian systems

3.2 Line, surface, and volume elements in different coordinate systems.

We will very often have occasion to use infinitesimal displacements along an arbitrary path $\vec{\mathbf{l}}$, represented by an infinitesimal vector $d\vec{\mathbf{l}}$, (oriented) infinitesimal patches of surface area from a larger surface $d\vec{\mathbf{S}}$, represented by $d\vec{\mathbf{S}}$, and infinitesimal bits of volume, represented by dV. Our line elements $d\vec{\mathbf{l}}$ simply point along the desired path. Surface elements are "oriented" in terms of a unit vector normal to their infinitesimal area, while the volume elements we deal with are simple scalar quantities.

These quantities are most transparent in a cartesian xyz system. A line element is simply an infinitesimal vector connecting the points (x, y, z) and $(x+dx, y+dy, z+dz)$, or $d\vec{\mathbf{l}} = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz$.

A surface element is just a square patch of area. If we have an infinitesimal patch of surface in the xy plane defined by dx and dy, its area is $dx\, dy$ and its "orientation" is a surface normal, $\hat{\mathbf{z}}$. For an arbitrarily-oriented surface, there is no simple general formula – one has to analyze the geometry of the situation and the orientation of the surface and define the infinitesimal area in terms of the given coordinates. Typically, one deals with high-symmetry surfaces, lying in a particular plane, so one may use simple constructions like $dy\, dz$ for surfaces in the yz plane, and so on.

The volume element is just a tiny cube of volume, bounded by dx, dy, and dz. Its volume is thus $dx\, dy\, dz$, and for our purposes it is a simple scalar quantity.

Line, surface, and volume elements in other coordinate systems are found by the coordinate transformations above, applied to infinitesimal displacements and rotations. For example, we can readily find the line, surface, and volume elements in spherical coordinates using the figure below.
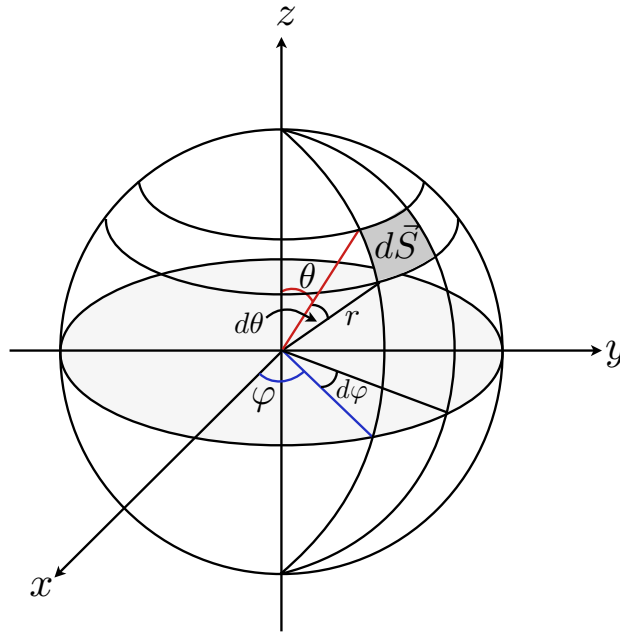

Figure 5: *Surface element in spherical coordinates*

The line element is found by connecting points (r, φ, θ) and $(r + dr, \varphi + d\varphi, \theta + d\theta)$. First, we move in the radial direction by dr. Next, we move along the $\hat{\boldsymbol{\varphi}}$ direction, in the xy plane. This means moving along a circle of radius $r \sin\theta$, the projection of the radial distance onto the xy plane, through an angle $d\varphi$. The length of this segment is the arclength $r \sin\theta \, d\varphi$. Finally, we move along the $\hat{\boldsymbol{\theta}}$ direction, which means moving along an arc of a circle of radius r through an angle $d\theta$. The length of this segment is the arclength $r \, d\theta$. Putting it all together, the vector describing this path is $d\vec{\mathbf{l}} = dr \, \hat{\mathbf{r}} + r \, d\theta \, \hat{\boldsymbol{\theta}} + r \sin\theta \, d\varphi \, \hat{\boldsymbol{\varphi}}$.

