University of Alabama
Department of Physics and Astronomy

PH 301 / LeClair Fall 2018

Problem Set 1
solutions

1. If $\mathbf{a}(t)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$ are functions of t, verify the following results:

$$\frac{d}{dt}\left[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\right] = \mathbf{a}\cdot\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right) + \mathbf{a}\cdot\left(\frac{d\mathbf{b}}{dt}\times c\right) + \frac{d\mathbf{a}}{dt}\cdot(\mathbf{b}\times\mathbf{c}) \tag{1}$$

$$\frac{d}{dt}\left[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\right] = \mathbf{a}\times\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right) + \mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times c\right) + \frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c}) \tag{2}$$

[*Hint:* Compute $\mathbf{b}\times\mathbf{c}$ first.]

Solution: Really just an application of the product rule, which works just fine with vector and scalar products:

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} \tag{3}$$

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt} \tag{4}$$

The generalization to a triple product is straightforward, and the desired result follows immediately.

2. Find the angle between a body diagonal of a cube and any one of its face diagonals. [*Hint:* Choose a unit cube with one corner at the origin and the opposite corner at point (1,1,1). Write down the vectors for the two diagonals and use the scalar product.]

Solution: The body diagonal is the vector $\mathbf{b} = (1,1,1)$. One face diagonal would be $\mathbf{f} = (1,1,0)$, it doesn't matter which one we pick. Their scalar product is

$$\mathbf{b}\cdot\mathbf{f} = (1,1,1)\cdot(1,1,0) = 2 = bf\cos\theta = \sqrt{3}\sqrt{2}\cos\theta \tag{5}$$

$$\cos\theta = \frac{2}{\sqrt{2}\sqrt{3}} = \sqrt{\frac{2}{3}} \qquad \theta \approx 35.3^\circ \tag{6}$$

3. **(a)** Prove that if $\mathbf{v}(t)$ is any vector that depends on time but which has constant *magnitude*, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. **(b)** Prove the converse that if $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$, then $|\mathbf{v}(t)|$ is constant. [*Hint:* Consider the derivative of $\mathbf{v}^2$.] This is a very handy result. It explains why, in two-dimensional polars, $d\mathbf{r}/dt$ has to be in the direction of $\hat{\boldsymbol{\varphi}}$ and vice versa. It also shows that the speed of a charged particle in a magnetic field is constant, since the acceleration is perpendicular to the velocity.

Solution: **(a)** If $|\mathbf{v}|$ is constant, then so is $\mathbf{v}^2$, and its time derivative must be zero.

$$\frac{d\mathbf{v}^2}{dt} = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{7}$$

Since the dot product is commutative, $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$,

$$\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 2\dot{\mathbf{v}}\cdot\mathbf{v} = 0 \tag{8}$$

$$\implies \quad \dot{\mathbf{v}}\cdot\mathbf{v} = 0 \tag{9}$$

Hence, $\mathbf{v}$ and $\dot{\mathbf{v}}$ must be orthogonal.

(b) Just work backwards. We know $\mathbf{v}\cdot\dot{\mathbf{v}} = 0$. Proceeding,

$$\mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{10}$$

$$2\mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{11}$$

$$2\mathbf{v}\cdot\dot{\mathbf{v}} = \dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{12}$$

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{d\mathbf{v}^2}{dt} = \dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}} = 0 \tag{13}$$

$$\implies \quad \mathbf{v}^2 = \text{const} \tag{14}$$

Since $\mathbf{v}^2 = |\mathbf{v}|^2$ is constant, clearly $|\mathbf{v}|$ is constant.

4. When a baseball flies through the air, the ratio $f_{\text{quad}}/f_{\text{lin}}$ of the quadratic to the linear drag force is given by

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{\gamma D}{\beta}v = \left(1.6\times 10^3\,\text{s/m}^2\right) Dv \tag{15}$$

Given that a baseball has a diameter of about 7 cm, find the approximate speed v at which the two drag forces are equally important. For what range of speeds is it safe to treat the drag force as purely quadratic? Under normal conditions is it a good approximation to ignore the linear term? Answer the same questions for a golf ball of diameter 4.3 cm.

