University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Exam 1 redux - solution

1. A particle of mass m traveling along a horizontal line with constant velocity v_o suddenly experiences a resistive force $f = -bv^3$. **(a)** Find an expression for $v(t)$. *Hint: it is always the same method.* **(b)** Does the object eventually halt, either at finite time or as $t \to \infty$? If so, under what condition, and if not, why?

bonus +1: What is the time constant of the $v(t)$ decay? That is, your result should have a term that looks like t/τ, what is τ?

Solution: The only force present is f, so we just need to set up the equation of motion, separate variables, and integrate.

$$m\ddot{x} = m\dot{v} = m\frac{dv}{dt} = f = -bv^3 \tag{1}$$

$$\frac{dv}{v^3} = -\frac{b}{m}\,dt \tag{2}$$

$$\int_{v_o}^{v} \frac{dv}{v^3} = -\int_0^t \frac{b}{m}\,dt = -\frac{b}{m}t \tag{3}$$

$$-\frac{b}{m}t = -\frac{1}{2v^2}\bigg|_{v_o}^{v} = -\frac{1}{2}\left(\frac{1}{v^2} - \frac{1}{v_o^2}\right) \tag{4}$$

$$\frac{2bt}{m} = \frac{1}{v^2} - \frac{1}{v_o^2} \tag{5}$$

$$\frac{1}{v^2} = \frac{2bt}{m} + \frac{1}{v_o^2} \tag{6}$$

$$v = \sqrt{\frac{1}{\frac{2bt}{m} + \frac{1}{v_o^2}}} \tag{7}$$

One can see that $v \to 0$ as $t \to \infty$, so the object does halt eventually. We can also see that v is a function of $2bt/m$. Comparing to the usual form t/τ, we identify the time constant for velocity decay as $\tau = m/2b$.

2. A rectangular block of height h and area A floats in water. If the density of the block is ρ_b and the density of the water is ρ_w, find the frequency of small oscillations when the block bobs up and down on the surface of the water. *Hints: The buoyant force is the weight of the displaced fluid. If the object is floating in equilibrium, what is the net force if it is displaced by some δx?*

bonus, +2: show that the frequency of oscillation is equivalent to that of a pendulum of length d_o, where d_o is the submerged depth of the block in equilibrium.

Solution: First let's find the equilibrium condition. In equilibrium, the buoyant force (weight of displaced water) equals the block's weight. Let x be length of the block that is below the surface of the water. Then

$$F_B = mg \tag{8}$$
$$\rho_w Axg = \rho_b Ahg \tag{9}$$
$$x_{\text{eq}} = \frac{\rho_b}{\rho_w} h \tag{10}$$

Now, if we push the block down an additional distance δx from equilibrium, there will be a net force due to the new volume of water displaced ($A\delta x$):

$$\delta F = -\rho_w Ag\delta x = ma = \rho_b Aha \tag{11}$$
$$a = -\frac{\rho_w g}{\rho_b h}\delta x = -\omega^2 \delta x \qquad \Longrightarrow \qquad \omega = \sqrt{\frac{\rho_w g}{\rho_b h}} \tag{12}$$

We end up with simple harmonic motion about the equilibrium point. Noting that $x_{\text{eq}} = \frac{\rho_b}{\rho_w} h$, we can write $\omega = \sqrt{g/x_{\text{eq}}}$, meaning the block behaves like a pendulum whose length is equal to the equilibrium depth.

3. **(a)** Write down the total energy E for a mass on a spring in 1D in terms of $\dot{x}$ and x. **(b)** Show, as is the case for any conservative 1D system, that you can obtain the equation of motion for the coordinate x by differentiating the equation $E = \text{const}$. It should be the familiar Newton's 2nd law result.

bonus $+1$: Suppose the energy were not constant, but you had it on good authority that energy was lost to the environment at a rate proportional to velocity squared, i.e,. $\dot{E} \propto -\dot{x}^2$. With the result of (b), show that this is in fact just a damped harmonic oscillator. *Hint: consider a constant of proportionality* 2β *to produce a familiar-looking result.*

Solution: We know the energy well enough. Remember the chain rule and $\dot{E}$ is found readily.

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{13}$$
$$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x} = 0 \tag{14}$$
$$m\ddot{x} + kx = 0 \tag{15}$$

Equation of motion recovered. If $\dot{E} = -2\beta\dot{x}^2$,

$$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x} = -2\beta\dot{x}^2 \tag{16}$$
$$0 = m\dot{x}\ddot{x} + 2\beta\dot{x}^2 + kx\dot{x} \tag{17}$$
$$0 = m\ddot{x} + 2\beta\dot{x} + kx \tag{18}$$

This is the equation for a damped harmonic oscillator we are familiar with.