The surface element is, as mentioned above, not readily defined for the general case, but we *can* define an infinitesimal patch of area on the surface of a sphere, for instance, which is quite handy.[viii] The shaded area in the figure above is then $d\vec{\mathbf{S}} = r^2 \sin\theta \, d\theta \, d\varphi \hat{\mathbf{r}}$, being the product of the two arclengths specified above. Its "orientation" is in the direction of increasing r, viz., $\hat{\mathbf{r}}$.

The volume element is unique, and is the product of a patch of area on the surface of a sphere translated in the radial direction by dr, or a little parallelepiped of cross-sectional area $|d\vec{\mathbf{S}}|$ and

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[viii]For a sphere we fix r. We could also pick a cone, where we fix θ, for instance.

thickness dr. Thus, $dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$.

Cartesian

$$\vec{dl} = \hat{\mathbf{x}} \, dx + \hat{\mathbf{y}} \, dy + \hat{\mathbf{z}} \, dz \tag{41}$$

$$d\vec{S} = \begin{cases} dy \, dz \, \hat{\mathbf{x}} & \text{constant } x \\ dx \, dz \, \hat{\mathbf{y}} & \text{constant } y \\ dx \, dy \, \hat{\mathbf{z}} & \text{constant } z \end{cases} \tag{42}$$

$$dV = dx \, dy \, dz \tag{43}$$

Cylindrical

$$\vec{dl} = \hat{\mathbf{R}} \, dR + \hat{\boldsymbol{\varphi}} \, R \, d\varphi + \hat{\mathbf{z}} \, dz \tag{44}$$

$$d\vec{S} = \begin{cases} R \, d\varphi \, dz \, \hat{\mathbf{R}} & \text{constant } R \\ dR \, dz \, \hat{\boldsymbol{\varphi}} & \text{constant } \varphi \\ R \, dR \, d\varphi \, \hat{\mathbf{z}} & \text{constant } z \end{cases} \tag{45}$$

$$dV = R \, dR \, d\varphi \, dz \tag{46}$$

Spherical

$$\vec{dl} = dr \, \hat{\mathbf{r}} + r \, d\theta \, \hat{\boldsymbol{\theta}} + r \sin\theta \, d\varphi \, \hat{\boldsymbol{\varphi}} \tag{47}$$

$$d\vec{S} = \begin{cases} r^2 \sin\theta \, d\theta \, d\varphi \, \hat{\mathbf{r}} & \text{constant } r \\ r \sin\theta \, dr \, d\varphi \, \hat{\boldsymbol{\theta}} & \text{constant } \theta \\ r \, dr \, d\theta \, \hat{\boldsymbol{\varphi}} & \text{constant } \varphi \end{cases} \tag{48}$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi \tag{49}$$

3.3 Partial Derivatives

In introductory calculus, one of the first topics we are acquainted with is the derivative. What you are only told much later, however, is that there are really *two kinds* of basic derivatives: the *total derivative*, which is what you have encountered in introductory calculus, and the *partial derivative*.

The need for two types of derivatives comes when we consider functions of multiple variables, or functions of variables which themselves depend on other variables. Suppose we have a function $f(x)$, but x itself is a function of t, $x = x(t)$. In order to take the derivative of f with respect to t,

we would need to apply the chain rule:

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} \tag{50}$$

You might have encountered this when studying parametric curves. What if we have a function of two variables, both of which are also functions of another variable, like $z = (x(t), y(t))$? If we want to find the derivative of z with respect to t, we have two things to consider now. Our function z depends on t via $x(t)$, but also through $y(t)$. A variation in t causes a variation in x and y, both of which end up affecting z. Besides all of this, z might depend on t independently of x and y. Thus, our derivative should have three terms, and we might imagine something like this:

$$\frac{dz}{dt} = \frac{dz}{dt} + \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt} \qquad \text{INCORRECT} \tag{51}$$

We have missed something subtle, but crucial: what happens to y while we calculate dz/dx? While we find dz/dt, can x and y vary (since t apparently is)? Can y have any old value? How about t? The problem is the interrelationship between the variables: if we are to find dz/dx, we cannot let t vary while we do it, because that would change x. Similarly, we can't let y vary, because that means t must be varying. We have to find the influence of x on z while holding all other variables constant, that is, we should find how z depends on x and *only x*. This is where partial derivatives come in: a partial derivative of a function of several variables is just the derivative with respect to one of those variables while *all others are held constant*. This is what we want: find the derivative of z with respect to x while y and t stay put!