Solution: For $f_{\text{quad}} = f_{\text{lin}}$ with $D = 7$ cm, we have $v = 0.9$ cm/s (about 0.15mph). Clearly the linear drag is negligible at any reasonable everyday speed.

A golf ball is roughly 1.5 times smaller than a baseball, meaning its critical speed will be about 1.5 times larger, around 1.4 cm/s. In both cases, linear drag is utterly negligible for all practical situations.

5. A projectile is launched with initial velocity $\mathbf{v}_i$ from the start of a ramp, with the ramp making an angle φ with respect to the horizontal. The projectile is launched with an angle $\theta > \varphi$ with respect to the horizontal. **(a)** At what position along the ramp does the projectile land? **(b)** What angle θ maximizes the distance the particle makes it along the ramp (your answer will be in terms of the angle φ? Note no numeric solution is required.

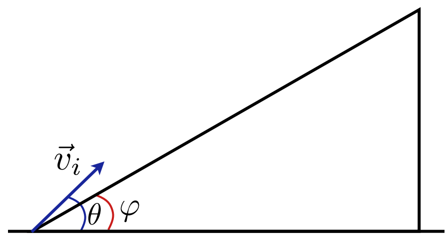


Figure 1: *A projectile is launched onto a ramp.*

Solution: Let the origin be at the projectile's launch position, with the x and y axes of a cartesian coordinate system aligned as shown below.

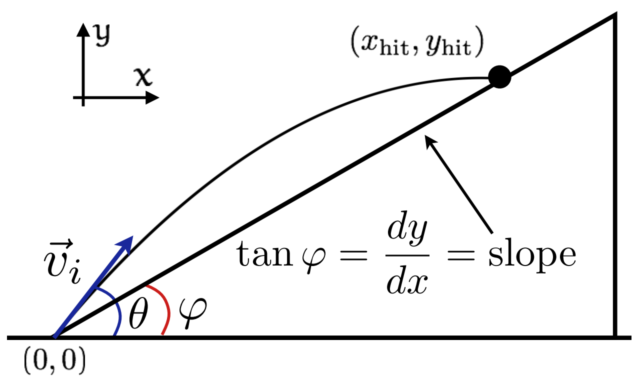


Figure 2: *Where does the projectile hit the ramp?*

Thus ramp begins at position $(0, 0)$, and the projectile is launched from $(0, 0)$. We seek the intersection of the projectile's trajectory with the surface of the ramp at position $(x_{\text{hit}}, y_{\text{hit}})$, subject to the conditions that $x_{\text{hit}}, y_{\text{hit}} \geq 0$ to have a sensible solution.

We already know the trajectory $y(x)$ for a projectile launched from the origin:

$$y_p = x\tan\theta - \frac{gx^2}{2|\mathbf{v}_i|^2\cos^2\theta} \tag{16}$$

The ramp itself can be described by as a line. We know the slope m of the ramp is $m = \Delta x/\Delta y =$

$\tan\varphi$, and we know it intersects the point $(x_o, y_o) = (0, 0)$. This is sufficient to derive an equation of the line describing the ramp's surface, $y_r(x)$, using point-slope form:

$$\begin{aligned} y_r - y_o &= m\,(x - x_o) \\ y_r &= (\tan\varphi)\,x \end{aligned}$$

The distance l the projectile goes along the ramp surface is found simply from $(x_{\text{hit}}, y_{\text{hit}})$ or x_{hit} and φ:

$$l = \sqrt{x_{\text{hit}}^2 + y_{\text{hit}}^2} \qquad \text{or} \qquad l = \frac{x}{\cos\varphi} \tag{17}$$

