Exam 2 Formula Sheet

Lagrange:

$$\mathcal{L} = T - U \qquad \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad [i = 1, \ldots, n] \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Central Forces:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad U_{\text{eff}}(r) = U(r) + U_{\text{cf}} r) = U(r) + \frac{l^2}{2\mu r^2}$$

$$u = \frac{1}{r} \qquad u''(\varphi) = -u(\varphi) - \frac{\mu}{l^2 u(\varphi)^2} F$$

$$F = \frac{G m_1 m_2}{r^2} = \frac{\gamma}{r^2} \qquad r(\varphi) = \frac{c}{1 + \epsilon \cos\varphi} \qquad c = l^2/\gamma\mu \qquad E = \frac{\gamma^2 \mu}{2 l^2}(\epsilon^2 - 1) \qquad \epsilon = \frac{A l^2}{\gamma \mu}$$

$$\tau^2 = \frac{4\pi^2}{G M_s} a^3 \qquad \epsilon = 0\,\text{(circle)} \qquad 0 < \epsilon < 1\,\text{(ellipse)} \qquad \epsilon = 1\,\text{(parabola)} \qquad \epsilon > 1\,\text{(hyperbola)}$$

$$E > 0\,\text{(hyperbola)} \qquad E = 0\,\text{(parabola)} \qquad E = 0\,\text{(circle, ellipse)}$$

Oscillations:

$$\underline{\underline{\boldsymbol{M}}}\ddot{\mathbf{q}} = -\underline{\underline{\boldsymbol{K}}}\mathbf{q} \qquad T = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k \qquad U = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_k$$

$$\mathbf{q}(t) = \Re\big(\mathbf{a} e^{i\omega t}\big) \qquad (\underline{\underline{\boldsymbol{K}}} - \omega^2 \underline{\underline{\boldsymbol{M}}})\mathbf{a} = 0 \qquad \text{normal modes}$$

Things you know but can't quite remember:

$$g = 9.81\,\mathrm{m/s^2} \qquad \text{sphere} \quad V = \frac{4}{3}\pi r^3 \qquad ax^2 + bx^2 + c = 0 \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sin(\theta \pm \varphi) = \sin\theta\cos\varphi \pm \cos\theta\sin\varphi \qquad \cos(\theta \pm \varphi) = \cos\theta\cos\varphi \mp \sin\theta\sin\varphi$$

$$2\cos\theta\cos\varphi = \cos(\theta+\varphi) + \cos(\theta-\varphi) \qquad 2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$$

$$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi) \qquad c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$$

$$\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) \qquad \cos\theta - \cos\varphi = 2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

$$\sin\theta + \sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right) \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$$

Those other ones:

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) = \cos iz \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}) = \sin iz \qquad \tanh z = \frac{\sinh z}{\cosh z} \qquad \cosh^2 z - \sinh^2 z = 1$$

"What will this ever be good for:"

$$\frac{d}{dx}\sin ax = a\cos ax \qquad \frac{d}{dx}\cos ax = -a\sin ax \qquad \frac{d}{dx}\sinh ax = a\cosh ax \qquad \frac{d}{dx}\cosh ax = a\sinh ax$$

$$\frac{d}{dx}\tan z = \sec^2 z \qquad \frac{d}{dx}\tanh z = \mathrm{sech}^2\, z \qquad \frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

$$\int u\,dv = uv - \int v\,du \qquad \int \frac{dx}{1+x^2} = \arctan x \qquad \int \frac{dx}{1-x^2} = \mathrm{arctanh}\, x$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x \qquad \int \frac{dx}{\sqrt{1+x^2}} = \mathrm{arcsinh}\, x \qquad \int \tan(x)\,dx = -\ln\cos x \qquad \int \tanh x\,dx = \ln\cosh x$$

$$\int \frac{dx}{x+x^2} = \ln\left(\frac{x}{1+x}\right) \qquad \int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2) \qquad \int \frac{dx}{\sqrt{x^2-1}} = \mathrm{arccosh}\, x \qquad \int \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \arccos(1/x) \qquad \int \frac{\sqrt{x}\,dx}{\sqrt{1-x}} = \arcsin(\sqrt{x}) - \sqrt{x(1-x)} \qquad \int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{(1+x^2)^{1/2}}$$

$$\int \ln(x)\,dx = x\ln(x) - x \qquad \int \frac{dx}{x} = \ln x$$

Arrows:

$$|\mathbf{F}| = \sqrt{F_x^2 + F_y^2} \quad \text{magnitude} \qquad \theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \quad \text{direction} \qquad \mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = (a_y b_z - a_z b_y)\,\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\,\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\,\hat{\mathbf{z}}$$

Almost, $x \ll 1$

$$f(z) = f(a) + f'(a)(z-a) + \frac{1}{2!}f''(a)(z-a)^2 + \frac{1}{3!}f'''(z-a)^3 + \ldots$$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad \tan x \approx x + \frac{1}{3}x^3 \qquad \sin x \approx x - \frac{1}{3!}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \qquad \ln(1+z) \approx z - \frac{1}{2}z^2$$

should your calculator be in degrees or radians? ...just saying.