This is distinct from the *total derivative* we are used to, where we take the derivative with respect to one variable and all other variables are allowed to vary however they want. We use special notation for partial derivatives to make sure we know when we are holding all other variables constant and when we aren't. When it is a partial derivative, we'll use a ∂ symbol:

$\dfrac{\partial z}{\partial x}$ derivative of z with respect to x while all other variables are held constant

$\dfrac{dz}{dx}$ derivative of z with respect to x, other variables can change

Now we can find the derivative of z with respect to t taking into account the variable interrelationships properly:

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{52}$$

What this says is that the t variation of z has three parts: the first term is the explicit dependence of z on t, which we find while holding x and y constant; the second term is the dependence of z on t via x, which means we first find the dependence of z on x holding everything else constant, and multiply that by the rate x changes with t; and third term is analogous to the second.

Taking a slightly different example, suppose that f is a function of three variables x, y, and z. Normally these variables are assumed to be independent. However, in some situations they may be dependent on each other. For example, y and z could be functions of x. In this case the partial derivative of f with respect to x does not give the true rate of change of f with respect to x, because it does not take into account the dependency of y and z on x. The total derivative is a way of taking such dependencies into account.

For example, take $f(x, y, z) = xyz$. The rate of change of f with respect to x is normally determined by taking the partial derivative of f with respect to x, which is, in this case, $\partial f/\partial x = yz$. However, if y and z are not truly independent but depend on x as well this does not give the right answer. Suppose y and z are both equal to x. Then $f = xyz = x^3$ and so the (total) derivative of f with respect to x is $df/dx = 3x^2$. Notice that this is not equal to the partial derivative $yz = x^2$.

3.3.1 Example

Really, all we have done is apply the chain rule, but with special attention paid to interdependent variables. A physical example might help illustrate the difference. First, an example of the normal chain rule you are hopefully used to. From the Wikipedia,[ix]

> Suppose that a mountain climber ascends at a rate of 0.5 kilometers per hour. The temperature is lower at higher elevations; suppose the rate by which it decreases is 6°C per kilometer. To calculate the decrease in air temperature per unit time that the climber experiences, one multiplies 6°C per kilometer by 0.5 kilometer per hour, to obtain 3°C per hour. This calculation is a typical chain rule application.

Call temperature T, height h, and time t. The climber's height is a function of time $h(t)$, and the temperature is a function of height, $T(h)$. This problem says

$$\frac{dT}{dt} = \frac{dT}{dh}\frac{dh}{dt} \tag{53}$$

Here we have no problem using plain old total derivatives, because our variables are not truly interrelated – the temperature change with height is going to be true regardless of what the climber is doing. Put another way, the climber's progress does not affect the temperature gradient in the slightest. On the other hand, we can take a slightly different problem: let's say we want to find out

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[ix]Entry 'Chain Rule'.

how the volume of a cylinder changes as we change its radius. We know the volume of a cylinder of radius r and height h is

$$V(r, h) = \pi r^2 h \tag{54}$$

Now the partial and total derivatives have different meaning: $\partial V/\partial r$ means finding the rate at which volume changes with radius *as we hold height constant*, whereas dV/dr means finding the rate of change *while height is allowed to vary*. The former is found by differentiating V with respect to r while treating h as a *constant*:

$$\frac{\partial V}{\partial r} = 2\pi r h \qquad \text{volume change with } r \text{ at constant } h \tag{55}$$

Similarly, we can find the volume change with h at constant r:

$$\frac{\partial V}{\partial h} = \pi r^2 \qquad \text{volume change with } h \text{ at constant } r \tag{56}$$

The *total* derivative of volume with respect to r must take into account both effects: changing r changes V directly, but at the same time h might vary as well. The volume change with r in this case is found by differentiating V with respect to r while treating h as *a variable which might depend on r by itself:*

$$\frac{dV}{dr} = \frac{\partial V}{\partial r} + \frac{\partial V}{\partial h}\frac{dh}{dr} = 2\pi r h + \pi r^2 \frac{dh}{dr} \qquad \text{volume change with } r \text{ if } h \text{ varies} \tag{57}$$