The point of intersection is $y_r = y_p$, resulting x value is the x_{hit} we desire.

$$y_r = x\tan\varphi = y_p = x\tan\theta - \frac{gx^2}{2|\mathbf{v}_i|^2\cos^2\theta} \qquad \text{note} \quad x = l\cos\varphi \tag{18}$$

$$l\cos\varphi\tan\varphi = l\cos\varphi\tan\theta - \frac{gl^2\cos^2\varphi}{2|\mathbf{v}_i|^2\cos^2\theta} \tag{19}$$

$$\tan\varphi = \tan\theta - \frac{gl\cos\varphi}{2|\mathbf{v}_i|^2\cos^2\theta} \tag{20}$$

$$l = \frac{2|\mathbf{v}_i|^2\cos^2\theta}{g\cos\varphi}(\tan\theta - \tan\varphi) = \frac{2|\mathbf{v}_i|^2\cos\theta}{g\cos\varphi}(\sin\theta - \tan\varphi\cos\theta) \tag{21}$$

$$l = \frac{2|\mathbf{v}_i|^2\cos\theta}{g\cos^2\varphi}(\sin\theta\cos\varphi - \sin\varphi\cos\theta) = \frac{2|\mathbf{v}_i|^2\cos\theta\sin(\theta-\varphi)}{g\cos^2\varphi} \tag{22}$$

As we should expect, the distance up the ramp depends on the relative angle between the ramp and launch, $\theta - \varphi$. We could have found this result a bit more quickly if we had noted the identity below (which we basically just derived).[i]

$$\tan\theta - \tan\varphi = \frac{\sin(\theta-\varphi)}{\cos\theta\cos\varphi} \tag{23}$$

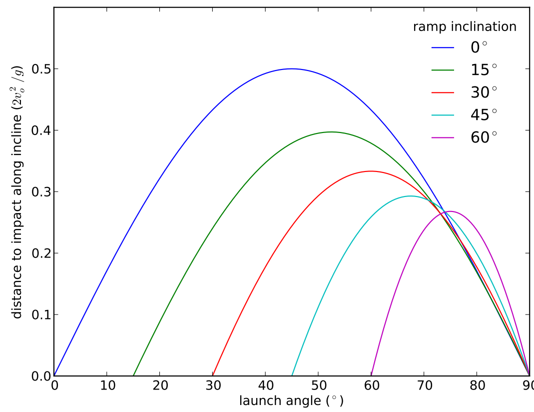
In order to maximize the distance up the ramp with respect to launch angle, we require $dl/d\theta = 0$. This is simple enough:

$$\frac{dl}{d\theta} = \frac{d}{d\theta}\frac{2|\mathbf{v}_i|^2\cos\theta\sin(\theta-\varphi)}{g\cos^2\varphi} = \frac{2|\mathbf{v}_i|^2}{g\cos^2\varphi}\cos(\varphi - 2\theta) = 0 \tag{24}$$

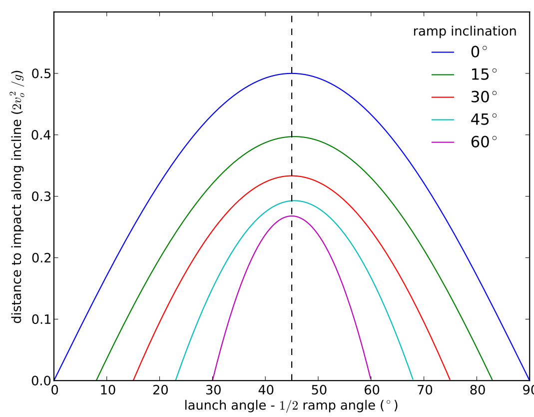
If we restrict ourselves to sensible ramp angles, $\varphi \in [0 - \pi/2]$, this implies $\varphi - 2\theta = \frac{\pi}{2}$. In the limiting case of no ramp ($\varphi = 0$, launch over flat ground), this predicts $\theta = \pi/4$ for maximum range, which we know to be correct. Below are plots of the distance along the incline as a function of launch