Revenge of the Arrows:

$$\Sigma \mathbf{F} = \mathbf{F}_{\text{net}} = m\mathbf{a} \qquad \Sigma F_x = ma_x \qquad \Sigma F_y = ma_y, \text{ etc.} \qquad F_{\text{grav}} = mg = \text{weight} \qquad \mathbf{F}_{12} = -\mathbf{F}_{21}$$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \approx \frac{\Delta(m\mathbf{v})}{\Delta t} \quad \text{direction} \ldots \qquad f_s \leq \mu_s n \qquad f_k = \mu_k n \qquad \mathbf{F}_c = -\frac{mv^2}{r}\,\hat{\mathbf{r}}$$

$$\mathbf{F} = m\ddot{\mathbf{r}} = \begin{cases} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \\ F_z &= m\ddot{z} \end{cases} \text{(Cartesian } x,y,z) = \begin{cases} F_r &= m(\ddot{r} - r\dot{\varphi}^2) \\ F_\varphi &= m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \end{cases} \text{(2D polar } r,\varphi)$$

$$\mathbf{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} = q_2 \mathbf{E}_1 \qquad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mathbf{F}_B = q\mathbf{v} \times \mathbf{B} \qquad \mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

The Arrows Strike Back:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \qquad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\boldsymbol{\nabla} f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \qquad \text{Cartesian} \qquad \boldsymbol{\nabla} f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi} \qquad \text{Spherical}$$

$$\boldsymbol{\nabla} \times \mathbf{F} = \hat{\mathbf{x}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \qquad \text{Cartesian}$$

$$\boldsymbol{\nabla} \times \mathbf{F} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\varphi \sin\theta) - \frac{\partial F_\theta}{\partial \varphi}\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\varphi)\right] + \hat{\boldsymbol{\varphi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right] \qquad \text{Spherical}$$

Blowback:

$$f(v) = f_{\text{lin}} + f_{\text{quad}} \qquad f_{\text{lin}} = bv \qquad f_{\text{quad}} = cv^2 \qquad \text{separate and integrate}$$

Space Oddity:

$$\frac{dx}{dt} = \dot{x} \qquad \frac{d^2x}{dt^2} = \frac{d\dot{x}}{dt} = \ddot{x} \qquad m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}$$

Centering and Rotation:

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}_\alpha = \frac{m_1\mathbf{r}_1 + \cdots + m_N\mathbf{r}_N}{M} \qquad \mathbf{l} = \mathbf{r} \times \mathbf{p} \qquad \mathbf{L} = \sum_{\alpha=1}^{N} \mathbf{l}_\alpha = \sum_{\alpha=1}^{N} \mathbf{r}_\alpha \times \mathbf{p}_\alpha \qquad \dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$$

Work, meh:

$$\Delta T = T_2 - T_1 = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \equiv W(1 \to 2) \qquad U(\mathbf{r}) = -W(\mathbf{r}_o \to \mathbf{r}) \equiv -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' \qquad \mathbf{F} = -\boldsymbol{\nabla} U$$

central force: $\mathbf{F} = f(\mathbf{r})\,\hat{\mathbf{r}}$ central force is spherically symmetric [f($\mathbf{r}$)=f(r)] if and only if conservative

Interacting is difficult (conservative forces):

$$U = U^{\text{int}} + U^{\text{ext}} = \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta} + \sum_\alpha U_\alpha^{\text{ext}} \qquad (\text{net force on } \alpha) = -\boldsymbol{\nabla}_\alpha U$$

Wiggling:

$$F = -kx \Longleftrightarrow U = \frac{1}{2}kx^2 \qquad \ddot{x} = -\omega^2 x \Longleftrightarrow x(t) = A\cos\left(\omega t - \delta\right)$$

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0 \Longleftrightarrow x(t) = Ae^{-\beta t}\cos\left(\omega_1 t - \delta\right) \qquad \beta = b/2m \quad \omega_o = \sqrt{\frac{k}{m}} \quad \omega_1 = \sqrt{\omega_o^2 - \beta^2} \quad (\beta < \omega_o)$$

$$\text{driven } A^2 = \frac{f_o^2}{\left(\omega_o - \omega\right)^2 + 4\beta^2\omega^2} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

Basic physics you should really know at this point, so I am printing it super small.