If it turns out h is independent of r, the partial and total derivatives are the same, $\partial V/\partial r = dV/dr$, and this illustrates the main distinction between the two derivatives: partial derivatives hold all other variables constant, while total derivatives assume all variables depend on all others (and hence the chain rule must be applied). For completion, we can also find dV/dh:

$$\frac{dV}{dh} = \frac{\partial V}{\partial h} + \frac{\partial V}{\partial r}\frac{dr}{dh} = \pi r^2 + 2\pi r h \frac{dr}{dh} \qquad \text{volume change with } h \text{ if } r \text{ varies} \tag{58}$$

3.3.2 Tangents to 3D curves

Let's say we have a function of more than one variable that represents a surface in three dimensions, e.g.,

$$z = f(x, y) = x^4 + x^2 y^2 + y^4 \tag{59}$$

How would we describe the derivative of this function at a single point? Our usual single-variable approach is useless: at any given point, there are an infinite number of tangent lines to a point on the curve – we really need a tangent *plane* not a tangent line. We can clarify the situation with partial derivatives: if we hold one of the variables constant at some particular value, then there is only one such tangent line left. For example, we can choose y to be constant, which picks a tangent line $z(x, y)$ parallel to the x axis.

If we wish to find the tangent to the surface above at $(1, 1, 3)$ which is parallel to the x axis, we can treat y as a constant, which means finding $\partial z / \partial x$:

$$\frac{\partial z}{\partial x} = 4x^3 + 2xy^2 \tag{60}$$

Notice that the last term in z containing only y^4 is gone – it has no dependence on x, and y is just a constant when we take a partial derivative. Since we desire a tangent line at $(1, 1, 3)$, we know $y=1$ and $x=1$, and our slope is

$$\left. \frac{\partial z}{\partial x} \right|_{x=y=1} = 6 \tag{61}$$

Thus, at $(1, 1, 3)$ our curve has a tangent line parallel to the x axis with slope 6, or we might say "the partial derivative of z with respect to x at $(1, 1, 3)$ is 6."

3.3.3 Equality of second-order partial derivatives

As an aside, what happens when you take higher-order partial derivatives? For a function $f(x, y)$ there are four possibilities:

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 f}{\partial y \, \partial x} \qquad \frac{\partial^2 f}{\partial x \, \partial y} \tag{62}$$

The first two are sensible – we can simply differentiate twice with respect to either x or y. What about the last two? Does it matter if we first differentiate with respect to x and then y, or vice

versa? As it turns out, so long as the second partial derivatives are continuous (as nearly all well-behaved functions we will deal with are), the order does not matter, and we can exchange the order of partial differentiation at will. That is

$$\frac{\partial^2 f}{\partial y\, \partial x} \frac{\partial^2 f}{\partial x\, \partial y} \tag{63}$$

Going back to our example of the volume of a cylinder, we can verify this equality:

$$\frac{\partial^2 V}{\partial r\, \partial h} = \frac{\partial}{\partial r} \frac{\partial V}{\partial h} = \frac{\partial}{\partial r} \left(\pi r^2 \right) = 2\pi r \tag{64}$$

$$\frac{\partial^2 V}{\partial h\, \partial r} = \frac{\partial}{\partial h} \frac{\partial V}{\partial r} = \frac{\partial}{\partial h} \left(2\pi r h \right) = 2\pi r \tag{65}$$

3.3.4 Anti-partial-derivatives

What if we have a partial derivative and wish to reconstruct the original function? Let's say we have a function f and we know that $\partial f/\partial x = 2x + y$. We can still integrate this with respect to x, but since we have a partial derivative, we still treat y as a constant:

$$f = \int \frac{\partial f}{\partial x}\, dx = x^2 + xy + g(y) + (\text{const}) \tag{66}$$

Unlike "normal" integration, we do not have a constant of integration, but *a function of y*. Since our partial differentiation of f with respect to x treated y as a constant, we can add any function which only depends on y, $g(y)$, and we would get the same $\partial f/\partial x$! We have to take this into account when we "undo" the partial differentiation, and thus given only $\partial f/\partial x$ we cannot determine f uniquely, but only up to an overall additive function of y (and a possible constant term). Any functions of the form $f(x,y) = x^2 + xy + g(y) + (\text{const})$ satisfy $\partial f/\partial x = 2x + y$.