[i] $\tan x \pm \tan y = \frac{\sin x}{\cos x} \pm \frac{\sin y}{\cos y} = \frac{\sin x\cos y \pm \cos x\sin y}{\cos x\cos y} = \frac{\sin(x \pm y}{\cos x\cos y}$

angle for various ramp angles, with the distance in units of $2v_i^2/g$ for convenience:



Since the *maximum* distance for a given φ depends on $\varphi - 2\theta$, if we instead plot the distance l versus $\theta - \varphi/2$ all the curves should line up symmetrically about $\theta - \varphi/2 = \pi/4$. This is an important conclusion: just as the angle for maximum range over level ground was $\pi/4$, for maximum range up the ramp the launch angle minus half the ramp angle should be $\pi/4$. I've included the python code for generating the first plot at the end of this solution set in case you are curious.



Incidentally, if you solve the more general problem of launching the projectile a distance d before the start of the ramp (see picture below), this is the result:

$$l = \left[\frac{|\mathbf{v}_i|^2 \cos^2\theta}{g\cos\varphi}\right]\left[\left(\tan\theta - \tan\varphi\right) + \sqrt{(\tan\varphi - \tan\theta)^2 + \frac{2d\tan\varphi}{|\mathbf{v}_i|^2\cos^2\theta}}\right] - \frac{d}{\cos\varphi} \tag{25}$$

$$= \left[\frac{|\mathbf{v}_i|^2 \cos\theta}{g\cos^2\varphi}\right]\left[\sin(\theta - \varphi) + \sqrt{\sin^2(\theta - \varphi) + \frac{2d\sin\varphi\cos\varphi}{|\mathbf{v}_i|^2\cos^2\theta}}\right] - \frac{d}{\cos\varphi} \tag{26}$$

You can verify that for $d=0$ you recover our previous result. This solution presumes you have a launch velocity sufficient to reach the ramp in the first place. That requirement you can find by setting the projectile's range equal to d, and you find $|\mathbf{v}_i|^2\sin 2\theta \geq gd$.

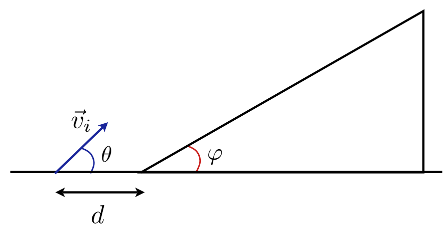


Figure 3: *Launching the projectile before the start of the ramp.*

6. The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. **(a)** Assuming the projectile has cross-sectional area A (normal to its velocity) and speed v, and that the density of the fluid is ρ, show that the rate at which the projectile encounters fluid (mass/time) is $\rho A v$. **(b)** Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is $\rho A v^2$. (It is certainly not true that all the fluid the projectile encounters is accelerated to the full speed v, and one might guess the actual force has a correction factor $\kappa < 1$, so that $f_{\text{quad}} = \kappa \rho A v^2$, with κ depending on the shape of the body, smaller for more streamlined objects).

Solution: In a time Δt, our cross-sectional area A crosses a distance $v\Delta t$, meaning it has swept out a volume Avt. Assuming a constant fluid density, this implies the mass encountered is $\Delta m = \rho A v \Delta t$. The rate is then

$$\dot{m} = \frac{\Delta m}{\Delta t} = \frac{\rho A v \Delta t}{\Delta t} = \rho A v \tag{27}$$

If the mass encountered is Δm, then accelerating it to velocity v implies the fluid gains a momentum $(\Delta m)v$, and conservation of momentum dictates that the object must change its momentum by $-(\Delta m)v$. The time rate of change of this momentum is the drag force.

$$\Delta p = (\Delta m)v \tag{28}$$

$$F = \frac{\Delta p}{\Delta t} = v\frac{\Delta m}{\Delta t} = \rho A v^2 \tag{29}$$

