University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Exam 1 Solution

1. A mass m is constrained to move along the x axis subject to a force $F(v) = -F_o e^{-v/u}$, where F_o and u are constants. **(a)** Find $v(t)$ if the initial velocity is $v_o > 0$ at $t = 0$. **(b)** At what time does it come instantaneously to rest?

Solution: With $F(v) = -F_o e^{-v/u}$, write down the second law, then separate variables.

$$m\frac{dv}{dt} = F(v) = -F_o e^{-v/u} \tag{1}$$

$$e^{v/u}\,dv = -\frac{F_o}{m}\,dt \tag{2}$$

Now we integrate, and that's that.

$$\int_0^t -\frac{F_o}{m}\,dt = -\frac{F_o t}{m} = \int_{v_o}^{v} e^{v/u}\,dv = u\left(e^{v/u}\right)\Big|_{v_o}^{v} = u\left(e^{v/u} - e^{v_o/u}\right) \tag{3}$$

$$-\frac{F_o t}{mu} = e^{v/u} - e^{v_o/u} \tag{4}$$

$$e^{v/u} = e^{v_o/u} - \frac{F_o t}{mu} \tag{5}$$

$$v = u\ln\left(e^{v_o/u} - \frac{F_o t}{um}\right) \tag{6}$$

The particle will come to rest when the velocity is zero, which will be true when when the argument of the $\ln()$ function in the equation above is equal to 1, i.e.,

$$1 = e^{v_o/u} - \frac{F_o t}{um} \tag{7}$$

$$\frac{F_o t}{um} = e^{v_o/u} - 1 \tag{8}$$

$$t = \frac{um}{F_o}\left(e^{v_o/u} - 1\right) \tag{9}$$

2. To perform a rescue, a lunar landing craft needs to hover just above the surface of the moon, which has a gravitational acceleration of $g/6 \approx 1.62\,\mathrm{m/s^2}$. The exhaust velocity is $2000\,\mathrm{m/s}$, but fuel amounting to only 20 percent of the total mass may be used. How long can the landing craft hover?

Solution: Let $g' = g/6$ for convenience. Our rocket equation is $m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}$, where in this case $F^{\text{ext}} = -mg'$. If the rocket is to hover, we require $\dot{v} = 0$, so $\dot{m}v_{\text{ex}} = -mg'$. Writing the derivatives explicitly, separate variables and integrate, noting the mass can go from m to λm with $\lambda = 0.8$ such that 20% of the total mass is burned.

$$\frac{dm}{dt}v_{\text{ex}} = -mg' \tag{10}$$

$$dt = -\frac{v_{\text{ex}}}{g'}\frac{dm}{m} \tag{11}$$

$$t = -\frac{v_{\text{ex}}}{g'}\int_{m}^{\lambda m}\frac{dm}{m} = -\frac{v_{\text{ex}}}{g'}\ln m\Big|_{m}^{\lambda m} = -\frac{v_{\text{ex}}}{g'}\ln\lambda \tag{12}$$

With $\lambda = 0.8$, $g' = g/6$, and $v_{\text{ex}} = 2000\,\text{m/s}$, you should find $t \approx 273\,\text{s}$.

3. Consider a gun of mass M (when unloaded) that fires a shell of mass m with muzzle speed v. (That is, the shell's speed *relative to the gun* is v.) Assuming the gun is completely free to recoil (no external forces on the gun or shell), use conservation of momentum, find the shell's speed relative to the ground in terms of v, m, and M.

Solution: Let v_s and v_g be the speeds of the shell and gun *with respect to the ground*, respectively. Then conservation of momentum tells us $mv_s = Mv_g$. The velocity given v is the *relative* velocity of the shell and gun, $v = v_s + v_g$ (where clearly one velocity is negative since they move in opposite directions). Using these two equations, we just want to eliminate v_g.

$$mv_s = Mv_g = M(v - v_s) = Mv - Mv_s \tag{13}$$

$$(m+M)\,v_s = Mv \tag{14}$$

$$v_s = \frac{M}{m+M}v = \frac{v}{1+m/M} \tag{15}$$

4. Near the surface of planet X, the gravitational force on a particle of mass m is vertically downward (along $-\hat{\mathbf{y}}$) but has magnitude $m\gamma y^2$, where γ is a constant and y is the mass' height above horizontal ground. **(a)** Find the work done by gravity on a mass m moving from $\mathbf{r}_1$ to $\mathbf{r}_2$. **(b)** Use this result to show that the force, though unusual, is conservative (hint: what is the work done in going from y_i to y_i?). **(c)** If I release a mass from height h above the planet's surface, how fast will it be going just before it reaches the ground?

Solution: The work done is readily calculated, we integrate $\mathbf{F}\cdot d\mathbf{r}$, with $\mathbf{r}$ going from (x_1, y_1) to (x_2, y_2).