Given all the partial derivatives of a function (the gradient, discussed below), one can reconstruct the original function up to an overall constant. Say we have a function $\varphi(x, y, z)$, and we know all three partial derivatives of φ:

$$\frac{\partial \varphi}{\partial x} = x \qquad \frac{\partial \varphi}{\partial y} = y \qquad \frac{\partial \varphi}{\partial z} = z^2$$

Integrating each equation separately (i.e., integrating $\partial \varphi/\partial x$ with respect to x, etc.), we come up with three expressions for φ:

$$\varphi = \frac{1}{2}x^2 + f(y,z) + C_1 = \frac{1}{2}y^2 + g(x,z) + C_2 = \frac{1}{3}z^3 + h(y,x) + C_3$$

where the C_i are constants. The partial differentiation over x is insensitive to a function $f(y,z)$, while the partial differentiation over y is insensitive to a function $g(x,z)$, etc., and we must add these possible functions back to φ after each integration. If the function φ is to be unique, all three descriptions must match. This leads us to

$$f(y,z) = \frac{1}{2}y^2 + \frac{1}{3}z^3$$
$$g(x,z) = \frac{1}{2}x^2 + \frac{1}{3}z^3$$
$$h(y,x) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$
$$C_1 = C_2 = C_3 \equiv C$$

All three expressions for φ are consistent, and when substituted into the previous equation for φ, give the same unique function (up to the overall constant):

$$\varphi(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^3 + C$$

Thus, given all partial derivatives of a function, you can "match" the antiderivatives up to give back the original function, up to an overall constant. How to get the constant? For this you need a *boundary condition*, i.e., knowledge of the function itself at at least one point per variable.

3.3.5 Common Notations

There are many notations for partial derivatives. For instance, for first-order derivatives, all of the examples below represent the first partial derivative of a function f with respect to x:

$$\frac{\partial f}{\partial x} = f'_x = \partial_x f \tag{67}$$

The first is analogous to the usual Leibniz notation, where we replace the usual d for a total derivative with ∂. The second is also analogous to notation you have probably seen for the total derivative of a single-variable function, $df/dx = f'$, with the addition of a subscript x to clarify which variable we are differentiating on. The third is more commonly seen in physics, and it is really just a compact way of writing the first example. We can extend all three examples to higher order derivatives:

$$\frac{\partial^2 f}{\partial x^2} = f''_{xx} = \partial_x f \tag{68}$$

$$\frac{\partial^2 f}{\partial x \, \partial y} = f''_{xy} = \partial_{xy} f \tag{69}$$

At different times, we might use all three of these notations (though we will not mix them) depending on which is more convenient and less "cluttered." They all mean the same thing, it really is a stylistic choice most of the time.

3.4 The Gradient

Let us say we have a function $f(x)$ which takes an x coordinate and returns a y coordinate, $y = f(x)$. This function takes a scalar, and returns a scalar. One example in three dimensions might be the temperature in a room as a function of position: $T(x, y, z)$. This scalar function – a *field* describing the temperature at all points in the room – tells us everything we need to know about the temperature in the room. On the other hand, what if we were at a position (x, y, z) and wanted to know the direction in which to move in order to reach the warmest point in the room (the 'hot and cold' game you might have played)? Equivalently, our scalar function might be the height of a mountain as a function of lateral position, $z = h(x, y)$. In what direction should we travel to have steepest ascent toward the peak? What we desire is a vector pointing in the direction of the greatest rate of change. Using the example of temperature, the change in temperature dT with an infinitesimal change in distance from (x, y, z) to $(x + dx, y + dy, z + dz)$ is

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \tag{70}$$

We notice now that this looks a lot like a scalar product. Let us define a vector

$$\vec{\boldsymbol{\nabla}} T = \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \tag{71}$$

Then we write the temperature difference as a scalar product of this new vector and the infinitesimal displacement:

$$dT = \vec{\boldsymbol{\nabla}} T \cdot d\vec{\mathbf{l}} = |\vec{\boldsymbol{\nabla}} T||d\vec{\mathbf{l}}| \cos \theta \tag{72}$$

The vector $\vec{\boldsymbol{\nabla}} T$ we call the *gradient* of the scalar function $T(x, y, z)$, and it represents the rate of change of $T(x, y, z)$. Moreover, its magnitude $|\vec{\boldsymbol{\nabla}} T|$ is the *maximal* rate of change or slope of T with position, and its direction corresponds is the direction of maximal change in T with position.

In the mountain example, $\vec{\nabla} h$ points toward the quickest way up the mountain.