7. Problem 2.7 from your textbook is about a class of 1-D problems that can always be reduced to doing an integral. Specifically, if F is a function of v alone ($F = F(v)$), then you can show

$$t = m \int_{v_o}^{v} \frac{dv'}{F(v')} \tag{30}$$

Here is another class of problems. Show that if the net force on a 1-D particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by

$$v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,dx' \tag{31}$$

[*Hint:* Use the chain rule to prove the following handy relationship (the "$v\,dv/dx$ rule"): if you regard v as a function of x, then $\dot{v} = v dv/dx = (1/2)dv^2/dx$. Use this to rewrite Newton's second law in the separated form $md(v^2) = 2F(x)\,dx$ and then integrate from x_o to x.] Comment on the result for the case that $F(x)$ is actually a constant. (You may also recognize your solution as a statement about kinetic energy and work.)

Solution: Start with our rule:

$$a = \dot{v} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\dot{x} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx} \tag{32}$$

Now separate variables, multiply by $m/2$ to recover recognizable quantities, and integrate.

$$2a\,dx = dv^2 \tag{33}$$

$$\frac{1}{2}m\,dv^2 = ma\,dx = F\,dx \tag{34}$$

$$\int_{v_o}^{v} \frac{1}{2}m\,d(v')^2 = \int_{x_o}^{x} F(x')\,dx' \tag{35}$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_o^2 = \int_{x_o}^{x} F(x')\,dx' \qquad \text{or} \qquad v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,dx' \tag{36}$$

The first expression in the last line is a statement of the work-energy theorem. If F is constant,

$$v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,dx' = v_o^2 + \frac{2F}{m}(x - x_o) \tag{37}$$

Using $F = ma$, we recover the familiar result from introductory mechanics $v^2 = v_o^2 + 2a(x - x_o)$.

8. Using the result of the previous problem, consider a mass m constrained to move on the x axis and subject to a force $F = -kx$, where k is a positive constant. The mass is released from rest at $x = x_o$ at time $t = 0$. First find the speed using equation 31, and then using $v = dx/dt$, separate the equation and integrate. You should recognize this as one way – not the easiest – to solve the simple harmonic oscillator.

Solution: Let $F = -kx$.

$$v^2 = v_o^2 + \frac{2}{m}\int\limits_{x_o}^{x} F(x')\,dx' = v_o^2 + \frac{2}{m}\int\limits_{x_o}^{x} -kx'\,dx' = v_o^2 + \frac{2}{m}\left(-\frac{1}{2}kx^2 + \frac{1}{2}kx_o^2\right) \tag{38}$$

$$v^2 = v_o^2 + \frac{k}{m}\left(x_o^2 - x^2\right) \qquad \text{note } v_o = 0 \tag{39}$$

$$v^2 = \frac{k}{m}\left(x_o^2 - x^2\right) \qquad \text{or} \qquad v = \dot{x} = \sqrt{\frac{k}{m}\left(x_o^2 - x^2\right)} = \pm\omega\sqrt{x_o^2 - x^2} \tag{40}$$

Now we separate variables and integrate:

$$\frac{dx}{dt} = \pm\omega\sqrt{x_o^2 - x^2} \tag{41}$$

$$\omega\,dt = \frac{dx}{\sqrt{x_o^2 - x^2}} \tag{42}$$

$$\int\limits_0^t \omega\,dt = \int\limits_{x_o}^{x} \frac{dx'}{\sqrt{x_o^2 - (x')^2}} \tag{43}$$

We can proceed in 2 ways: trig substitution, or brute force. First, trig substitution, where we let $x = x_o \sin\theta$ and $dx = x_o \cos\theta\,d\theta$.

