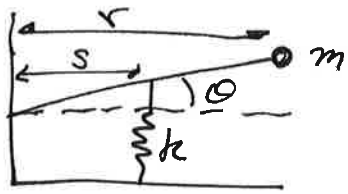
Ph301 exam2
F18

1.



$\Delta x \simeq r\theta$

$v \simeq r\dot\theta$

$$T = \tfrac{1}{2} m (r\dot\theta)^2 = \tfrac{1}{2} m r^2 \dot\theta^2$$

$$U_g = mgr\theta \qquad U_s = \tfrac{1}{2} k (s\theta)^2 = \tfrac{1}{2} k s^2 \theta^2$$

$$\mathcal{L} = \tfrac{1}{2} m r^2 \dot\theta^2 - \tfrac{1}{2} k s^2 \theta^2 - mgr\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{d}{dt}\left(m r^2 \dot\theta\right) = m r^2 \ddot\theta \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -ks^2\theta - mgr$$

$$\Rightarrow m r^2 \ddot\theta = -ks^2\theta - mgr$$

$$\ddot\theta = -\frac{ks^2}{mr^2}\theta - \frac{g}{r} \leftarrow$$ can define away by shift of origin

$$\Rightarrow \omega = \sqrt{\frac{ks^2}{mr^2}}$$

check: $s = r$, $\omega = \sqrt{\frac{k}{m}}$ just a mass-spring then

2. In geosynchronous, $\tau = 24\,\text{hr} = 86400\,\text{s}$

Kepler: $\tau = \frac{4\pi^2}{GM_e} a^3 \Rightarrow a^3 = \frac{\tau G M_e}{4\pi^2}$

circular orbit: $F_G = F_{\text{centr}}$ $\quad \frac{GM_e m}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM_e}{r}$

$$\Delta E = \Delta K + \Delta U = \tfrac{1}{2} m (v_1^2 - v_2^2) - GM_e m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\Delta E = \tfrac{1}{2} GM_e m\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - GM_e m\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = -\tfrac{1}{2} GM_e m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\Delta E \simeq 1.2 \times 10^{10}\,\text{J}$$

3. $F_1 = -k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2$

$F_2 = -k_2(x_2 - x_1) = m_2 \ddot{x}_2 = k_2 x_1 - k_2 x_2$

$m_1 = m_2$ $\quad \underline{\underline{M}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \underline{\underline{k}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$

$$\underline{\underline{K}} - \omega^2 \underline{\underline{m}} = \begin{bmatrix} k_1 + k_2 - \omega^2 m & -k_2 \\ -k_2 & k_2 - \omega^2 m \end{bmatrix}$$

$$\det(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) = k_2(k_1 + k_2) - (k_1 + k_2) m\omega^2 - k_2 \omega^2 m + \omega^4 m^2 - k_2^2 = 0$$

$$m^2 \omega^4 - (k_1 + 2k_2) m\omega^2 + k_1 k_2 = 0$$

$$\omega^2 = \frac{m(k_1 + 2k_2) \pm \sqrt{(k_1 + 2k_2)^2 m^2 - 4m^2 k_1 k_2}}{2m^2}$$

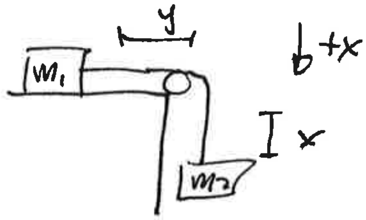
$$\omega^2 = \frac{k_1 + 2k_2 \pm \sqrt{k_1^2 + 4k_1 k_2 + 4k_2^2 - 4k_1 k_2}}{2m} = \frac{k_1 + 2k_2 \pm \sqrt{k_1^2 + 4k_2^2}}{2m}$$

check: $k_1 = k_2 \Rightarrow \omega^2 = \dfrac{3k \pm \sqrt{5}\,k}{m}$ as we had before

2 modes: 1) move together → →, ← ← lower ω

2) opposite → ←, ← → higher ω

4.



$\dot{y} = \dot{x}$ if rope is taut

$U = -mgx$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 g x$$

$$\frac{\partial \mathcal{L}}{\partial x} = m_2 g = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\left[(m_1 + m_2)\dot{x}\right] = (m_1 + m_2)\ddot{x}$$

$$\Rightarrow \ddot{x} = \frac{m_2 g}{m_1 + m_2}$$ in agreement w/ Newtonian result