What if the gradient is zero? If $\vec{\nabla} h = 0$, we are in a locally flat region, or a contour of constant height (a hilltop, a valley, or a saddle point/shoulder). This is nothing more than the generalization of "derivative=slope" to three-dimensional surfaces. The symbol $\vec{\nabla}$ is a vector *operator*, which operates on a scalar function and returns a vector whose direction is that of maximum increase, and whose magnitude is the rate of increase. If our scalar field (function) $T(x, y, z)$ is one describing the temperature in a room, the *gradient* $\vec{\nabla} T$ points from our position (x, y, z) toward the direction where temperature rises most quickly, and $|\vec{\nabla} T|$ is the rate of temperature increase per meter of displacement.

As an example, let us take a simple scalar function $r = \sqrt{x^2 + y^2 + z^2}$, describing distance from the origin.

$$\vec{\nabla} r = \frac{\partial r}{\partial x}\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\hat{\mathbf{z}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\vec{\mathbf{r}}}{r} = \hat{\mathbf{r}} \tag{73}$$

The gradient of r is $\hat{\mathbf{r}}$, which makes sense: radial distance from the origin increases as at a constant rate, and most rapidly in the radial direction! The rate of increase is $|\hat{\mathbf{r}}| = 1$, as we expect for a normal Euclidian space! We can think of $\vec{\nabla}$ as *operating* on a scalar function, and returning a vector corresponding to the rate and direction of steepest ascent for that function. It is crucial that we remember that $\vec{\nabla}$ is an *operator* on a function, and not a vector by itself – indeed, it makes no sense on its own:

$$\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} + \frac{\partial \varphi}{\partial z}\hat{z} \tag{74}$$

The $\vec{\nabla}$ operator acts on a scalar function taken as an argument, and returns a vector. In fact, we know one example already, which you probably have only seen in one dimension: the gradient of potential energy gives force:

$$\vec{\nabla} U = -\vec{\mathbf{F}} \tag{75}$$

This equation, which we studied in one dimension (where $dU/dx = -F_x$), tells us that potential energy is minimum when force is zero, or that forces push objects toward the steepest *decrease* in potential energy (due to the minus sign). This is a handy thing, getting a full-blown vector function (field) from a simple scalar function. One caveat: our expression for the gradient is valid only for cartesian coordinates. Using our coordinate transformations, we can define the gradient in other

systems as well. We must take care, however, that unit vectors in spherical coordinates depend on position, and thus their spatial derivatives are nonzero. In spherical coordinates,

$$\vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{76}$$

and in cylindrical coordinates,

$$\vec{\nabla} = \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\varphi}}\frac{1}{R}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z} \tag{77}$$

3.4.1 Vector Functions

In fact, we can do more with our little operator. If we pretend for the moment that $\vec{\nabla}$ behaves like a vector under scalar and vector products, it could operate on functions in two other ways:

$$\text{operates on scalar function directly:} \quad \vec{\nabla}T = \text{gradient} \tag{78}$$
$$\text{operates via scalar product on a vector function:} \quad \vec{\nabla}\vec{v} = \text{divergence} \tag{79}$$
$$\text{operates via vector product on a vector function:} \quad \vec{\nabla} \times \vec{v} = \text{curl} \tag{80}$$

What does it mean to use our derivative operator on a vector function? Before we move on to that question, we should review vector functions a bit and see how they differ from scalar functions. An obvious generalization of our scalar functions is to consider a function $\vec{\mathbf{F}}(x)$ which takes a scalar argument and returns a vector. Such a function can be described by a set of scalar functions paired with the unit vectors in our basis:

$$\vec{\mathbf{F}}(x,y,z) = F_x(x,y,z)\hat{\mathbf{x}} + F_y(x,y,z)\hat{\mathbf{y}} + F_z(x,y,z)\hat{\mathbf{z}} \tag{81}$$

Our vector function (or *field*) is a set of three functions F_x, F_y, and F_z which describe the components of $\vec{\mathbf{F}}$. The F_i themselves are scalar functions, like our $f(x)$. Basically, the function $\vec{\mathbf{F}}$ is a vector whose $\hat{\mathbf{x}}$ component is determined by the value of $F_x(x,y,z)$, and so on, and $\vec{\mathbf{F}}$ in principle represents a different vector at every point in space, which is why we call it a vector *field*. A concrete example might be the gravitational force between two point masses, described in polar or cartesian coordinates:

$$\vec{\mathbf{F}}(x, y, z) = \frac{-Gm_1m_2}{x^2 + y^2 + z^2} \left(\frac{x\hat{\mathbf{x}}}{\sqrt{x^2 + y^2 + z^2}} + \frac{y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2 + z^2}} + \frac{z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \right) \tag{82}$$