$$\int\limits_0^t \omega\,dt = \omega t = \int\limits_{x_o} x\frac{dx'}{\sqrt{x_o^2 - (x')^2}} = \int\limits_{x_o}^{x} \frac{\cos\theta\,d\theta}{\sqrt{1 - \sin^2\theta}} = \int\limits_{x_o}^{x} d\theta \tag{44}$$

$$\omega t = \theta\Big|_{x_o}^{x} = \sin^{-1}\left(\frac{x}{x_o}\right)\Big|_{x_o}^{x} = \sin^{-1}\left(\frac{x}{x_o}\right) - \sin^{-1} 1 = \sin^{-1}\left(\frac{x}{x_o}\right) - \frac{\pi}{2} \tag{45}$$

$$x = x_o \sin\left(\omega t + \frac{\pi}{2}\right) = x_o \cos\omega t \tag{46}$$

And the brute force method, in which we just look up the integral:

$$\omega t = \int\limits_{x_o}^{x} \frac{dx'}{\sqrt{x_o^2 - (x')^2}} = \sin^{-1}\left(\frac{x}{x_o}\right)\Big|_{x_o}^{x} = \sin^{-1}\left(\frac{x}{x_o}\right) - \frac{\pi}{2} \tag{47}$$

$$\sin\left(\omega t + \frac{\pi}{2}\right) = \cos\omega t = \frac{x}{x_o} \tag{48}$$

$$x = x_o \cos\omega t \tag{49}$$

9. A mass m has speed v_o at the origin and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the "$v\,dv/dx$ rule" above to write the equation of motion in separated form $md(v^2) = 2F(x)\,dx$, and then integrate both sides to give x in terms of v (or vice versa). Show that it will eventually travel a distance $2m\sqrt{v_o}/c$.

Solution: Starting with our favorite rule,

$$m\ddot{x} = m\dot{v} = \frac{1}{2}m\frac{dv^2}{dx} = -cv^{3/2} \tag{50}$$

$$dx = -\frac{m\,dv^2}{2cv^{3/2}} = -\frac{m\,dv^2}{2c(v^2)^{3/4}} \tag{51}$$

$$\int_0^x dx' = x = \int_{v_o}^{v} -\frac{m\,d(v')^2}{2c[(v')^2]^{3/4}} = \frac{-m}{2c}4(v^2)^{\frac{1}{4}}\Bigg|_{v_o}^{v} = \frac{2m}{c}\left(\sqrt{v_o}-\sqrt{v}\right) = x(v) \tag{52}$$

The net distance traveled is after the particle finally comes to rest and $v=0$: $x_{\text{net}} = \frac{2m\sqrt{v_o}}{c}$.

10. A basketball has mass $m = 0.6\,\text{kg}$ and diameter $D = 0.24\,\text{m}$. **(a)** What is its terminal speed? (See example 2.5 ...) **(b)** If it is dropped from a 30-m tower, how long does it take to hit the ground and how fast is it going when it does so? [*Hint:* review the "vertical motion with quadratic drag" section.] Compare with the corresponding numbers in a vacuum

Solution: From equation (2.59) in the text, and noting $\gamma = 0.25\,\text{Ns}^2/\text{m}^4$,

$$v_t = \sqrt{\frac{mg}{\gamma D^2}} \approx 20.2\,\text{m/s} \tag{53}$$

From (2.58) in the text, with $y = 30\,\text{m}$

$$y = \frac{v_t^2}{g}\ln\left[\cosh\left(\frac{gt}{v_t}\right)\right] \tag{54}$$

$$e^{\frac{gy}{v_t^2}} = \cosh\left(\frac{gt}{v_t}\right) \tag{55}$$

$$t = \frac{v_t}{g}\cosh^{-1}\left(e^{\frac{gy}{v_t^2}}\right) \approx 2.78\,\text{s} \tag{56}$$

From (2.57) in the text,

$$v = v_t\tanh\left(\frac{gt}{v_t}\right) \approx 17.65\,\text{m/s} \tag{57}$$

In vacuum, one expects $v = \sqrt{2gy} \approx 24.3\,\text{m/s}$ and $t = \sqrt{2y/g} \approx 2.47\,\text{s}$.