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_1}\mathbf{F}\cdot d\mathbf{r} \tag{16}$$

Now in this case $\mathbf{F} = F_y\hat{\mathbf{y}}$, and $d\mathbf{r} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$, so $\mathbf{F}\cdot d\mathbf{r} = F_y\,dy$.

$$W = \int_{y_1}^{y_2}F_y\,dy = \int_{y_1}^{y_2}-m\gamma y^2\,dy = \frac{1}{3}m\gamma\left(y_1^3 - y_2^3\right) \tag{17}$$

The work is clearly independent of the path followed since it depends only on the starting and ending coordinates, and it is also clear that the work done around a closed path is zero, $W(\mathbf{r}_1 \to \mathbf{r}_1) = \frac{1}{3}m\gamma\left(y_1^3 - y_1^3\right) = 0$

The speed of the dropped object is found from conservation of energy. The change in potential energy is $\Delta U = -W(0 \to \mathbf{r}) = \frac{1}{3}m\gamma y^3$, the change in kinetic energy is $\frac{1}{2}mv^2$ as always. Thus,

$$\frac{1}{2}mv_f^2 = \frac{1}{3}m\gamma y^3 \tag{18}$$

$$v = \sqrt{\frac{2}{3}\gamma h^3} \tag{19}$$

5. An undamped oscillator has period $\tau_o = 1$ second. When weak damping is added, it is found that the amplitude of the oscillator drops by 50% in one period $\tau_1 = 2\pi/\omega_1$. **(a)** How big is β compared to ω_o? **(b)** What is τ_1? *Hint: recall* $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$ *and* $\omega_1^2 = \omega_o^2 - \beta^2$ *for weak damping.*

We know $\tau_o = 1\,\text{s}$, and $A(\tau_1)/A(0) = \frac{1}{2}$. First we can look at the amplitude. Given $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, the maximum amplitude is $Ae^{-\beta t}$. Further, we can note that the cos function must have equivalent values at two times exactly one period apart. Thus,

$$\frac{A(\tau_1)}{A(0)} = \frac{1}{2} = \frac{e^{-\beta\tau_1}}{e^0} = e^{-\beta\tau_1} \tag{20}$$

$$-\beta\tau_1 = \ln\frac{1}{2} \qquad \left(\text{recall: } -\ln a = \ln\frac{1}{a}\right) \tag{21}$$

$$\beta\tau_1 = \ln 2 \tag{22}$$

We have at least a relationship between β and τ_1. Now note that

$$\omega_1^2 = \omega_o^2 - \beta^2 \tag{23}$$

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2} \tag{24}$$

$$\Longrightarrow \qquad \tau_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega_o^2 - \beta^2}} \tag{25}$$

$$\Longrightarrow \qquad \beta\tau_1 = \frac{2\pi\beta}{\sqrt{\omega_o^2 - \beta^2}} = \frac{2\pi}{\sqrt{(\omega_o/\beta)^2 - 1}} = \ln 2 \tag{26}$$

Solving the last equation for β/ω_o,

$$\left(\frac{2\pi}{\ln 2}\right)^2 = \left(\frac{\omega_o}{\beta}\right)^2 - 1 \tag{27}$$

$$\frac{\beta}{\omega_o} = \frac{1}{\sqrt{1 + (2\pi/\ln 2)^2}} \approx 0.1097 \tag{28}$$

$$\beta \approx 0.11\omega_o \tag{29}$$

Now that we have β,

$$\frac{\tau_1}{\tau_o} = \frac{\omega_1}{\omega_o} = \frac{\omega_o}{\sqrt{\omega_o^2 - \beta^2}} = \frac{1}{\sqrt{1 - \beta^2/\omega_o^2}} \approx 1.006 \tag{30}$$

$$\tau_1 \approx 1.006\tau_1 \approx 1.006\,\text{s} \tag{31}$$

6. An approximation for the potential energy of two ions as a function of their separation is

$$U(r) = -\frac{ke^2}{r} + \frac{b}{r^9} \tag{32}$$

The first term is the usual Coulomb interaction, while the second term is introduced to account for the repulsive effect of the two ions at small distances. **(a)** Find b as a function of the equilibrium spacing r_o (recall that the equilibrium spacing is when U is minimum). **(b)** Find the frequency of small oscillations about $r = r_o$, assuming the molecule oscillates according to its effective mass m. *Hint: look at the definition of a Taylor series. Use your result from the first part in the second part to simplify.*

Solution: The equilibrium spacing will be characterized by the net force between the ions being zero, or equivalently, the potential energy being zero:

$$F(r_o) = -\left.\frac{dU}{dr}\right|_{r=r_o} = 0 = \frac{ke^2}{r_o^2} - \frac{9b}{r_o^{10}} \tag{33}$$

$$ke^2 r_o^8 = 9b \tag{34}$$

$$b = \frac{1}{9} ke^2 r_o^8 \tag{35}$$

Substituting this result back into our potential energy expression, we can find the potential energy at equilibrium, how much energy is gained by the system of ions condensing into a crystal. This gives the potential energy at the equilibrium position r_o:

$$U(r) = -\frac{ke^2}{r_o} + \frac{ke^2 r_o^8}{9r_o^9} = -\frac{8ke^2}{9r_o} \tag{36}$$

The frequency of small oscillations can be found by Taylor expanding the potential about equilibrium for small displacements from equilibrium:

$$U(r - r_o) \approx U(r_o) + U'(r_o)(r - r_o) + \frac{1}{2}U''(r_o)(r - r_o)^2 \tag{37}$$

The first term in the expansion is just the potential energy at equilibrium which we found above. The second term is proportional to the force, $F = -U'$, and therefore must vanish at equilibrium (which is exactly the condition we enforced to find b, after all). The third term is quadratic in displacement, just as it would be for a simple harmonic oscillator, $U = \frac{1}{2}k(r - r_o)^2$. Thus, the coefficient of the quadratic term must be $\frac{1}{2}k$, which means the frequency of small oscillations is $\omega = \sqrt{k/m}$, where m is the effective (reduced) mass of the system. That is, the diatomic molecule looks like two masses coupled by a spring.

$$\frac{1}{2}k = \frac{1}{2}U''(r_o) \tag{38}$$

$$k = U''(r_o) = -\frac{2ke^2}{r_o^3}\frac{90b}{r_o^{11}} = \frac{8ke^2}{r_o^3} \tag{39}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8ke^2}{mr_o^3}} \tag{40}$$