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \frac{-Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{83}$$

Our vector function can be described completely by a set of three scalar functions and our unit vectors, taking an argument of either a position (x, y, z) or an equivalent radial vector $\vec{\mathbf{r}}$. At any point in space, $\vec{\mathbf{F}}$ is the (vector) force acting on a particle of mass m_1 due to a mass m_2.

3.5   Divergence

In cartesian coordinates, given a vector function $\vec{\mathbf{F}}(x, y, z)$:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{84}$$

3.6   Curl

In cartesian coordinates, given a vector function $\vec{\mathbf{F}}(x, y, z)$:

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{85}$$

4   Miscellaneous: Solving systems of linear equations

Say we have three equations and three unknowns, and we are left with the pesky problem of solving them. There are many ways to do this, we will illustrate two of them. Take, for example, three equations that result from applying Kirchhoff's rules to a particular multiple loop dc circuit:

$$I_1 - I_2 - I_3 = 0 \tag{86}$$
$$R_1I_1 + R_3I_3 = V_1 \tag{87}$$
$$R_2I_2 - R_3I_3 = -V_2 \tag{88}$$

The first way we can proceed is by substituting the first equation into the second:

$$V_1 = R_1 I_1 + R_3 I_3 = R_1 \left(I_2 + I_3 \right) + R_3 I_3 = R_1 I_2 + \left(R_1 + R_3 \right) I_3 \tag{89}$$

$$\implies \quad V_1 = R_1 I_2 + \left(R_1 + R_3 \right) I_3 \tag{90}$$

Now our three equations look like this:

$$I_1 - I_2 - I_3 = 0 \tag{91}$$

$$R_1 I_2 + \left(R_1 + R_3 \right) I_3 = V_1 \tag{92}$$

$$R_2 I_2 - R_3 I_3 = -V_2 \tag{93}$$

The last two equations now contain only I_1 and I_2, so we can solve the third equation for I_2 ...

$$I_2 = \frac{I_3 R_3 - V_2}{R_2} \tag{94}$$

... and plug it in to the second one:

$$V_1 = R_1 I_2 + \left(R_1 + R_3 \right) I_3 = R_1 \left(\frac{I_3 R_3 - V_2}{R_2} \right) + \left(R_1 + R_3 \right) I_3 \tag{95}$$

$$V_1 + \frac{V_2 R_1}{R_2} - \left(R_1 + R_3 + \frac{R_1 R_3}{R_2} \right) I_3 = 0 \tag{96}$$

$$I_3 = \frac{V_1 + \frac{V_2 R_1}{R_2}}{R_1 + R_3 + \frac{R_1 R_3}{R_2}} \tag{97}$$

$$I_3 = \frac{V_1 R_2 + V_2 R_1}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{98}$$

Now that you know I_3, you can plug it in the expression for I_2 above.

What is the second way to solve this? We can start with our original equations, but in a different order:

$$I_1 - I_2 - I_3 = 0 \tag{99}$$

$$R_2 I_2 - R_3 I_3 = -V_2 \tag{100}$$

$$R_1 I_1 + R_3 I_3 = V_1 \tag{101}$$

The trick we want to use is formally known as 'Gaussian elimination,' but it just involves adding these three equations together in different ways to eliminate terms. First, take the first equation

above, multiply it by $-R_1$, and add it to the third:

$$[-R_1I_1 + R_1I_2 + R_1I_3] = 0 \tag{102}$$

$$+ \qquad R_1I_1 + R_3I_3 = V_1 \tag{103}$$

$$\Longrightarrow \quad R_1I_2 + (R_1 + R_3)\,I_3 = V_1 \tag{104}$$

Now take the second equation, multiply it by $-R_1/R_2$, and add it to the new equation above:

$$-\frac{R_1}{R_2}\left[R_2I_2 - R_3I_3\right] = -\frac{R_1}{R_2}\left[-V_2\right] \tag{105}$$

$$+ \qquad R_1I_2 + (R_1 + R_3)\,I_3 = V_1 \tag{106}$$

$$\Longrightarrow \quad \left(\frac{R_1R_3}{R_2} + R_1 + R_3\right)I_3 = \frac{R_1}{R_2}V_2 + V_1 \tag{107}$$

Now the resulting equation has only I_3 in it. Solve this for I_3, and proceed as above.