11. [Bonus Question; Computer] The differential equation for the skateboard in example 2, $\ddot{\varphi} = -\frac{g}{R}\sin\varphi$ cannot be solved in terms of elementary functions, but is easily solved numerically. **(a)** Do this for the case $\varphi_o = 20°$, with $R = 5\,\text{m}$ and $g = 9.8\,\text{m/s}^2$. Make a plot of $\varphi(t)$ for two or three periods. **(b)** On the same plot, show the approximate solution given by (1.57) with the same $\varphi_o = 20°$. Comment on the two graphs. **(c)** Repeat for $\varphi_o = \pi/2$.

Solution: It is fairly easy to solve the problem in Mathematica, which UA students have free access to. Below is a screenshot of a Mathematic notebook to perform the calculation, followed by the plots generated for the two initial angles. For $\varphi_o = 20°$, the agreement is essentially perfect for the first two cycles. For $\varphi_o = \pi/2$, in spite of how large the initial angle is, the small-angle approximation does pretty well for short times. The approximation oscillates too quickly, as one would expect - for large amplitudes the true period is a little longer. I'm also attaching some example Python code to do the same job using the built-in ODE solvers. (The ODE solvers only handle first-order equations, so one has to recast the given second-order equation into two first-order equations.)

```
In[52]:= g = 9.81;
         R = 5.0;
         ϕ0 = π / 9

Out[54]= π/9

In[55]:= eqs = {ϕ''[t] == -(g / R) * Sin[ϕ[t]], ϕ'[0] == 0, ϕ[0] == ϕ0}

Out[55]= {ϕ''[t] == -1.962 Sin[ϕ[t]], ϕ'[0] == 0, ϕ[0] == π/9}

In[56]:= Q = NDSolve[eqs, ϕ, {t, 0, 15}]

Out[56]= {{ϕ → InterpolatingFunction[ Domain: {{0., 15.}} Output: scalar ]}}

In[57]:= Plot[{Evaluate[ϕ[t]] /. Q, ϕ0 Cos[t * Sqrt[g / R]]}, {t, 0, 12}, PlotStyle → Thick,
          Frame → True, FrameLabel → {"time", "ϕ(t)  (rad)"}, RotateLabel → True,
          PlotLegends → {"Exact", "Approximate"}]
```