1-D motion:

$$v(t) = \frac{d}{dt}x(t) \qquad a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t)$$

$$v(t) = \int_0^t a\,dt \qquad x(t) = \int_0^t v\,dt$$

↑ const. acc.

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x \Delta x$$

$$v_f = v_i + at$$

Projectile motion:

$$v_x(t) = v_i \cos\theta \qquad v_y(t) = v_i \sin\theta - gt$$

$$x(t) = x_i + v_x t \qquad y(t) = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

over level ground:

$$\text{max height } = H = \frac{v_i^2 \sin^2\theta_i}{2g}$$

$$\text{Range } = R = \frac{v_i^2 \sin 2\theta_i}{g}$$

2-D motion:

$$\mathbf{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$$

$$x(t) = x_i + v_{ix}t + \frac{1}{2}a_x t^2$$

$$y(t) = y_i + v_{iy}t + \frac{1}{2}a_y t^2$$

$$\mathbf{v} = v_x(t)\,\hat{\imath} + v_y(t)\,\hat{\jmath}$$

$$v_x(t) = \frac{dx}{dt} = v_{xi} + a_x t$$

$$v_y(t) = \frac{dy}{dt} = v_{yi} + a_y t$$

$$\mathbf{a} = a_x(t)\,\hat{\imath} + a_y(t)\,\hat{\jmath}$$

$$a_x(t) = \frac{dv_x}{dt} \qquad a_y(t) = \frac{dv_y}{dt}$$

$$a_c = \frac{v^2}{r} \qquad T = \frac{2\pi r}{v} \quad \text{circ.}$$

Force:

$$\Sigma\mathbf{F} = \mathbf{F}_{\text{net}} = m\mathbf{a} \qquad \Sigma F_x = ma_x \qquad \Sigma F_y = ma_y$$

$$F_{\text{grav}} = mg = \text{weight}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \approx \frac{\Delta(m\mathbf{v})}{\Delta t} \quad \text{direction} \ldots$$

$$f_s \le \mu_s n \qquad f_k = \mu_k n$$

$$\mathbf{F}_c = -\frac{mv^2}{r}\,\hat{\mathbf{r}}$$

Work-Energy:

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

$$\Delta K = K_f - K_i = W$$

$$W = \int F(x)\,dx = -\Delta U \quad F = -\frac{dU}{dx} \quad \text{in equil.} \quad F = 0$$

$$U_g(y) = mgy \qquad U_s(x) = \frac{1}{2}kx^2$$

$$K_i + U_i = K_f + U_f + W_{\text{ext}} = K_f + U_f + \int F_{\text{ext}}\,dx$$

Rotation: we use radians

$$s = \theta r \quad \leftarrow \text{arclength}$$

$$\omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$a_t = \alpha r \quad \text{tangential} \qquad a_r = \frac{v^2}{r} = \omega^2 r \quad \text{radial}$$

$$I = \sum_i m_i r_i^2 \Rightarrow \int r^2\,dm = kmr^2$$

$$I_z = I_{com} + md^2 \quad \text{axis } z \text{ parallel, dist } d$$

$$\tau_{net} = \sum \boldsymbol{\tau} = I\boldsymbol{\alpha} = \frac{d\mathbf{L}}{dt}$$

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} \qquad |\boldsymbol{\tau}| = rF\sin\theta_{rF}$$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = I\boldsymbol{\omega} \qquad L_i = L_f$$

$$K = \frac{1}{2}I\omega^2 = L^2/2I$$

$$\Delta K = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W = \int \tau\,d\theta$$

$$P = \frac{dW}{dt} = \tau\omega$$

Gravitation:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{r^2}\,\hat{\mathbf{r}}_{12} = -\frac{dU_g}{dr}\,\hat{\mathbf{r}}$$

$$g = \frac{GM_e}{R_e^2}$$

$$U_g(r) = -\int F(r)\,dr = \frac{GMm}{r}$$

$$K + U_g \ge 0 \quad \text{escape} \qquad K + U_g < 0 \quad \text{bound}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} \qquad T^2 = \left(\frac{4\pi^2}{GM}\right)r^3$$

$$E_{\text{orbit}} = \frac{-GMm}{2a} \quad \text{elliptical; } a \to r \text{ for circular}$$

Oscillations:

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad T = \frac{1}{f}$$

$$x(t) = x_m \cos(\omega t + \varphi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 x \quad \text{for SHM}$$

$$\omega = \sqrt{k/m} \quad \text{spring}$$

$$U = -\frac{1}{2}kx^2 \quad F = -\frac{dU}{dx} = ma$$

$$T = \begin{cases} 2\pi\sqrt{I/\kappa} & \text{torsion pendulum} \\ 2\pi\sqrt{L/g} & \text{simple pendulum} \\ 2\pi\sqrt{I/mgd_{\text{cm}}} & \text{physical pendulum} \end{cases}$$