Optional: There is one more way to solve this set of equations using matrices and Cramer's rule,[x] if you are familiar with this technique. If you are not familiar with matrices, you can skip to the next problem - you are not required or necessarily expected to know how to do this. First, write the three equations in matrix form:

$$\begin{bmatrix} R_1 & 0 & R_3 \\ 0 & R_2 & -R_3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \\ 0 \end{bmatrix} \tag{108}$$

$$\mathbf{aI} = \mathbf{V} \tag{109}$$

The matrix $\mathbf{a}$ times the column vector $\mathbf{I}$ gives the column vector $\mathbf{V}$, and we can use the determinant of the matrix $\mathbf{a}$ with Cramer's rule to find the currents. For each current, we construct a new matrix, which is the same as the matrix $\mathbf{a}$ except that the the corresponding column is replaced the column vector $\mathbf{V}$. Thus, for I_1, we replace column 1 in $\mathbf{a}$ with $\mathbf{V}$, and for I_2, we replace column 2 in $\mathbf{a}$ with $\mathbf{V}$. We find the current then by taking the new matrix, calculating its determinant, and dividing that by the determinant of $\mathbf{a}$. Below, we have highlighted the columns in $\mathbf{a}$ which have

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[x]See 'Cramer's rule' in the Wikipedia to see how this works.

been replaced to make this more clear:

$$I_1 = \frac{\begin{vmatrix} V_1 & 0 & R_3 \\ -V_2 & R_2 & -R_3 \\ 0 & -1 & -1 \end{vmatrix}}{\det \mathbf{a}} \qquad I_2 = \frac{\begin{vmatrix} R_1 & V_1 & R_3 \\ 0 & -V_2 & -R_3 \\ 1 & 0 & -1 \end{vmatrix}}{\det \mathbf{a}} \qquad I_3 = \frac{\begin{vmatrix} R_1 & 0 & V_1 \\ 0 & R_2 & -V_2 \\ 1 & -1 & 0 \end{vmatrix}}{\det \mathbf{a}} \tag{110}$$

Now we need to calculate the determinant of each new matrix, and divide that by the determinant of $\mathbf{a}$.[xi] First, the determinant of $\mathbf{a}$.

$$\det a = -R_1 R_2 - R_1 R_3 + 0 - 0 + 0 - R_2 R_3 = -(R_1 R_2 + R_2 R_3 + R_1 R_3) \tag{111}$$

We can now find the currents readily from the determinants of the modified matrices above and that of $\mathbf{a}$ we just found:

$$I_1 = \frac{-V_1 R_2 - V_1 R_3 + 0 - 0 + V_2 R_3 - 0}{-(R_1 R_2 + R_2 R_3 + R_1 R_3)} = \frac{V_1 (R_2 + R_3) - V_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{112}$$

$$\tag{113}$$

$$I_2 = \frac{R_1 V_2 - 0 - V_1 R_3 - 0 + 0 + R_3 V_2}{-(R_1 R_2 + R_2 R_3 + R_1 R_3)} = \frac{R_3 V_1 - V_2 (R_1 + R_3)}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{114}$$

$$\tag{115}$$

$$I_3 = \frac{0 - R_1 V_2 + 0 - 0 + 0 - V_1 R_2}{-(R_1 R_2 + R_2 R_3 + R_1 R_3)} = \frac{R_1 V_2 + R_2 V_1}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{116}$$

These are the same results you would get by continuing on with either of the two previous methods. Both numerically and symbolically, we can see from the above that $I_1 = I_2 + I_3$:

$$I_2 + I_3 = \frac{R_3 V_1 - V_2 (R_1 + R_3) + R_1 V_2 + R_2 V_1}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{117}$$

$$= \frac{V_1 (R_2 + R_2) + V_2 (R_1 - R_1 - R_3)}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{118}$$

$$= \frac{V_1 (R_2 + R_2) - V_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} = I_1 \tag{119}$$

[xi]Again, the Wikipedia entry for 'determinant' is quite instructive.