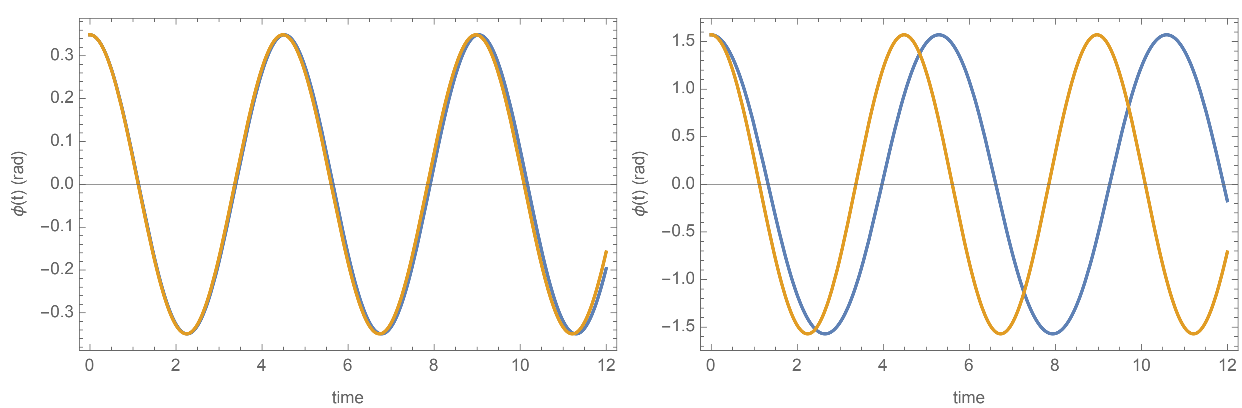


Figure 4: *Exact and approximate solutions to* $\ddot{\varphi} = -\frac{g}{R}\sin\varphi$ *for a starting angle of* **(left)** $\varphi_o = 20°$ *and* **(right)** $\varphi_o = \pi/2$. *Exact solutions are in blue, the analytic approximation is in orange.*

```
from scipy.integrate import odeint
import matplotlib.pyplot as plt
import numpy as np

#Taylor Ch. 2 example 2
#the solution for a particle in a spherical bowl is identical to that of a pendulum

#see https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.odeint.html

# our equation, if we include a damping term, is
# theta''(t) + b*theta'(t) + c*sin(theta(t)) = 0
# where b and c are positive constants, and a prime (') denotes a derivative. To solve
#this second order equation with odeint, we must first convert it to a system of first
#order equations. By defining the angular velocity omega(t) = theta'(t), we obtain the
#system:

# theta'(t) = omega(t)
# omega'(t) = -b*omega(t) - c*sin(theta(t))
# let y = vector [theta,omega]

#b is the damping parameter [damping proportional to angular velocity], c is the square of
#the angular frequency, (g/L) for a pendulum or (g/R) for the bowl

def pendulum(y, t, b, c):
        theta, omega = y
        dydt = [omega, -b*omega - c*np.sin(theta)]
        return dydt

g = 9.81
L = 5.0

b = 0.0 #damping
c = g/L  #g/R

#initial conditions: start at angle theta_o at rest and release
theta_o = np.pi/2
y0 = [theta_o, 0.0]  #=[theta,theta']

#simulate over 3 periods on a grid with 100 points per period
T = 3*(2*np.pi/np.sqrt(g/L))
t = np.linspace(0, T, 100*T+1)

sol = odeint(pendulum, y0, t, args=(b, c))

#The solution is an array with shape (10*T+1, 2). The first column is theta(t), and the
#second is omega(t). The following code plots both components and the usual small angle
#solution

plt.plot(t, sol[:, 0], 'b', label='theta(t)')
#plt.plot(t, sol[:, 1], 'g', label='omega(t)')
plt.plot(t,(theta_o)*np.cos(t*np.sqrt(g/L)),'r',label='small angle approx')
plt.legend(loc='upper right')
plt.xlabel('t')
plt.ylim([-1.5*theta_o,1.5*theta_o])
plt.grid()
plt.show()
```

Appendix: Python code for projectile-ramp plot

```python
import math
import numpy as np
import matplotlib.pyplot as plt

p = [0, 15, 30, 45, 60]      # just simulate for a few interesting phi values
labels = ['0$^\circ$', '15$^\circ$', '30$^\circ$', '45$^\circ$', '60$^\circ$']
colors = ['r', 'g', 'b', 'k', 'o']

def x(t,p):
    return (np.cos(t)*np.sin(t-p))/(np.cos(p))**2
    #for technical reasons, use the sin() and cos() defined by numpy, not math
    #the latter can't handle array inputs, the former can.


#specify a range of launch angles for the plot
t = np.arange(0, 90, 0.1)
plt.axis([0.0,90, 0.0,0.6])

ax = plt.gca()
ax.set_autoscale_on(False)

for i in range(len(p)) :
        plt.plot(t, x(np.radians(t),np.radians(p[i])),label=labels[i])


plt.xlabel('launch angle ($^\circ$)')
plt.ylabel('distance to impact along incline ($2v_o^2/g$)')
plt.legend(title="ramp inclination",frameon=0)

#plt.show()  #write to screen
plt.savefig('projectile-ramp-plots.pdf',bbox_inches='tight') #write to file
```